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MODULATIONAL INSTABILITIES AND LANDAU DAMPING WITHIN THE THERMAL WAVE MODEL DESCRIPTION OF HIGH–ENERGY CHARGED–PARTICLE BEAM DYNAMICS

D. Anderson[1], R. Fedele[2], V.G. Vaccaro[2],

M. Lisak[1], A. Berntson[1], S. Johansson[1]

[1]*Department of Electromagnetics, Chalmers University of Technology, S-41296 Göteborg, Sweden*

[2]*Dipartimento di Scienze Fisiche, Università di Napoli "Federico II" and INFN Sezione di Napoli, Complesso Universitario di M.S. Angelo, Via Cintia, I-80126 Napoli, Italy*

Abstract

Within the framework of the thermal wave model (TWM), it is shown that the longitudinal coherent instability in the presence of a non-negligible resistive part of the coupling impedance can be described in terms of a modulational instability associated with the nonlinear Schrödinger equation (NLSE). Furthermore, by using the Wigner transform to carry out the analysis in phase-space, the role of Landau damping is considered in connection with the above instability, showing that TWM is capable of reproducing all the results of the conventional theory of the coherent instability as well as of predicting new results (in particular, the possible existence of a *quantum-like Landau damping*), connected with the crucial role of thermal noise introduced by the emittance in the resonance condition between waves and particles in the beam.

1 Introduction

Nonlinear collective effects that take place in particle beam dynamics constitute, at the present time, a very large body of the accelerator phenomenology [1, 2] . In particular, coherent instabilities have already been recognized to be very important for the practical design of accelerating machines in their pioneering investigations [3] . Today, accelerator physics provides for a very well established theory and very powerful methodologies for describing coherent instabilities [4] as well as designing the machines (f.i., see [5]). In this context, very important subjects of both plasma theory (such as Landau damping [6] and plasma wave instabilities [7, 8]) and control system theory (such as Nyquist diagrams [2, 9, 10]) are regularly applied.

Furthermore, recently discovered new phenomena, deeply connected with nonlinear collective phenomenology of particle beams [11, 12] , represent new important insights to be included in the above theoretical descriptions.

On the other hand, recently, new approaches for describing the nonlinear collective particle beam dynamics in this more extended way have been proposed [13, 14, 15, 16] , and it seems that, in principle, they may include the above new phenomena. In particular, the Thermal Wave Model (TWM) [17] , which basically is a *quantum-like* description of the *classical* charged-particle beam dynamics, seems to be suitable for correctly describing coherent instabilities [16, 18] but which also yields new insights and predictions that go beyond the present conventional theory.

TWM has been formulated in a way which is fully similar to the one that Gloge and Marcuse [19] used to *transit* from electromagnetic optics to wave optics. In fact, using the classical correspondence between electromagnetic optics, electron optics, and quantum mechanics, the TWM formulation has been developed to *transit* from *geometrical electron optics* [20] to a *wave-like* (or *quantum-like*) electron optics, and has been applied to a number of problems of beam transport and dynamics in both conventional and plasma-based accelerators [16, 18, 21, 22, 23] . In the TWM description, the beam properties are described by a complex valued beam wave function (BWF) which satisfies a Schrödinger-like evolution equation where the beam emittance plays the role of Planck's constant. The square modulus of the BWF represents the beam density profile.

The beam instabilities occurring (for certain combinations of parameters) when the coupling impedance between the beam and its surroundings is purely reactive, has been recovered within the TWM approach as the classical modulational instability [16, 18] of the corresponding nonlinear Schrödinger equation (NLSE).

However, the more general case, when the resistive part of the coupling impedance becomes important, has not yet been analyzed within the framework of the TWM.

In this paper, we use the TWM to give a quantum-like description of the longitudinal coherent instabilities, taking into account the resistive part of the coupling impedance. According to the TWM, this is done assuming an evolution equation for the BWF which is a sort of generalized NLSE including a nonlinear integral term in addition to the classical nonlinear cubic one. In fact, the reactive part of the coupling impedance corresponds to a nonlinear potential term in the above Schrödinger-like equation which is proportional to the square modulus of BWF (i.e. the cubic nonlinearity of the NLSE) [16] , whilst the resistive part corresponds to a potential term which is proportional to the integral of the squared modulus of BWF. We show that coherent instabilities can be described in terms of the modulational instability also in the case of non negligible resistive part of the coupling impedance. The results obtained in the present work are in full agreement with previous results for the coherent instability based on conventional approaches and provide for further proof of the usefulness and consistency of the TWM approach but also give new insight which may be connected with recent new phenomena in particle beam dynamics [11, 12] .

