

ISTITUTO NAZIONALE DI FISICA NUCLEARE

Sezione di Napoli

INFN/TC-97/31
13 Ottobre 1997

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PACS: 03.65.ca, 41.85.-p, 42.50.-p

SIS-Pubblicazioni
dei Laboratori Nazionali di Frascati

Quantum-like Corrections and Semiclassical Description of Charged-Particle Beam Transport

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Abstract

It is shown that the standard classical picture of charged-particle beam transport in paraxial approximation may be conveniently replaced by a Wigner-like picture in *semi-classical approximation*. In this *effective* description the classical Liouville equation is replaced by a *Von Neumann-like equation*, where the transverse emittance plays the role of \hbar . Relevant remarks concerning the quantum-like corrections for an arbitrary potential in comparison with the standard classical description of the beam transport are given.

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1 Introduction

Quantum formalism for describing a number of macroscopic systems, such as plasmas, linear and nonlinear electromagnetic (e.m.) radiation beam propagation (for instance, optical fibers, transmission lines), e.m. traps, charged-particle beam transport, etc., have received a great deal of attention during the last two decades [1]. For these non-proper quantum systems, it is appropriate to say *quantum-like* description instead of the proper quantum one, because the physics involved, which is basically classical, can be fully described by formally replacing the Planck's constant with a suitable fundamental parameter of the particular system considered. A quantum-like theory of light rays was, for example, constructed by Gloge and Marcuse [2] in order to recover wave optics starting from a formal quantization of geometrical optics based on Fermat's principle. In particular, this procedure has allowed to recover, in paraxial approximation, the Schrödinger-like equation for the e.m. field, the so-called Fock-Leontovich equation [3], widely used in linear and nonlinear e.m. radiation optics [4]-[6]. The *transition* from geometrical optics (the analogous of classical mechanics) to wave optics (the analogous of wave mechanics) is performed by introducing some correspondence rules, fully similar to the Bohr's ones, in which \hbar is replaced by $\lambda/2\pi$, the inverse of the wavenumber ($\lambda/2\pi \equiv 1/k$). In particular, in this context the paraxial approximation (the analogous of the non relativistic approximation of quantum mechanics) describes the radiation beam transport in an arbitrary medium and the corresponding quantum-like formalism (a quantum-like uncertainty principle included) and Fock-Leontovich equation can be fully recovered by formally replacing \hbar with $\lambda/2\pi$ in the non relativistic quantum mechanics [2]. This fruitful procedure has provided for transferring algorithms and many solutions of quantum mechanics to radiation beam physics, especially for optical fibers [7, 8], coherent and squeezed states theories [9]-[14], Schrödinger cat states [15, 16], and phase-space investigations within a Wigner-like picture [17] in which a quasi-classical distribution, fully similar to quantum Wigner transform [18] governs the paraxial e.m. ray evolution.

More recently, a procedure *ala Gloge and Marcuse* has allowed to construct a quantum-like model of charged-particle beam transport in both real space and phase space, called *Thermal Wave Model* (TWM) [19]. This model has been applied to a number of problems of charged-particle beam optics and dynamics [20]-[25]. It assumes that the particle beam evolution is governed by a Schrödinger-like equation for a complex function, the so-called beam wave function (BWF) whose squared modulus is proportional to the beam density where Planck's constant is replaced by the *beam emittance* [26]. In particular, in TWM framework, a Wigner-like transform seems to be useful and appropriate to give the quantum-like phase-space description of particle beams [25].

In this paper, we want to suggest an approach alternative to the one *ala Gloge and Marcuse* given by TWM. By starting from the electronic ray concept given in electron optics, we review the standard electronic ray approach to charged-particle beam optics and dynamics and introduce an effective description of the transverse beam dynamics which takes into account the thermal spreading among the electronic rays. In the following subsections we start from the electronic ray concept and introduce the paraxial-ray approximation. In section 2, the paraxial-ray equation is solved for the case of a linear lens (Hill's equation), while in section 3 the statistical description of electronic rays allows us to obtain some important results such as the virial description of the beam and a quantum-like uncertainty relation. A Liouvillian description of the electronic rays

is performed in section 4, where, in paraxial approximation, we show that an *effective* description can be given in terms of a *quasi-distribution* in the phase-space which plays the role analogous to the one played in quantum mechanics by Wigner function for pure states [18]. An analysis of the *quantum-like corrections* that the above effective approach gives is presented in section 5 where a comparison with the classical approach up to the 4th-order moment-description of the system for an arbitrary potential is performed. It is shown that for dilute beams and in paraxial approximation the discrepancies are negligible. Finally, in section 6 we summarize the conclusions and give some remarks that are relevant for charged-particle and e.m. beam transport as well as for quantum optics and very recent investigations in constructing positive definite distribution functions such as the one used in symplectic tomography [27, 28, 29].

1.1 The concept of electronic rays

It is well known that *electron optics* [30] has been developed by using the similarity between charged-particle motion and the behaviour of the light rays in geometrical optics. For nonrelativistic particle motion, this analogy shows that potential energy and particle trajectories play the role fully similar to the ones played by refractive index and light rays, respectively. In particular, this similarity allows us to introduce the concept of *electronic rays*. On the basis of this optical language, refraction and reflection laws for electronic rays can be introduced and their formulation is fully similar to the one that is used for light rays. The basic electron optics concepts have been developed in connection with the first experimental investigations of charged-particle motion (ions and electrons) in oscilloscopes and mass spectrometers. However, electron optics have been rapidly developed and applied to electron microscopy [31], electro-optical transducers [32], particle accelerators [33, 34], etc. .

The general statement from which we recover the above optical description is the so-called Hamilton's principle:

$$\delta \int p \cdot dq = 0 \quad , \quad (1)$$

fully similar to Fermat's principle of geometric optics:

$$\delta \int n ds = 0 \quad . \quad (2)$$

1.2 The paraxial electronic-ray approximation

When the potential is a function of the coordinates, it corresponds to an inhomogeneous refractive index, and the electron trajectory through this inhomogeneous potential region corresponds to a light ray through an inhomogeneous medium.

In case we have several particles moving together in an arbitrary potential, each particle trajectory is an electronic ray.

In order to consider a charged particle beam as a special case of the above particle system, we introduce the so-called *paraxial electronic ray approximation* [33]. In this case, the system has a special direction, the instantaneous propagation direction, say z , and the following conditions hold:

$$\dot{x} \equiv \frac{dx}{dz} \ll 1 \quad , \quad \dot{y} \equiv \frac{dy}{dz} \ll 1 \quad , \quad (3)$$

where x and y are the transverse (with respect to z) coordinates in the comoving frame. In other words, paraxial approximation corresponds to a very small deviation of the electronic rays from the propagation direction. Note that in principle the beam particles may have a relativistic motion along z (longitudinal motion) but, in order to be consistent with the paraxial approximation, their transverse motion must be non relativistic ($v_{\perp} \equiv \sqrt{v_x^2 + v_y^2} \ll c$). At instant t a surface orthogonal to all the electronic rays can be constructed. This surface is analogous to the one that in e.m. optics is obtained by taking constant the phase at each time: in fact, at each instant, it is orthogonal to all the light rays, and the initial surface transforms during the e.m. propagation. Correspondingly, the surface orthogonal to the electronic rays transforms during the particle motion: let us call it *eikonal surface* of the particle system. If at $t = 0$ the eikonal surface has finite curvature radius ρ ($0 < \rho < \infty$), thus the electron ray will converge (diverge) in correspondence of surface concavity turned forward (backward) the propagation direction. Consequently, the beam will be focussed (defocussed) and ρ plays essentially the role of *focal length* of the device which produced the initial electronic ray convergence (divergence). Such devices are usually called *lenses* and typically they produce electric forces (electrostatic lens) or magnetic forces (magnetic lens) on the beam particles [30, 33, 34].

Let us consider a beam so dilute that the space charge effects can be considered negligible. If the thermal spreading of the particle velocity is negligible, in the case of aberrationless focusing, the particle converge in one point F only (focal point). Consequently, in the limit of $\rho \rightarrow \infty$, without thermal spreading, the beam remains unchanged (not focussed, not defocussed).

Of course, if the thermal spreading is taken into account, the above circumstances will be modified. In fact, the beam will not focus at only one point and, with the initial condition $\rho = \infty$, the electron rays will diverge and the beam naturally defocuses.

In order to go deep into the *thermal spreading among the electronic rays*, in the next section we consider the single particle motion in a linear lens and in the section later a statistical treatment of the electronic rays will be performed.

2 Single particle-motion (single electronic ray)

Let us consider for simplicity the particle motion in the 2-D case: for instance, the y -component of the particle motion is neglected. Typically, the hamiltonian for the x -component motion of a single charged-particle with rest mass m_0 is given in the following dimensionless form in the comoving frame:

$$H = \frac{p^2}{2} + U(x, z) \quad , \quad (4)$$

where $p = \dot{x}$ is the canonical conjugate momentum. Note that (4) describe a 1-D motion (along x) of a classical particle when z plays the role of a time-like variable and U is an effective dimensionless potential energy, which can be expressed in terms of a polynomial form in x of arbitrary degree N as:

$$U(x, z) = \frac{k_0(z)}{1!}x + \frac{k_1(z)}{2!}x^2 + \frac{k_2(z)}{3!}x^3 + \frac{k_3(z)}{4!}x^4 + \dots = \sum_{n=0}^N \frac{k_n}{(n+1)!}x^{n+1} \quad . \quad (5)$$

U has been made dimensionless dividing the effective energy potential of the system by the relativistic longitudinal energy $m_0\gamma_0c^2 \equiv mc^2$ (γ_0 being the longitudinal relativistic

factor) . In particular, for a pure quadrupole-like potential (linear lens) (5) becomes

$$H = \frac{p^2}{2} + \frac{k_1(z)}{2}x^2 \quad . \quad (6)$$

Let us consider the equation of motion which follows from (6) (the Hill's equation [33, 35]):

$$\ddot{x} + k_1(z)x = 0 \quad , \quad (7)$$

where $\dot{p} = -k_1(z)x$. The general solution of (7) can be put in the following form

$$x = \sqrt{2}E(z) \cos(\phi(z) - \phi_0) \equiv \sqrt{2}E(z) \cos \Delta\phi(z) \quad , \quad (8)$$

where ϕ_0 is an arbitrary constant and $E(z)$ is a function defined unless an arbitrary constant factor. By imposing that (8) is a solution of (7), we easily obtain the following conditions

$$E^2 \Delta\dot{\phi} = \text{const.} \equiv \frac{I_0}{2} \quad , \quad (9)$$

and

$$\ddot{E} + k_1 E - \frac{I_0^2}{4E^3} = 0 \quad . \quad (10)$$

Moreover, it is easy to prove that x and p satisfy the following quadratic form:

$$\mathcal{J}(x, p, z) = \gamma(z)x^2 + 2\alpha(z)xp + \beta(z)p^2 = \frac{I_0}{2} \quad , \quad (11)$$

where

$$\gamma(z) = \frac{I_0}{4E^2} + \frac{\dot{E}^2}{I_0} \quad , \quad \alpha(z) = -\frac{E\dot{E}}{I_0} = -\frac{1}{2I_0} \frac{dE^2}{dz} \quad , \quad \beta(z) = \frac{E^2}{I_0} \quad , \quad (12)$$

are called *Twiss parameters* [35]. Note that $\mathcal{J}(x, p, z)$ is an invariant for the hamiltonian (6), namely:

$$\frac{\partial \mathcal{J}}{\partial z} + \{\mathcal{J}, H\} = 0 \quad , \quad (13)$$