In Section 2, the formulation of the problem that we want to solve in this paper is presented. In particular, we present the appropriate NLSE to be used for the instability analysis. In fact, in Section 3, the analysis of this equation is carried out in the conventional way i.e. first the appropriate stationary solution is found. A linear equation for small perturbations is then derived and it is shown that in the presence of a resistive part of the impedance, the perturbations are always unstable, in contrast to the purely reactive case where the perturbations are unstable only in certain parameter regimes. The instability results in the general case are summarized in a conventional form given as the contour plots in the (Z_r, Z_i) plane for constant instability growth rate, where Z_r and Z_i are the inductive and resistive parts, respectively, of the coupling impedance. In Section 4, the above analysis is also generalized to include the effects of a finite longitudinal width of the beam (bunched beams). This effect, which has not previously been consistently analyzed within the TWM formalism, is demonstrated to give rise to a stabilizing effect on the instability. The role of Landau damping is also considered, pointing out the difficulty to derive this effect, in the configuration space, by using the present mathematical methods suitable to describe the modulational instability of the NLSE. However, this difficulty is overcome in Section 5, where a transition to the phase space is performed by means of the Wigner transform. The phase-space analysis, fully equivalent to the one given by NLSE clearly show, in agreement with the conventional description, the existence of Landau damping and its stabilizing effect against the coherent (i.e. modulational) instability. Additionally, it is shown that there exists a *quantum-like effect* in the interaction between particles and collective modes in the beam which does not occur in the conventional description. This effect is due to the crucial role played by the thermal noise (through the emittance) in affecting the resonance condition in the wave-particle interaction leading to the Landau damping. Finally, the conclusions are summarized in Section 6.

2 The nonlinear Schrödinger equation for an arbitrary coupling impedance

Let us consider a charged-particle beam travelling in a circular accelerating machine. Neglecting the transverse dynamics as well as the radiation damping effects, the longitudinal evolution of the BWF, $\Psi(x,s)$, according to the TWM [17] , can in general be written as [16] :

$$i\epsilon\frac{\partial\Psi}{\partial s} = \frac{1}{2}\frac{\eta}{\beta^2}\epsilon^2\frac{\partial^2\Psi}{\partial x^2} + \frac{\Psi}{(E_0/q)\beta cT_0}\int_0^x \mathcal{U}(x's)\,dx' \quad , \tag{1}$$

where $\mathcal{U}(x,s)$ denotes the self consistent voltage describing in general the interaction of the beam with the surroundings (other notation is standard, see [2, 4, 24]) .

The voltage, $\mathcal{U}(x,s)$, is related to the charge line density, $\lambda(x,s)$, according to

$$\mathcal{U}(x,s) = e\beta cZ_r\lambda(x,s) + e\beta cE_0\frac{Z_i}{n}\frac{\partial\lambda}{\partial x} \quad , \tag{2}$$

and the system is self-consistently closed by the relation

$$\lambda(x,s) \;=\; \frac{N}{2\pi R_0}|\Psi(x,s)|^2 \quad , \tag{3}$$

(R_0 being the radius of the synchronous particle orbit). Thus, using Eq.s (1)-(3), we can write the equation for the BWF in the form:

$$i\frac{\partial\Psi}{\partial s} \;=\; \alpha\frac{\partial^2\Psi}{\partial x^2} \;+\; \kappa|\Psi|^2\Psi \;+\; \mu\;\Psi\;\int_0^x\;|\Psi(x',s)|^2\;dx' \quad , \tag{4}$$

where the coefficients in Eq.n (4) are given by

$$\alpha \;=\; \frac{\eta\epsilon}{\beta^2} \quad , \tag{5}$$

$$\kappa \;=\; \frac{q^2N}{2\pi\epsilon E_0T_0}\frac{Z_i}{n} \quad , \tag{6}$$

and

$$\mu \;=\; \frac{q^2N}{2\pi\epsilon E_0T_0R_0}\;Z_r \quad . \tag{7}$$

Our problem consists in making a stability (instability) analysis, in order to show that the modulational instability associated with the solutions of (4) coincides with the coherent instability, extending in this way what has been done in previous papers [16, 18] .

3 Instability analysis and its comparison with conventional approaches

According to the previous Section, in the case of $\mu = 0$, Eq.n (4) has already been investigated for the stability of small perturbations on a stationary background solution. Thus, we directly consider here the case $\mu \neq 0$. In this case (4) includes the resistive integral nonlinearity.

(i). It is easy to show that a stationary CW-solution of Eq.n (4) exists of the form

$$\Psi(x,s) \;=\; \Psi_s(x,s) \;=\; \Psi_0\;e^{i\phi_0(x,s)} \quad , \tag{8}$$

where the phase $\phi_0(x,s)$ varies as

$$\Phi_0(x,s) \;=\; \lambda_1 s \;+\; \lambda_3 s^3 \;+\; \nu x s \quad , \tag{9}$$

with

$$\lambda_1 = -\kappa\Psi_0^2, \;\; \lambda_3 = -\mu\Psi_0^2, \;\; \text{and} \;\; \nu = \frac{1}{3}\alpha\mu^2 = \frac{1}{3}\alpha\mu^2\Psi_0^4 \quad . \tag{10}$$

(ii). The dynamics of small perturbations on the stationary solution is investigated by writing

$$\Psi(x,s) \;=\; (\Psi_0 \;+\; \delta\Psi(x,s))\;e^{i\phi_0(x,s)} \quad . \tag{11}$$