where $\{\dots\}$ denotes the classical Poisson brackets [36]. It is worth noting that the invariant which is quadratic form in coordinates and momentum for parametric classical oscillator is known as Ermakov invariant [37] and its quantum analogous was found by Lewis [38] and discussed in [39]. It is easy to see that the determinant of the matrix associated with the quadratic form (11) is conserved:

$$\gamma\beta - \alpha^2 = \frac{1}{4} \quad . \quad (14)$$

Thus, from (12)-(14) we obtain the identity

$$\frac{I_0^2}{4} = \left(\dot{E}^2 + \frac{I_0^2}{4E^2} \right) E^2 - (\dot{E}E)^2 \quad , \quad (15)$$

and the following inequality, which will be used later

$$E \left(\dot{E}^2 + \frac{I_0^2}{4E^2} \right)^{1/2} \geq \frac{I_0}{2} \quad . \quad (16)$$

3 Statistical description of electronic rays

The results of the previous section can be used now to statistically describe the spreading among the electronic rays in a linear lens.

First of all, we observe that solution (8) is typically considered in particle accelerators for the case of a very smooth $k_1(z)$ compared to the variation of the *phase advance* $\Delta\phi(z)$ [33, 40]. Also the amplitude $E(z)$ is typically a very slow function compared to $\Delta\phi(z)$ [33, 40]. It is easy to see that in this circumstances the paraxial approximation is naturally satisfied. In fact, for an arbitrary initial transverse-space particle distribution, the most of particle trajectories remains confined in a limited region (if suitable stability conditions hold). Consequently, in the statistical description it can be assumed that this region represents a sort of *mean spread* for the generic particle position or, equivalently, a *mean spot* for a generic electronic ray corresponding to the most probable phase-space accessible region. This way, we can introduce also the average of an arbitrary observable. In particular, in order to estimate the above spot size we have to compute the following r.m.s. definition:

$$\sigma_x^2 \equiv \langle x^2 \rangle = \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x^2(t) dt \quad , \quad (17)$$

Since $x(z)$ contains a fast-period dependence on z , one can replace the (17) with an average on the phase

$$\sigma_x^2 = \langle x^2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} x^2 d\Delta\phi \quad , \quad (18)$$

which gives (the average is performed only on the fast time scale, where $E(z)$ is almost constant)

$$\sigma_x^2(z) = \langle x^2 \rangle = E^2(z) \quad . \quad (19)$$

Consequently, the instantaneous amplitude of solution (8) of the electronic ray equation in a linear lens corresponds to the statistical estimate of the transverse beam spot size σ_x . Similarly, we define the r.m.s. of the electronic ray slope $p \equiv dx/dz = \sqrt{2}\dot{E} \cos \Delta\phi - (I_0/\sqrt{2}E) \sin \Delta\phi$, obtaining:

$$\sigma_p^2(z) \equiv \langle p^2 \rangle = \dot{E}^2 + \frac{I_0^2}{4E^2} = \left(\frac{d\sigma_x}{dz} \right)^2 + \frac{I_0^2}{4\sigma_x^2} \quad . \quad (20)$$

For the observable xp the statistical average gives

$$\sigma_{xp} \equiv \langle xp \rangle = E\dot{E} = \frac{1}{2} \frac{d}{dz} \langle x^2 \rangle = \frac{1}{2} \frac{d\sigma_x^2}{dz} \quad , \quad (21)$$

and, finally, the mean value of the energy (6)

$$\mathcal{H}(z) \equiv \langle H \rangle = \frac{1}{2} \left(\dot{E}^2 + \frac{I_0^2}{4E^2} \right) + \frac{1}{2} k_1 E^2 = \frac{\sigma_p^2}{2} + \frac{k_1(z)}{2} \sigma_x^2 \quad , \quad (22)$$

or, equivalently

$$\mathcal{H}(z) = \frac{1}{2} \left(\frac{d\sigma_x}{dz} \right)^2 + \frac{I_0^2}{8\sigma_x^2} + \frac{1}{2} k_1(z) \sigma_x^2 \quad . \quad (23)$$

Consequently, the hamiltonian \mathcal{H} defined by (23) has now the meaning of averaged total energy associated with the transverse motion of the beam particles. It is very easy to prove the following very important relationships

$$\frac{d^2\sigma_x^2}{dz^2} + 4k_1(z)\sigma_x^2 = 4\mathcal{H} \quad . \quad (24)$$

and

$$\frac{d\mathcal{H}}{dz} = \left\langle \frac{\partial U}{\partial z} \right\rangle = \frac{1}{2} k_1 \sigma_x^2 \quad . \quad (25)$$

Remarkably, (24) and (25) describe statistically (virial description) the behaviour of the paraxial electronic rays in a linear lens of strength $k_1(z)$. But some additional information can be obtained from (19)-(21). In fact, the quantities $\langle x^2 \rangle$, $\langle p^2 \rangle$, and $\langle xp \rangle$ are the elements of the *diffusion matrix* whose determinant essentially defines the squared of the *diffusion coefficient*. Let us introduce the following quantity proportional to this coefficient and called *r.m.s. emittance* [41, 42]:

$$\frac{\epsilon}{2} = [\langle x^2 \rangle \langle p^2 \rangle - \langle xp \rangle^2]^{1/2} \quad . \quad (26)$$

Note, results (19)-(21) show us that both in the linear lens and in vacuo ϵ is an invariant and coincides with I_0 :

$$\frac{I_0^2}{4} = \langle x^2 \rangle \langle p^2 \rangle - \langle xp \rangle^2 \quad . \quad (27)$$

For an arbitrary potential, ϵ is not necessarily preserved. Remarkably, from (26) in particular we have:

$$\sigma_x \sigma_p \geq \frac{\epsilon}{2} \quad . \quad (28)$$

We would like to stress that (16) represents a tautology, whilst statistical form (28) actually represents a sort of *uncertainty relation* even if the particle beam is a *classical system*. Furthermore, it is clear that (28) defines the transverse beam emittance as the *minimum reachable uncertainty*. By using (19)-(25), it is easy to see that this minimum is reached at the equilibrium condition ($d\sigma_x/dz = 0$). At the equilibrium the phase-space distribution for a sufficiently dilute beam is Gaussian in both configuration and momentum space. Let us take this two equilibrium distribution for the dimensionless hamiltonian (6), namely given by

$$n_p^{(0)}(p) = n_{p0}^{(0)} \exp \left[-\frac{p^2}{2\sigma_{p0}^2} \right] \quad , \quad (29)$$

where $\sigma_{p0}^2 \equiv k_B T / (mc^2) \equiv \langle p^2 \rangle_{z=0}$, (k_B and T being the Boltzmann constant and the transverse temperature of the system, respectively), and

$$n_x^{(0)}(x) = n_{x0}^{(0)} \exp \left[-\frac{x^2}{2\sigma_{x0}^2} \right] \quad , \quad (30)$$

where $\sigma_{x0}^2 \equiv \langle x^2 \rangle_{z=0}$. Note that $\langle xp \rangle = \langle xp \rangle_{z=0} = 0$. Consequently, at the equilibrium, (27) gives

$$\langle x^2 \rangle_{z=0}^{1/2} \langle p^2 \rangle_{z=0}^{1/2} = \frac{\epsilon}{2} \quad , \quad (31)$$

which proves that the minimum product of the uncertainties is given at the equilibrium states and numerically coincides with half of the beam emittance, and now it is easy to prove that [33]:

$$\frac{\epsilon}{2} = \left(\frac{k_B T}{m c^2} \right)^{1/2} \langle x^2 \rangle_{z=0}^{1/2} = \frac{v_{th}}{c} \sigma_0 \quad , \quad (32)$$

which shows explicitly the thermal nature of the beam emittance; $v_{th} \equiv (k_B T/m)^{1/2}$ represents the transverse *thermal velocity* of the system. Consequently, ϵ scales as \sqrt{T} . The above results clearly show that, if the temperature of the system is not negligible, the electron rays are affected by a diffusion whose effect is to spread out them while the beam is propagating. This effect produce a *dispersion* among the electron rays whose tendency, due to the potential $U(x, z)$, is to be *ordered*. To see more evident this diffusion effect, let us consider the special case of $U(x, z) = 0$ (i.e. the beam is travelling in vacuo). In this case, (23) and (25) imply that \mathcal{H} is a positive constant given by

$$\mathcal{H} = \frac{1}{2} \left(\frac{d\sigma_x}{dz} \right)^2 + \frac{\epsilon^2}{8\sigma_x^2} = \text{constant} \quad , \quad (33)$$

and, consequently, (24) becomes

$$\frac{d^2\sigma_x^2}{dz^2} = 4\mathcal{H} = \text{constant} \quad , \quad (34)$$

which, for the initial condition $\sigma_x(z=0) \equiv \sigma_0$ and $\dot{\sigma}_x(z=0) = 0$, gives

$$\sigma_x^2(z) = \sigma_0^2 + 2\mathcal{H}z^2 = \sigma_0^2 + \frac{\epsilon^2}{2\sigma_0^2} z^2 \quad . \quad (35)$$

This means that, while the beam is travelling from $z = -|\bar{z}|$, the electronic rays will not focus in a one point only. Starting from an initial spread $\bar{\sigma} \equiv \sigma_x(-|\bar{z}|) = \sigma_0 \left(1 + \frac{\epsilon^2}{2\sigma_0^4} \bar{z}^2 \right)^{1/2}$, in case of focusing the electron rays will reach the minimum spot σ_0 and then the beam will diverge giving greater values of the spot. We want to point out that, since the particle are moving in vacuo, their trajectories must be straight. Even if the electron rays are straight, their mixing is due to the thermal spreading (diffusion) in such a way to produce the beam envelope described by (35) which represents a hyperboloid of rotation around the z -axis. The entity of this ray mixing is the order of $\frac{\epsilon^2}{4\sigma_0^2} \approx \frac{v_{th}^2}{c^2}$. It is easy to see that in the present 2-D case the eikonal surfaces reduces to a circular line (but in 3-D it is a spherical surface).

For particle beams in the accelerators, typically v_{th}/c is much less than 1. In fact, transverse particle motion is classical whilst the longitudinal one is relativistic. So, the condition $v_{th}/c \ll 1$ is thus equivalent to consider in vacuo the envelope function $E(z)$ slowly varying with respect to the oscillating term $\cos \Delta\phi$. This is in fact consistent with the above paraxial approximation.

4 Liouvillian description

The statistical description presented above, allows us to understand that, for particle beams with finite emittance (temperature), the determination of an electronic ray at the

arbitrary x -position of the transverse plane given at each z is affected by an intrinsic uncertainty that cannot be reduced to zero. Only when the transverse temperature is exactly zero, the electronic ray mixing (diffusion) disappears and finding an electronic ray at a given transverse position is a deterministic operation based on simple geometrical arguments.

However, for finite-beam emittance, the intrinsic uncertainty on the transverse position at each z cannot allow for resolving among two or more rays in the sense that they are indistinguishable within this uncertainty which must be the order of $\sigma_x(z)$. In particular, at the focal point, it would be σ_0 . Consequently, for a finite emittance, we need to assign a probability (in principle positive and finite) of finding an electronic ray at the transverse location x in the plane for given z . This probability distribution, say $P_x(x, z; \epsilon)$, would be both depending on the (transverse) emittance ϵ (i.e. transverse temperature) and normalized in the x -space, namely

$$\int_{-\infty}^{\infty} P_x(x, z; \epsilon) dx = 1 \quad , \quad (36)$$

with the following physical meaning. Multiplying $P_x(x, z; \epsilon)$ by the total number of the beam particles, one obtains the transverse particle beam density (i.e., the electronic ray density with respect the transverse direction).