Separating k into real and imaginary parts, according to $k = k_r + ik_i$, we obtain from Eq.n (23)

$$k_r^2 - k_i^2 = \left(\alpha\Omega^2\right)^2 \left(1 - \frac{2\kappa\Psi_0^2}{\alpha\Omega^2}\right) \quad , \tag{24}$$

and

$$k_r\, k_i = -\left(\alpha\Omega^2\right)^2 \frac{2\mu}{\alpha\Omega^3}\Psi_0^2 \quad , \tag{25}$$

from which we can solve for k_i to obtain

$$k_i^2 = \frac{\left(\alpha\Omega^2\right)^2}{2}\left[-\left(1 - \frac{2\kappa\Psi_0^2}{\alpha\Omega^2}\right) + \sqrt{\left(1 - \frac{2\kappa\Psi_0^2}{\alpha\Omega^2}\right)^2 + \left(\frac{4b\mu\Psi_0^2}{\alpha\Omega^3}\right)^2}\,\right] \quad . \tag{26}$$

We note that when $\mu = 0$, Eq.n (26) reduces to

$$k_i^2 = \frac{\left(\alpha\Omega^2\right)^2}{2}\left(\frac{2\kappa\Psi_0^2}{\alpha\Omega^2} - 1\right) \quad \text{when} \quad \frac{2\kappa\Psi_0^2}{\alpha\Omega^2} > 1 \quad , \tag{27}$$

and

$$k_i^2 = 0 \quad \text{when} \quad \frac{2\kappa\Psi_0^2}{\alpha\Omega^2} < 1 \quad , \tag{28}$$

as it should [16] . It follows directly from Eq.n (26) that $k_i^2 > 0$, *whenever* $\mu \neq 0$, *i.e. the CW-solution is always unstable when resistive effects are included.* This is in full agreement with the result obtained by the conventional approach for a coasting beam, see e.g. [16] .

Furthermore, if we consider the instability growth rate, k_i, as given, we can view Eq.n (26) as a relation between μ and κ, or equivalently, between Z_i and Z_r. Since $\kappa \sim Z_i$ and $\mu \sim Z_r$, it is convenient to introduce in Eq.n (26) the normalizations

$$\overline{Z_i} = \frac{2\kappa\Psi_0^2}{\alpha\Omega^2} \sim \frac{Z_i}{\eta n} \quad , \tag{29}$$

$$\overline{Z_r} = \frac{\mu\Psi_0^2}{\alpha\Omega^3} \sim \frac{Z_r}{\eta n} \quad , \tag{30}$$

and

$$\overline{k_i} = \frac{k_i}{\alpha\Omega^2} \quad , \tag{31}$$

Eq.n (26) can then be rewritten in the lucid form:

$$\overline{Z_i} = 1 + \overline{k_i}^2 - \frac{1}{4\overline{k_i}^2}\overline{Z_r}^2 \quad . \tag{32}$$

This implies that the curves for constant instability growth rate are parabolas in the ($\overline{Z_r}$, $\overline{Z_i}$) (or equivalently (Z_r, Z_i)) space.

For a comparison with results found for the coherent instability obtained by conventional techniques, it is convenient to rewrite the dispersion relation obtained by the TWM approach, Eq.n (23), as

$$k^2 = k_0^2 + k_1^2 \quad , \tag{33}$$

where

$$k_0^2 \;=\; \left(\alpha\Omega^2\right)^2 \quad , \tag{34}$$

and

$$k_1^2 \;=\; -2\alpha\Omega^2\left(\kappa \;+\; i\frac{\mu}{\Omega}\right)\Psi_0^2 \quad . \tag{35}$$

Using the fact that

$$\kappa \;+\; i\frac{\mu}{\Omega} \;=\; \frac{Nq^2}{2\pi\epsilon E_0 T_0 n}\left(Z_i \;+\; i\frac{n}{\Omega R_0}Z_r\right) \quad , \tag{36}$$

and noting that [2, 4] $\Omega = -n/R_0$, $\omega_0 = 2\pi/T_0$, and $R_0 = \beta c/\omega_0$, we can rewrite k_1^2 as

$$k_1^2 \;=\; \left(\frac{n\omega_0}{\beta c}\right)^2 \frac{qI_0}{2\pi\beta^2 E_0}\, i\frac{\eta Z}{n} \quad . \tag{37}$$

In the high current limit, when $k_1^2 \gg k_0^2$ (as studied in [16, 18]) , the dispersion relation, Eq.n (33), reduces to

$$k^2 \;\approx\; k_1^2 \quad , \tag{38}$$

which is identical to the result obtained by conventional techniques [2, 4] .

The present analysis includes the stabilizing influence of the linear dispersion on the modulational instability. However, in the appropriate limit of a high current beam, the predictions of the TWM approach and conventional approaches agree completely, not only for the case of a purely reactive impedance as shown previously [16] , but also in the general case when the impedance contains a resistive part as well as a reactive part. This result provides further proof that the TWM approach is a convenient and alternative description of the dynamics of high energy charged-particle beams in accelerators.