In order to give the transverse beam dynamics description in terms of this probability distribution, let us start from Liouville equation for the electronic rays. To this end, we introduce the phase-space density distribution $\rho(x, p, z)$ in such a way to have for a generic observable $f(x, p)$ the following average:

$$\langle f(x, p) \rangle = \int f(x, p) \rho(x, p, z) dx dp \quad , \quad (37)$$

provided that the following normalization condition holds

$$\int \rho(x, p, z) dx dp = 1 \quad . \quad (38)$$

By definition ρ is constant of motion, and consequently must obey to the following equation (Liouville equation) [43]:

$$\frac{\partial \rho}{\partial z} + \{\rho, H\} = 0 \quad , \quad (39)$$

where H is the hamiltonian for an arbitrary potential given by (4). By using the Hamilton's equations, (39) can be explicitly written in the following form:

$$\frac{\partial \rho}{\partial z} + p \frac{\partial \rho}{\partial x} - \left(\frac{\partial U}{\partial x} \right) \frac{\partial \rho}{\partial p} = 0 \quad . \quad (40)$$

Within describes a phase-space evolution of electronic rays.

By introducing the dimensionless variables:

$$\bar{z} \equiv \frac{z}{2\sigma_0} \quad , \quad \bar{x} \equiv \frac{x}{2\sigma_0} \quad , \quad (41)$$

Eq.n (40) assumes the form:

$$\frac{\partial \bar{\rho}}{\partial \bar{z}} + p \frac{\partial \bar{\rho}}{\partial \bar{x}} - \left(\frac{\partial \bar{U}}{\partial \bar{x}} \right) \frac{\partial \bar{\rho}}{\partial p} = 0 \quad , \quad (42)$$

where $\bar{\rho} \equiv \rho(x/2\sigma_0, p, z/2\sigma_0) \equiv \bar{\rho}(\bar{x}, p, \bar{z})$ and $\bar{U} \equiv U(x/2\sigma_0, z/2\sigma_0) \equiv \bar{U}(\bar{x}, \bar{z})$.

However, we want to give a more interesting, but approximate *effective* electronic-ray description, taking explicitly into account their thermal spreading. According to the results of the previous section, since for finite emittance the indistinguishability among two or more rays due to the thermal spreading is the order of $\eta \equiv \epsilon/2\sigma_0 = v_{th}/c \ll 1$, $\partial\bar{U}/\partial\bar{x}$ in (40) can be conveniently replaced by the following symmetrized Schwarz-like finite difference ratio:

$$\frac{\partial\bar{U}}{\partial x} \approx \frac{\bar{U}(\bar{x} + \eta/2) - \bar{U}(\bar{x} - \eta/2)}{\eta} . \quad (43)$$

This way, (40) must be replaced by the following equation for an effective distribution, say $\bar{\rho}_w(\bar{x}, p, \bar{z}; \eta)$:

$$\frac{\partial\bar{\rho}_w}{\partial\bar{z}} + p \frac{\partial\bar{\rho}_w}{\partial\bar{x}} - \frac{\bar{U}(\bar{x} + \eta/2) - \bar{U}(\bar{x} - \eta/2)}{\eta} \frac{\partial\bar{\rho}_w}{\partial p} = 0 . \quad (44)$$

The *transition* from (42) to (44), based on physical arguments, is partially a change of partial differential equation (i.e. (42)) to differential-difference equation (i.e. (44)) which may be considered as ansatz of a *deformation* of the Liouville equation.

Given the smallness of η , multiplying both numerator and denominator of the last term of the l.h.s. by the imaginary unit i , we have:

$$\frac{\bar{U}(\bar{x} + \eta/2) - \bar{U}(\bar{x} - \eta/2)}{i\eta} i \frac{\partial\bar{\rho}_w}{\partial p} \approx \frac{\bar{U}(\bar{x} + \frac{i\eta}{2} \frac{\partial}{\partial p}) - \bar{U}(\bar{x} - \frac{i\eta}{2} \frac{\partial}{\partial p})}{i\eta} \bar{\rho}_w , \quad (45)$$

Thus, going back to the old variables x and z , (44) assumes formally the look of a Von Neumann equation [18, 44] (let us say Von Neumann-like equation):

$$\left\{ \frac{\partial}{\partial z} + p \frac{\partial}{\partial x} + \frac{i}{\epsilon} \left[U \left(x + i \frac{\epsilon}{2} \frac{\partial}{\partial p} \right) - U \left(x - i \frac{\epsilon}{2} \frac{\partial}{\partial p} \right) \right] \right\} \rho_w = 0 , \quad (46)$$

Where $\rho_w \equiv \bar{\rho}_w(2\sigma_0\bar{x}, p, 2\sigma_0\bar{z}; 2\sigma_0\eta) \equiv \rho_w(x, p, z; \epsilon)$. Eq.n (46) shows that in the framework of this effective description, the phase-space evolution equation for electronic rays is a quantum-like phase-space equation where \hbar and the time t are replaced by the emittance ϵ and the propagation coordinate z , respectively.

However, some considerations are in order.

(i). Approximation (43) is due both to the smallness of η and the fact that evaluation of \bar{U} -variation around the location \bar{x} does not make sense within an interval of size η . This, in fact, corresponds to the intrinsic uncertainty produced among the rays by the finite-temperature spreading. In other words, thermal mixing of electronic rays affects the evaluation of U -variation with respect to x . Thus, (44) represents a possible way to take into account the ray-mixing in this evaluation.