4 Effects of finite beam energy spread on the modulational instability

When analyzing the effect of finite beam energy spread on the coherent modulational instability, it is more convenient to use a slightly different approach. Instead of the approach used in Section 4, we will start by separating $\Psi(x,s)$ into real amplitude and phase according to:

$$\Psi(x,s) \;=\; A(x,s)\, e^{i\Theta(x,s)} \quad . \tag{39}$$

Inserting this ansatz into Eq.n (4), and separating real and imaginary parts, one obtains the following system for A and Θ:

$$A_s \;=\; \alpha\left(2A_x\Theta_x \;+\; A\Theta_{xx}\right) \quad , \tag{40}$$

$$-\Theta_s \;=\; \alpha\left(\frac{A_{xx}}{A} \;-\; \Theta_x^2\right) \;+\; \kappa A^2 \;+\; \mu\int_0^x A^2\, dx' \quad . \tag{41}$$

In this case, the zero-order solutions for A_0 and Θ_0 are determined by

$$A_{0s} \;=\; \alpha\left(2A_{0x}\Theta_{0x} \;+\; A_0\Theta_{0xx}\right) \quad , \tag{42}$$

$$-\Theta_{0s} \;=\; \alpha\left(\frac{A_{0xx}}{A_0} \;-\; \Theta_{0x}^2\right) \;+\; \kappa A_0^2 \;+\; \mu \int_0^x A_0^2 \, dx' \quad . \tag{43}$$

In the CW-case, $A_0 = constant \Rightarrow A_{0x} = A_{0xx} = A_{0s} = 0$ and $\Theta_0(x,s)$ must satisfy simultaneously two equations, viz

$$\Theta_{0xx} \;=\; 0 \quad , \tag{44}$$

$$-\Theta_{0s} \;=\; -\alpha\Theta_{0x}^2 \;+\; \kappa A_0^2 \;+\; \mu \; x \; A_0^2 \, dx' \quad . \tag{45}$$

Integrating Eq.n (44) we obtain

$$\Theta_0(x,s) \;=\; \lambda_1 s \;+\; \lambda_3 s^3 \;+\; \nu x s \quad , \tag{46}$$

where λ_1, λ_3, and ν are given by Eq.s (10). The equation for the perturbations read

$$\delta A_s \;=\; \alpha\,(2\delta A_x \Theta_{0x} \;+\; A_0 \delta\Theta_{xx}) \quad , \tag{47}$$

$$-\delta\Theta_s \;=\; \alpha\left(\frac{\delta A_{xx}}{A_0} \;-\; 2\Theta_{0x}\delta\Theta_x\right) \;+\; 2\kappa A_0 \delta A \;+\; 2\mu A_0 \int_0^x \delta A \, dx' \quad . \tag{48}$$

Since $\Theta_{0x} = \nu s$, we can assume

$$\delta A \;\;,\;\; \delta\Theta \;\sim\; e^{-i\Omega x} \quad , \tag{49}$$

which implies that Eq.s (47) and (48) can be written as

$$\delta A_s \;=\; -\alpha\left(2is\nu\Omega\delta A \;+\; A_0\Omega^2\delta\Theta\right) \quad , \tag{50}$$

$$-\delta\Theta_s \;=\; \alpha\left(-\frac{\Omega^2}{A_0}\delta A \;-\; 2i\nu s\Omega\delta\Theta\right) \;+\; 2\kappa A_0 \delta A \;+\; 2i\,\frac{\mu A_0}{\Omega}\,\delta A \quad . \tag{51}$$

The terms proportional to $2is\nu\Omega\alpha$ can be transformed away, cf Eq.s (20), and the remaining system becomes

$$\delta A_s \;=\; -\alpha A_0 \Omega^2 \delta\Theta \quad , \tag{52}$$

$$-\delta\Theta_s \;=\; \left(-\frac{\alpha\Omega^2}{A_0} \;+\; 2\kappa A_0 \;+\; 2i\,\frac{\mu A_0}{\Omega}\right)\,\delta A \quad , \tag{53}$$

and finally assuming s-variation according to $\exp(iks)$ we obtain the dispersion relation:

$$k^2 \;=\; \alpha\Omega^2\left(\frac{\alpha\Omega^2}{A_0} \;-\; 2\kappa A_0^2 \;-\; 2i\,\frac{\mu A_0^2}{\Omega}\right) \quad , \tag{54}$$

i.e. the same result as in Eq.n (23). In the case of finite beam energy spread, new terms appear in the linearized equations for δA and $\delta\Theta$, Eq.ns (50) and (51). One stabilizing effect is proportional to $A_{0xx}/A_0 = -F/a^2$, where a is the characteristic longitudinal width of the beam and F is a form factor of order unity) which depends on the actual longitudinal density profile. Including this term, the dispersion relation becomes:

$$k^2 \;=\; \alpha^2\Omega^2\left(\Omega^2 \;+\; \frac{F}{a^2} \;-\; 2\frac{\kappa}{\alpha}A_0^2 \;-\; 2i\,\frac{\mu A_0^2}{\Omega}\right) \quad . \tag{55}$$

Eq.n (55) can be written in the form as Eq.n (33), i.e. as

$$k^2 \;=\; k_0^2 \;+\; k_1^2 \quad , \tag{56}$$

where k_1^2 is defined as before, Eq.n (35), but where k_0^2 now includes a contribution from the beam energy spread and is given by

$$k^2 \;=\; \left(\alpha^2\Omega^2\right)^2 \left(1 \;+\; \frac{F}{a^2\Omega^2}\right) \quad . \tag{57}$$

This implies that the curves in the (Z_r, Z_i) plane corresponding to constant growth rate, k_i, still are parabolas, viz

$$\overline{Z_i} \;=\; \Gamma^2 \;+\; \overline{k_i}^2 \;-\; \frac{1}{4\overline{k_i}^2}\overline{Z_r}^2 \quad , \tag{58}$$

but where however now

$$\Gamma^2 \;=\; 1 \;+\; \frac{F}{a^2\Omega^2} \quad . \tag{59}$$

The result expressed by Eq.s (58) and (59) which implies that the finite longitudinal length of the beam provides a stabilizing effect on the instability which manifests itself by an upward shift of the parabolic level curves, thus extending the stability region upward along the $\overline{Z_i}$ axis.

However, contrary to the results reviewed in [16, 18] , the level curves remain parabolic and no two-dimensional stability region appears around the origin in the $(\overline{Z_r}, \overline{Z_i})$ plane. The reason for this discrepancy is that the present analysis of NLSE, which properly recovers the coherent instability in terms of the modulational instability in the CW case, does not reproduce the Landau damping for the case of a finite beam length. In fact, for moderate instability growth rates, Landau damping will be strong enough to deform the level curves away from parabolic form and even create a region of stability around the origin of the (Z_r, Z_i) plane. However, for stronger instabilities, Landau damping will be negligible and the level curves regain their parabolic shape and conform with the present predictions. This explanation is amply confirmed by a comparison with the contour plots presented in [2, 4] . Nevertheless, in the next section, we show that the present quantum-like description is indeed capable of reproducing also Landau damping, provided that the instability analysis is carried out in the phase space. This is done using the Wigner transform [25] . It should be emphasized that what is done in the phase space in terms of this transformation should be fully equivalent to the analysis in the configuration space in terms of the NLSE. A more rigorous analysis of the modulational instability for the generalized NLSE will be given in a forthcoming work.

5 The quantum-like Landau damping and its role in the instability analysis

We want to transit from the longitudinal instability description in configuration space, with the equation

$$i\epsilon\eta\frac{\partial\Psi}{\partial s} \;=\; -\frac{\epsilon^2\eta^2}{2}\frac{\partial^2\Psi}{\partial x^2} \;+\; U(x,s)\Psi \quad , \tag{60}$$

where

$$U(x,s) \;=\; \frac{q\eta}{E_0\beta c T_0}\int_0^x \mathcal{U}(x',s)\;dx' \qquad (61)$$

(see also Eq.n (1)), to the one that may be carried out in the phase space. To this end, let us introduce the following Wigner-like function [26] :

$$\rho_w(x,p,s) = \frac{1}{2\pi\epsilon|\eta|}\int_{-\infty}^{\infty}\Psi^*\left(x+\frac{y}{2},s\right)\;\Psi\left(x-\frac{y}{2},s\right)\;\exp\left(i\frac{py}{\epsilon\eta}\right)\;dy \quad , \qquad (62)$$

where $p \equiv dx/ds = -\eta q\mathcal{U}(x,s)/(2\pi E_0 R_0)$ is the momentum conjugated to x. The following normalization condition is also assumed:

$$\int \rho_w(x,p,s)\;dx\;dp = 1 \quad . \qquad (63)$$

We observe that, if Ψ satisfies the (60), thus ρ_w satisfies the following von Neumann-like equation [26] :

$$\left\{\frac{\partial}{\partial s} + p\frac{\partial}{\partial x} + \frac{i}{\epsilon\eta}\left[U\left(x+i\frac{\epsilon\eta}{2}\frac{\partial}{\partial p}\right) - U\left(x-i\frac{\epsilon\eta}{2}\frac{\partial}{\partial p}\right)\right]\right\}\rho_w \;=\; 0 \quad , \qquad (64)$$

which can be cast in the form:

$$\frac{\partial\rho_w}{\partial s} + p\frac{\partial\rho_w}{\partial x} = \sum_{\alpha=0}^{\infty}\frac{(-1)^\alpha}{(2\alpha+1)!}\left(\frac{\epsilon\eta}{2}\right)^{2\alpha}\frac{\partial^{2\alpha+1}U}{\partial x^{2\alpha+1}}\frac{\partial^{2\alpha+1}\rho_w}{\partial p^{2\alpha+1}} \quad . \qquad (65)$$