(ii). Since $\bar{U}(\bar{x} + \frac{i\eta}{2} \frac{\partial}{\partial p}) - \bar{U}(\bar{x} - \frac{i\eta}{2} \frac{\partial}{\partial p}) = \frac{\partial\bar{U}}{\partial\bar{x}} i\eta \frac{\partial}{\partial p} + O\left(\eta^3 \frac{\partial^3}{\partial p^3}\right)$, approximation (45) is equivalent to assume that terms $O\left(\eta^3 \frac{\partial^3}{\partial p^3}\right)$ are small corrections compared to the lower-order ones, according to the paraxial approximation. Consequently, from the quantum-like point of view, approximation (45) plays the role analogous to the one played by *semi-classical approximation* [45].

(iii). While the distribution $\rho(x, p, z)$ involved in (40) is introduced in a classical framework and it is positive definite, the function $\rho_w(x, p, z; \epsilon)$ is introduced in a quantum-like

framework which plays the role of an *effective* description taking into account the thermal spreading among the electronic rays. In addition, in this context $\rho_w(x, p, z; \epsilon)$ cannot be used to give information within the phase-space cells with size smaller than ϵ , due to the intrinsic uncertainty exhibited by the system for finite temperatures, i.e. due to the indistinguishability among the electronic rays. Consequently, we would expect that ρ_w violates the positivity definiteness within some phase-space regions. On the other hand, even with the limitations given by the point (i). and (ii)., it is clear from the Von Neumann-like equation (46) that ρ_w is a sort of *Wigner-like* function. Thus, it is not positive definite, due to the quantum-like uncertainty principle given in section 3. This means that, in analogy with quantum mechanics, $\rho_w(x, p, z; \epsilon)$ can be defined a *quasi distribution* even its x -projection and p -projection are actually configuration-space distribution and momentum-space distribution, respectively. In particular, within the framework of the above effective description of the electronic ray evolution, we assume that the probability $P_x(x, z; \epsilon)$, introduced above, is:

$$P_x(x, z; \epsilon) = \int \rho_w(x, p, z; \epsilon) dp \quad , \quad (47)$$

provided that also ρ_w is normalized over the phase space.

Note that, for arbitrary U :

$$\lim_{\epsilon \rightarrow 0} \rho_w(x, p, z; \epsilon) = P^{(0)}(x, z) \delta(p - V^{(0)}(x, z)) \equiv \rho_0(x, p, z) \quad , \quad (48)$$

which describes the (transverse) phase-space motion of a *cold* beam. Multiplying the total number of particles by $P^{(0)}(x, z)$ we obtain the transverse space-density of the electronic rays at each z for a cold beam. Furthermore, $V^{(0)}(x, z)$ is the (transverse) current velocity which in this case obeys, with $P^{(0)}(x, z)$, to the following equations

$$\frac{\partial P^{(0)}}{\partial z} + \frac{\partial}{\partial x} (P^{(0)} V^{(0)}) = 0 \quad , \quad (49)$$

(continuity equation),

$$\left(\frac{\partial}{\partial z} + V^{(0)} \frac{\partial}{\partial x} \right) V^{(0)} = - \frac{\partial U}{\partial x} \quad , \quad (50)$$

(fluid motion equation). Note that in the above limit the local slope of the electronic rays $p = dx/dz$ is determined only by the gradient of U . In particular, in vacuo ($U = 0$) a cold uniform beam has phase-space density of the form $P_0 \delta(p - V_0)$, with P_0 and V_0 constants. With the language of particle accelerator physics, this kind of beam is called *monochromatic beam*. It is easy to see that all the electron rays of a monochromatic beam have the same slope.

The above results allow us to write that, for an arbitrary potential, we have

$$\lim_{\epsilon \rightarrow 0} P_x(x, z; \epsilon) = P^{(0)}(x, z) \quad . \quad (51)$$

Remarkably, from the above results it follows that it may exist a complex function, say $\Psi(x, z)$ such that

$$P_x(x, z; \epsilon) = \Psi(x, z) \Psi^*(x, z) \quad , \quad (52)$$

used also for description of pure quantum states, and the following *quantum-like density matrix*

$$G(x, x', z) = \Psi(x, z)\Psi^*(x', z) \quad , \quad (53)$$

used also for description of mixed quantum states, connected with ρ_w by means of the following Wigner-like transformation:

$$\rho_w(x, p, z; \epsilon) = \frac{1}{2\pi\epsilon} \int_{-\infty}^{\infty} G\left(x + \frac{y}{2}, x - \frac{y}{2}, z\right) \exp\left(i\frac{py}{\epsilon}\right) dy \quad , \quad (54)$$

or, for pure states

$$\rho_w(x, p, z; \epsilon) = \frac{1}{2\pi\epsilon} \int_{-\infty}^{\infty} \Psi^*\left(x + \frac{y}{2}, z\right) \Psi\left(x - \frac{y}{2}, z\right) \exp\left(i\frac{py}{\epsilon}\right) dy \quad . \quad (55)$$

Consequently, $\Psi(x, z)$ must obey to the following Schrödinger-like equation:

$$i\epsilon \frac{\partial \Psi}{\partial z} = -\frac{\epsilon^2}{2} \frac{\partial^2}{\partial x^2} \Psi + U(x, z)\Psi \quad . \quad (56)$$

This equation has been the starting point to construct the quantum-like approach of charged-particle beams which is known in the literature as Thermal Wave Model (TWM). It has been applied to a number of problems in particle accelerators and plasma physics [19]-[25]. TWM assumes that the transverse (longitudinal) dynamics of a charged particle beam, interacting with the surroundings, is governed by a Schrödinger-like equation for a complex function in which Planck's constant is replaced by the transverse (longitudinal) beam emittance. This complex function, called beam wave function (BWF) has the following meaning: its squared modulus is proportional to the transverse (longitudinal) beam density. This way the beam as a whole is thought as a single quantum-like particle whose *diffraction-like* spreading due to the emittance accounts for the *thermal spreading*.