By using (61), (65) becomes:

$$\frac{\partial\rho_w}{\partial s} + p\frac{\partial\rho_w}{\partial x} = \left(\frac{q\eta}{2\pi E_0 R_0}\right)\sum_{\alpha=0}^{\infty}\frac{(-1)^\alpha}{(2\alpha+1)!}\left(\frac{\epsilon\eta}{2}\right)^{2\alpha}\frac{\partial^{2\alpha+1}\mathcal{U}}{\partial x^{2\alpha+1}}\frac{\partial^{2\alpha+1}\rho_w}{\partial p^{2\alpha+1}} \quad . \qquad (66)$$

The beam current is now introduced as follows [2] :

$$I(x,s) \equiv q\beta c\lambda_0\int_{-\infty}^{\infty}\rho_w(x,p,s)\;dp \quad , \qquad (67)$$

where λ_0 is a positive constant. Note that the Fourier transform of $\mathcal{U}(x,s)$ and $I(x,s)$ are connected by the coupling impedance Z.

Linearizing around the equilibrium state (i.e., $\rho_w = \rho_0(p)$, $\mathcal{U} = \mathcal{U}_0 = 0$, and $I = I_0 = 0$):

$$\rho_w(x,p,s) = \rho_0(p) + \rho_1(x,p,s) \quad , \qquad (68)$$

$$\mathcal{U}(x,s) \;=\; \mathcal{U}_1(x,s) \quad , \qquad (69)$$

$$I(x,s) \;=\; I_1(x,s) \quad , \qquad (70)$$

we have:

$$\frac{\partial\rho_1}{\partial s} + p\frac{\partial\rho_1}{\partial x} = \left(\frac{q\eta}{2\pi E_0 R_0}\right)\sum_{\alpha=0}^{\infty}\frac{(-1)^\alpha}{(2\alpha+1)!}\left(\frac{\epsilon\eta}{2}\right)^{2\alpha}\frac{\partial^{2\alpha+1}\mathcal{U}_1}{\partial x^{2\alpha+1}}\rho_0^{(2\alpha+1)} \quad , \qquad (71)$$

$$I_1(x,s) \equiv e\beta c\lambda_0\int_{-\infty}^{\infty}\rho_1(x,p,s)\;dp \quad , \qquad (72)$$

where $\rho_0^{(2\alpha+1)} \equiv d^{2\alpha+1}\rho_0/dp^{2\alpha+1}$. By assuming for ρ_1, U_1, and I_1 solutions of the form:

$$\rho_1(x,p,s) = \widetilde{\rho_1}(\chi,p,\omega)\ \exp\left(i\chi x - i\omega s\right) \quad , \tag{73}$$

$$U_1(x,s) = \widetilde{U_1}(\chi,\omega)\ \exp\left(i\chi x - i\omega s\right) \quad , \tag{74}$$

$$I_1(x,s) = \tilde{I}_1(\chi,\omega)\ \exp\left(i\chi x - i\omega s\right) \quad , \tag{75}$$

respectively, and introducing the impedance definition, we finally get the following dispersion relation:

$$1 \;=\; i\alpha_0 Z(\chi,\omega) \int_{-\infty}^{\infty} \frac{\rho_0\left(p+\epsilon\eta\chi/2\right) \;-\; \rho_0\left(p-\epsilon\eta\chi/2\right)}{\epsilon\eta\chi}\, \frac{dp}{\chi p - \omega} \quad , \tag{76}$$

where $\alpha_0 \equiv q^2\beta c\eta\lambda_0/(2\pi E_0 R_0)$.

A preliminary analysis of this dispersion relation can be carried out as follows.

- We can take the limit of small χ, but keeping ϵ and ω finite (f.i., $v_{ph}/c \equiv \omega/\chi >> 1$). Since in this case

  $$\frac{\rho_0\left(p+\epsilon\eta\chi/2\right) \;-\; \rho_0\left(p-\epsilon\eta\chi/2\right)}{\epsilon\eta\chi} \approx d\rho_0/dp \quad , \tag{77}$$

  Eq.n (76) becomes:

  $$1 \;=\; i\alpha_0 Z(\chi,\omega) \int_{-\infty}^{\infty} \frac{d\rho_0/dp}{\chi p - \omega}\, dp \quad , \tag{78}$$

  which coincides with the dispersion relation of the conventional theory [2] *and, thus, reproduces all the coherent instability results for coasting beams of the conventional theory for small χ but including the Landau damping.*

- Eq.n (76) shows the existence of *new effects* which should be significant for large χ that are not included in the conventional theory. In this preliminary analysis we only point out that *Landau damping is intrinsically included in the TWM description of coherent instability.* Generalizing the conventional theory, TWM seems to show the existence of a quantum-like Landau damping. However, this very novel subject will be investigated more carefully in a forthcoming work.

- In order to recover, as an example, some of the results given in the previous sections in terms of NLSE in the configuration space, let us consider the case of monochromatic beam, which means:

  $$\rho_0(p) \;\propto \delta(p) \quad . \tag{79}$$

  In this case, although the instability is present, Landau damping is not working due to the absence of the momentum spread. In fact, the dispersion relation (76) becomes now:

  $$1 \;=\; -\frac{i\alpha_0 Z(\chi,\omega)}{\epsilon\eta\chi} \left[\frac{1}{\epsilon\eta\chi^2/2+\omega} + \frac{1}{\epsilon\eta\chi^2/2+\omega}\right] \quad , \tag{80}$$

  which can be cast in the form:

  $$\omega^2 \;=\; \frac{\epsilon^2\eta^2\chi^4}{4} + i\alpha_0\chi Z \quad . \tag{81}$$

Expressing both ω and Z in their complex representations, viz

$$\omega \;=\; \omega_R \,+\, i\omega_I \;\; and \;\; Z \;=\; Z_R \,+\, iZ_I \quad , \tag{82}$$

we get the following relation:

$$Z_I \;=\; -\frac{\delta_0}{4\omega_I^2} Z_R^2 \,+\, \frac{\omega_I^2}{\delta_0} \,+\, \frac{\epsilon^2\eta^2\chi^2}{4} \quad , \tag{83}$$

where $\delta_0 \equiv \alpha_0\chi = q^2\beta c\eta\chi\lambda_0/(2\pi E_0 R_0)$, which is formally identical to (32) given in Section 4. In particular, for small χ (83) becomes, still in accordance with Section 4, the following relation:

$$Z_I \;\approx\; -\frac{\delta_0}{4\omega_I^2} Z_R^2 \,+\, \frac{\omega_I^2}{\delta_0} \quad , \tag{84}$$

which is in full agreement with the corresponding instability equation for a monochromatic coasting beams given by conventional description [2, 4] .

6 Conclusions, remarks, and perspectives

In this paper, an investigation of longitudinal coherent instability has been carried out within the context of TWM. The interaction of the beam with its surroundings (and with itself) has been expressed in terms of a (nonlinear) potential in a Schrödinger-like equation.

The corresponding equation for the beam wave function constitutes a new generalized Schrödinger equation, which, as far as we know, has not been analyzed before. In this paper, this equation has been analyzed for the stability of small perturbations of a constant amplitude backgroud beam. The above investigation is inserted in a field already explored of several nonlinear phenomena described by different kinds of NLSE. In particular, we have reviewed the main results concerning the longitudinal coherent instability for a coasting beam in the conventional machines when the interaction between the beam and the surroundings is modelled in terms of a purely reactive impedance [16, 18] . In this case the instability reduces to the classical modulational instability of the conventional Schrödinger equation for cubic nonlinearity.

However, in this paper we have extended this problem to the more general case of non-negligible resistive part of the coupling impedance for coasting beam as well as for beams with finite size. In the first case, we have found that the perturbation are always unstable and the instability growth rate is found in terms of the real and the imaginary parts of the coupling impedance (Z_R and Z_I, respectively).

The results are summarized in terms of curves in the (Z_r, Z_i) plane, corresponding to constant instability growth rate. These curves are found to be parabolas, in full qualitative as well as quantitative agreement with results of previous conventional techniques for analyzing coherent instabilities of high-energy charged-particle beams.

In the second case, the analysis has been extended to include the effects of a longitudinal extent of the background beam. This is shown to give rise to a stabilizing effect on the modulational instability, but does not, within the present analysis, change qualitatively the form of the curves for constant instability growth rate, which remains parabolic.

On the other hand, in the conventional approaches, it has been found that for a finite energy spread of the beam, Landau damping will become an important effect and will deform the level curves for small instability growth rates, even to the point of creating a two-dimensional region of stability around the origin in the (Z_r, Z_i) plane. However, for stronger instabilities, Landau damping becomes negligible and the level curves regain their parabolic form.

However, if the analysis within configuration space, as expresses by the NLSE, is generalized into a phase space description by means of a Wigner-like formalism, the TWM approach predicts new important results concerning coherent instabilities for the beam dynamics. In fact, we have shown that the results of instability given by the conventional theory can be recovered also for finite energy spread, provided to transit to the phase space. The resulting phase-space description is fully equivalent to the one given by the NLSE in the configuration space, although it seems to be simpler in the phase space. This way, we have given a preliminary phase-space description of the collective interaction of the beam with the surroundings, modelled in terms of an arbitrary coupling impedance. With this analysis we have obtained a linear dispersion relation which shows the existence of a more general Landau damping (we have called it *quantum-like Landau damping*) which for the case of small χ reproduces all the results of the conventional theory.

Remarkably, in the limit of $\epsilon \to 0$, the dispersion relation (76) coincides exactly, for arbitrary χ, with the one given by the conventional theory. However, ϵ cannot in principle be reduced to zero, because it accounts for the thermal noise which is very important and competes with the resonance in the wave-particle interaction (i.e. Landau damping). Consequently, TWM takes realistically into account, due to the thermal noise, eventual displacements from the exact resonance condition. Since the inhomogeneity wave parameter is χ, the variation of $\rho_0(p)$ cannot be estimated in regions of momentum space with size smaller than $\epsilon|\eta|\chi$. This limitation transforms the usual derivative appearing in the conventional theory, as given by (78), into the finite difference ratio as give in (76).