5 Quantum-like corrections

In this section we analyze the quantum-like corrections [46] that the *effective* electronic ray description presented above gives with respect to the *pure* classical treatment. In other words, we make a comparison between the quantum-like description given by (46) and the one given by (40). To this end, we first observe that, if the beam is in a quadrupole (linear lens), (46) collapses in (40) and no quantum-like corrections are present. One can calculate the set of moment equations associated with (40) and (46) , respectively. Defining the following *Liouville operator*

$$\hat{\mathcal{L}} \equiv \frac{\partial}{\partial z} + p \frac{\partial}{\partial x} - \left(\frac{\partial U}{\partial x} \right) \frac{\partial}{\partial p} \quad , \quad (57)$$

U being an arbitrary potential which can be expanded in Taylor series with respect to x . It is easy to see that (40) and (46) become, respectively

$$\hat{\mathcal{L}}\rho_w = 0 \quad . \quad (58)$$

and

$$\hat{\mathcal{L}}\rho_w = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{\epsilon}{2} \right)^{2k} \frac{\partial^{2k+1} U}{\partial x^{2k+1}} \frac{\partial^{2k+1} \rho_w}{\partial p^{2k+1}} \quad , \quad (59)$$

By introducing the ν -order (ν being a non-negative integer) moment of $\hat{\mathcal{L}}$ as

$$\mathcal{M}^{(\nu)}(x, z) \equiv \int_{-\infty}^{\infty} p^\nu \hat{\mathcal{L}} \rho_w dp \quad , \quad (60)$$

the classical equation (58) leads to

$$\mathcal{M}^{(\nu)}(x, z) = 0 \quad \forall \nu \geq 0 \quad , \quad (61)$$

which, in turn, gives

– for $\nu = 0$, the continuity equation

$$\frac{\partial P_x}{\partial z} + \frac{\partial}{\partial x} (P_x V) = 0 \quad , \quad (62)$$

– for $\nu = 1$, the motion equation

$$\left(\frac{\partial}{\partial z} + V \frac{\partial}{\partial x} \right) V = -\frac{\partial U}{\partial x} - \frac{1}{P_x} \frac{\partial \Pi}{\partial x} \quad , \quad (63)$$

– for $\nu = 2$, the energy equation

$$\mathcal{M}^{(2)}(x, z) = 0 \quad , \quad (64)$$

and so on, where

$$V(x, z) = \frac{1}{P_x} \int_{-\infty}^{\infty} p \rho_w dp \quad , \quad (65)$$

is the current velocity, which is experimentally the first order moment of ρ_w , and

$$\Pi(x, z) \equiv \int_{-\infty}^{\infty} (p - V)^2 \rho_w dp \quad (66)$$

is the kinetic pressure (divided by the total number of the particles) or the second order moment of ρ_w .

On the other hand, the quantum-like equation (59) gives

$$\mathcal{M}^{(\nu)}(x, z) = 0 \quad \nu = 0, 1, 2 \quad (67)$$

and

$$\mathcal{M}^{(\nu)}(x, z) = - \sum_{k=1}^{\infty} (-1)^k \binom{\nu}{2k+1} \left(\frac{\epsilon}{2} \right)^{2k} \frac{\partial^{2k+1} U}{\partial x^{2k+1}} \int_{-\infty}^{\infty} p^{\nu-2k-1} \rho_w dp \neq 0 \quad \forall \nu \geq 3 \quad . \quad (68)$$

Consequently, for an arbitrary potential and up to the energy equations, the two descriptions (the classical and the quantum-like) coincide. The discrepancy appears at the order equal or higher than the 3-rd one in the moment equations. In principle equations (58) and (59) are equivalent to an infinite set of their moment equations (61) and (67)-(68), respectively. The characteristic of these moment equations is that the one of ν -order is an evolution for the ν -order moment of ρ_w , but contains $(\nu + 1)$ -order moment of this function. Provided that a *closure equation* is introduced, which relates $(\nu + 1)$ -order moment with the lower-order ones, the truncated set of equations, consisting of moment

equations up to the ν -order plus the closure equation, is fully equivalent to (58) or (59), respectively.

Usually, the lowest order of truncation is introduced for $\nu = 1$, by introducing, for the transverse dynamics, the following ideal gas state equation [46] (isothermal approximation):

$$P_x \frac{k_B T}{m c^2} = \Pi \quad . \quad (69)$$

In fact, even if the beam propagates along z with relativistic motion, the transverse particle motion around z is classical. Consequently, the beam behaves transversally like a nonrelativistic ideal gas. Moreover, note that, denoting with N the total number of beam particles, the quantity $n \equiv N P_x$ is the transverse number density of the beam. At this level, we are describing our beam in terms of the fluid theory

$$\frac{\partial P_x}{\partial z} + \frac{\partial}{\partial x}(P_x V) = 0 \quad , \quad (70)$$

$$\left(\frac{\partial}{\partial z} + V \frac{\partial}{\partial x} \right) V = - \frac{\partial U}{\partial x} - \frac{v_{th}^2}{c^2} \frac{1}{P_x} \frac{\partial P_x}{\partial x} \quad . \quad (71)$$

It is obvious from (61)–(68) that *the classical and the quantum-like descriptions coincide at the level of the fluid theory for non-cold beams*. Note that, in particular, in the limit $\epsilon \rightarrow 0$, (70) and (71) recover (49) and (50), respectively.

Going on to $\nu = 2$, for the truncation a closure equation involving the moments of 3rd order and the lower ones have to be introduced. By virtue of (62)–(64) and (67), *the descriptions coincide also at this level if a suitable closure equation is chosen for both*.