In future work the explicit dispersion relation for the case of arbitrary χ should be found for several distribution functions $\rho_0(p)$.

We conclude that the presently obtained results further validates the TWM approach as a consistent alternative description of the dynamics of high-energy charged-particle beams in accelerators.

References

[1] M. Month and S. Turner (Ed.s), *Frontieres of Particle Beams; Observation, Diagnosis and correction.* Proc. Joint US-CERN School on Particle Accelerators. Capri, Italy, October 20-26, 1988 (Springer-Verlag, Berlin, 1989).

[2] J. Lawson, *The physics of charged particle beams*, (Clarendon Press, Oxford, 1988), 2nd edition, and references therein.

[3] C.E. Nielsen, A.M. Sessler, and K.R. Symon, *High Energy Accelerators and Instrumentation* (CERN, Geneva, 1959), p. 239; for a historical review on this subject see Ref. [2] and references therein.

[4] R. D. Ruth, *An overview of collective effects in circular and linear accelerators*, in Ref. [1] ; A. Hofman, *Single-beam coherent phenomena - longitudinal*, CERN 77-13, November 1976, p. 139

[5] S. Chattopadhyay, M. Cornacchia, and C. Pellegrini (Ed.s), Proc. of Workshop on *Nonlinear Dynamics in Particle Accelerators: Theory and Experiments*, Arcidosso, Italy, September 4-9,1994 (AIP Press, Woodbury, New York, 1995).

[6] L.D. Landau, *J. Phys. USSR*, **10**, 25 (1946).

[7] P. A. Sturrock, *Plasma Physics*, (Cambridge University Press, Cambridge, 1994).

[8] G. Schmidt, *Physics of High Temperature Plasmas*, (Academic Press, New York, 1979).

[9] J.M. Wang, *Beam Transfer and Landau Damping*, in Ref. [1] .

[10] M. Chanel, *Beam Instabilities and Their Cures*, in E.D. Maletić and A.G. Ruggiero (Ed.s), *Cristalline Beams and Related Issues* (World Scientific, Singapore, 1996); see also R.D. Kohaupt, *Cures for Instabilities*, in Ref. [1] .

[11] F. Ruggiero, *A Review of New Manifestations of Collective Effects*, Proc. of the EPAC98, Stockholm, 22-26 June, 1998 (Institute of Physics Publishing, Bristol and Phyladelphia, 1998), p. 18, and references therein.

[12] L.K. Spentzouris, J.F. Ostiguy, and P.L. Colestock, *Phys. Rev. Lett.* **76**, 620 (1996); P.L. Colestock and L.K. Spentzouris, in The Tamura Symposium Proc., Austin, Texas, 1994 AIP Conf. Proc. 356 (AIP, Woodbury, New York, 1996).

[13] H. Schamel, *Phys. Rev. Lett.* **79**, 2811 (1997); H. Schamel, Physica Scripta **T75**, 23 (1998).

[14] P.L. Colestock, L.K. Spentzouris, and S. Tzenov (1998), these Proceedings and references therein.

[15] R. Fedele and P.K. Shukla (Ed.s), *Quantum-like Models and Coherent Effects*, (World Scientific Publ., Singapore, 1995).

[16] R. Fedele, L. Palumbo, and V.G. Vaccaro, Proc. of EPAC92, Berlin, 24-28 March, 1992, H. Henke, H. Homeyer, and Ch. Petit-Jean-Genaz (Ed.s) (Editions Frontieres, Singapore, 1992) p. 762; R. Fedele, G. Miele, L. Palumbo, and V.G. Vaccaro, *Phys. Lett.* **A179**, 407 (1993).

[17] R. Fedele and G. Miele, *Il Nuovo Cimento* **D13**, 1527 (1991).

[18] M. Migliorati and L. Palumbo, *Coherent instabilities in particle accelerators: conventional and novel approaches*, in Ref. [15] ; R. Fedele, *Physica Scripta*, **T63** 162 (1996).

[19] D. Gloge and D. Marcuse, *J. Opt. Soc. Am.* **59**, 1629 (1969).

[20] P. A. Sturrock, *Static and Dynamic Electron Optics*, (Cambridge University Press, Cambridge, 1955).

[21] R. Fedele and P.K. Shukla, *Phys. Rev.* **A45**, 4045 (1992).

[22] R. Fedele P.K. Shukla, and V.G. Vaccaro *Journal de Physique IV, C6-II*, **5** 119 (1995).

[23] R. Fedele and V.G. Vaccaro *Physica Scripta*, **T52** 36 (1994).

[24] L. Palumbo and V.G. Vaccaro, *Wake Field and Mesurements*, in Ref. [1] .

[25] E. Wigner, *Phys. Rev.*, **40** 749 (1932).

[26] R. Fedele, F. Galluccio, V.I. Man'ko, and G. Miele, *Phys. Lett.* **A209**, 263 (1995); R. Fedele and V.I. Man'ko, *Phys. Rev.* **E58**, 992 (1998).