For orders $\nu \geq 3$, according to (68), the truncation cannot allow for having the equivalence between the classical and the quantum-like descriptions. In particular, the 3rd-order moment equation ($\nu = 3$) of (59) is

$$\mathcal{M}^{(3)} = \left(\frac{\epsilon}{2} \right)^2 \left(\frac{\partial^3 U}{\partial x^3} \right) P_x \quad , \quad (72)$$

and the one for 4th-order moment ($\nu = 4$) is

$$\mathcal{M}^{(4)} = 4 \left(\frac{\epsilon}{2} \right)^2 \left(\frac{\partial^3 U}{\partial x^3} \right) P_x V \quad . \quad (73)$$

The above analysis, allows us to conclude that *at the third-order moment description the discrepancy between the classical and the quantum-like descriptions appears as a very delicate effect*. In fact, from the experimental point of view, one can easily measure the moments up to the second-order, but the measure of the higher-order moments is a very hard task. Moreover, it is clear from (72) and (73) that, for arbitrary potentials and for given emittances, the discrepancy increases as the density of the beam. Thus, to make it evident very intense beams are necessary.

In addition, if U is a symmetric potential with respect to the propagation direction z , i.e. $U(-x, z) = U(x, z)$, the discrepancy corresponding to the 3rd- and the 4th-order moments are still negligible for beams that are mainly concentrated around z (x very close to zero), because in this case $\frac{\partial^3 U}{\partial x^3} \propto x \approx 0$.

6 Conclusions and remarks

In this paper, the charged-particle beam transport has been investigated with a quantum-like approach. By starting from the electronic-ray concept in paraxial approximation, we have given the statistical description of the electronic ray evolution which has allowed us for obtaining a quantum-like picture of a charged-particle beam transport, where a sort of quantum-like uncertainty principle holds for the spread of particle position distribution and the spread of particle momentum. This way we first introduced a sort of *Wigner-like* pictures behind the electronic ray evolution and then recovered the already known quantum-like description for charged-particle beam dynamics called Thermal Wave Model [19]-[25]. It is worth noting that, within the Wigner-like picture given in section 4, (56) has not been fully recovered, according to all the assumptions of TWM. In fact, (56) is fully consistent with (46) even when the terms $O\left(\eta^3 \frac{\partial^3}{\partial p^3}\right)$ are not small corrections, whilst the effective description developed in section 4 is consistent only with semi-classical approximation. Consequently, solution of (56) for BWF Ψ in semiclassical approximation can give solutions for the *deformed* equation (44) via Wigner-like transform (55).

Within the framework of the Wigner-like picture, the quantum-like corrections have been introduced and compared with the standard classical picture for arbitrary potentials, showing that the above quantum-like approach could be a useful tool for particle accelerator physics investigation. It is worth mentioning that this comparison is in agreement with a recent numerical phase-space analysis which compares the quantum-like Wigner function of charged-particle beam in a quadrupole with small sextupole and octupole aberrations with the results of a standard particle tracking code simulation [25].

Nevertheless, we want to remark that this approach could be relevant also for a wide spectrum of topics in e.m. radiation optics, general quantum mechanics and quantum optics for the considerations that are in order.

(1). Eq. (46) collapses in (40) in the case of quadrupole (harmonic oscillator). However, due to the Wigner-like picture, (46) describes some states that are not described by its classical-like counterpart. In other words, the similarity between ρ and ρ_w in the harmonic oscillator is not possible for all the states. This makes evident a *quantum-like effect* that ρ_w contains and that ρ does not contain. Of course, according to the investigation about the discrepancy given above, *this quantum-like effect is hardly measurable but not in principle negligible*. Additionally, note that also (42) and (44) become the same equation in the case of quadrupole, where ϵ does not appears explicitly. However, a possible normalized solution of this Liouville equation for harmonic oscillator is [25]

$$\frac{1}{\pi\epsilon} \exp\left[-\frac{2}{\epsilon} \left(\gamma(z)x^2 + 2\alpha(z)xp + \beta(z)p^2\right)\right] , \quad (74)$$

which explicitly depends on ϵ . Consequently, in principle, to recover classical solutions we do not need necessarily to take the limit $\epsilon \rightarrow 0$. In this limit we can recover the special family of classical solutions that describe the cold-beam solution only, as it has been pointed out in section 4. This means that (40) contains something more than the classical limit. In fact, solution (74), in which ϵ is a finite quantity, leads easily to the quantum-like uncertainty relation (28).

(2). From (59) it is clear that for finite emittance but in the case in which $\left(\frac{\epsilon}{2}\right)^2 \frac{\partial^2 \rho_w}{\partial p^2} \gg$

$\left(\frac{\epsilon}{2}\right)^s \frac{\partial^s \rho_w}{\partial p^s}$ for $s \geq 3$, (46) and (40) formally coincide for an arbitrary (anharmonic) potential. However, also in this case, ρ_w contains in principle the quantum-like effects that ρ does not contain.

(3). Since in quantum mechanics and in quantum optics the measuring of the states described by Wigner functions was recently reduced by means of tomographic procedures to measuring positive marginal distribution related to Wigner function by an integral transform (the Radon transform of optical tomography method [47, 48] or Fourier transform of symplectic tomography [27, 28]), we could state that in the above quantum-like approach there is a possibility to transit from Liouville equation to an equation for a positive marginal distribution of two types [28] which has standard classical features.

(4). Finally, we want to remark that, even if we have given a quantum-like picture for charged-particle beam transport, fully consistent with the quantum-like uncertainty relation (28), our description does not contradict classical mechanics. In fact, while \hbar is a fundamental, universal constant, ϵ has not such properties. Since the latter depends on the thermal noise, we can, *in principle*, arrange a series of experimental devices in which the temperature is progressively reduced. This way, we enhance the accuracy in finding the electronic ray location by reducing the thermal uncertainty more and more. Consequently, the quantum-like uncertainty *in principle* collapses into the *classical* independence between measuring of spot-size and momentum-spread. In this sense, our *effective* description is formally *quantum-like* but intrinsically *classical*. Of course, a *natural* limitation in reducing the thermal noise is established by the *proper* quantum uncertainty relation which states that quantum fluctuations are *unavoidable* and *intrinsic*. In fact, the nature of the physical systems is basically *quantum* and not *classical*, but *this is true for all the systems in nature and not only for charged-particle beams!*

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