ISTITUTO NAZIONALE DI FISICA NUCLEARE

Sezione di Milano

INFN/TC-97/11
16 Aprile 1997

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PACS: 41.75.-i; 41.85.-p; 29.27.-a; 29.27.-Bd

SIS–Pubblicazioni
dei Laboratori Nazionali di Frascati

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AN ENVELOPE TREATMENT OF SPACE CHARGE DOMINATED PROTON BEAMS IN RF LINACS

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Abstract

In this paper we provide a theoretical description of the transverse as well longitudinal envelope dynamics of space charge-dominated proton beams accelerated in radio-frequency linear accelerators. These results provide a basis for general analytical investigation of the beam dynamics of beams in the trans-relativistic regime, extending to heavy particles the previously developed envelope theory of space-charge dominated electron beams. We also investigate possible equilibrium solutions of the coupled longitudinal and transverse envelope equations in the laminar regime, as well as the stability of these solutions to perturbations provided by an alternating gradient focusing lattice. Application of these results for mitigating halo formation in high average current proton beams for nuclear fission and waste transmutation are discussed.

I. Introduction

In this work, we will be examining the behavior of trans-relativistic, bunched beams with nontrivial dynamical space charge effects in evidence, of the type which may be found in high-current proton radio-frequency (RF) linear accelerators (linacs). This work is completely general, however, in that it describes in a fully relativistic formalism, the envelope dynamics of any bunched beam accelerating in a RF linac. This treatment represents an extension of previous work on ultra-relativistic space-charge dominated electron beams in two fundamental ways: the evaluation of the forces in the more complicated trans-relativistic case, and the inclusion of a longitudinal envelope treatment. This second point is necessary for heavy particles, because their longitudinal dynamics are in general not simply ballistic as in the case of electrons in high-gradient RF linacs.

II. The Relativistic Envelope Equation under RF Acceleration and External Focusing

We begin by writing the envelope equations for a round Kapchinskii-Vladimirski beam, with uniformly charge distribution inside an ellipsoid of radius R and longitudinal semi-axis L, as

$$R'' + R'\frac{(\beta\gamma)'}{(\beta\gamma)} + \mathrm{K}_r R = \frac{2I/I_0}{R\,(\beta\gamma)^3} + \frac{\varepsilon_{n,th}^2}{R^3\,(\beta\gamma)^2} \tag{2.1}$$

$$L'' + \mathrm{K}_z L = \frac{Qc}{I_0\beta^2\gamma^4 R\;L} + \frac{\varepsilon_{n,z}^2}{\beta^2\gamma^6 L^3} \tag{2.2}$$

where $I_0 = 3.1 \cdot 10^7$ A, Q is the proton bunch charge, $I = \frac{3Q\beta c}{4L}$ is the peak current, c the speed of light, $\gamma = 1 + \frac{T}{m_p c^2}$ the normalized energy ($m_p c^2 = 938$ MeV) and $\beta \equiv \frac{v_z}{c} = \sqrt{1 - 1/\gamma^2}$ the normalized velocity along the linac z-axis. The two equations are valid in the paraxial approximation, with the reference axis being the independent variable z, so that $R' \equiv \frac{dR}{dz}$, $L' \equiv \frac{dL}{dz}$ and $\gamma' \equiv \frac{d\gamma}{dz} = \frac{eE_{acc}}{m_p c^2} \cos\varphi_0$, with E_{acc} the accelerating gradient. The two focusing gradients, transverse K_r and longitudinal K_z, are given respectively by:

$$K_r = -\frac{\overline{F}_r}{m_p c^2 \beta^2 \gamma} \frac{1}{r} \tag{2.3}$$

$$K_z = \frac{2k\gamma_0' \sin|\varphi_0|}{(\beta\gamma)^3} \tag{2.4}$$

where $\overline{F}_r$ is the average applied force by the external focusing system and k is the RF wave number $k \equiv 2\pi/\lambda_{RF}$, with λ_{RF} being the RF wavelength. As usual, the longitudinal (synchrotron) focusing gradient K_z assumes a negative accelerating phase φ_0 in order to achieve a stable motion in the longitudinal plane, if $\varphi_0 = 0$ is defined to be the phase of maximum acceleration, so that the actual energy evolution along the linac is assumed to be $\gamma = \gamma_0 + \gamma' z = \gamma_0 + \gamma_0' \cos\varphi_0 \cdot z$.

The average force $\overline{F}_r$ from the external focusing system will have, in general many contributions, i.e. $\overline{F}_r = \overline{F}_{RF} + \overline{F}_Q + \overline{F}_{SOL}$. The three separate terms are:

i) $\overline{F}_{SOL} = \Omega_L |qB_0| \frac{r}{2\gamma}$ $(\Omega_L \equiv \frac{|qB_0|}{2m_p})$, corresponding to an uniform solenoid of field B_0.

ii) $\overline{F}_Q = -\frac{m_p c^2 \beta^2 \gamma}{\beta_Q^2} r$, where $\beta_Q = L_{1at}/\phi_\beta$ is the average beta function of the quadrupole lattice, which has periodicity L_{1at} and betatron phase advance (in the absence of space-charge forces) per period of ϕ_β.

iii) $\overline{F}_{RF}$ is the average (over an RF cavity cell) force from the transverse components of the RF field, which we next delineate.

The RF field components $\{E_z, E_r, B_\theta\}$ of the $TM_{010\text{-}\pi}$ resonant mode in an indefinitely long multi-cell structure can be written as:

$$E_z(r,z,t) = \cos(\omega t + \varphi_0) \sum_{n,odd} a_n \cos(nk'z) I_0\left(r\sqrt{n^2k'^2 - k^2}\right)$$

$$E_r(r,z,t) = \cos(\omega t + \varphi_0) \sum_{n,odd} nk' a_n \sin(nk'z) \frac{I_1\left(r\sqrt{n^2k'^2 - k^2}\right)}{\sqrt{n^2k'^2 - k^2}} \qquad (2.5)$$

$$cB_\theta(r,z,t) = -\sin(\omega t + \varphi_0) \sum_{n,odd} k a_n \cos(nk'z) \frac{I_1\left(r\sqrt{n^2k'^2 - k^2}\right)}{\sqrt{n^2k'^2 - k^2}}$$

where $\omega \equiv k \cdot c$, $k' \equiv \frac{\pi}{d} = k\frac{\lambda_{RF}}{2d}$ and d is the RF cell length. The synchronous particle speed β_S is then easily seen to be $\beta_S = \frac{2d}{\lambda_{RF}} = \frac{k}{k'}$. The linear expansion off-axis of the fields in Eq.1.5 becomes (E_z is actually expanded up to second order in r):

$$E_z^{II} = \cos(\omega t + \varphi_0) \sum_{n,odd} a_n \cos\left(n\frac{kz}{\beta_S}\right)\left[1 + \frac{(kr)^2}{4}\left(n^2/\beta_S^2 - 1\right)\right]$$

$$E_r^I = \cos(\omega t + \varphi_0)\frac{kr}{2\beta_S}\sum_{n,odd} n a_n \sin\left(n\frac{kz}{\beta_S}\right) \qquad (2.6)$$

$$cB_\theta^I = -\sin(\omega t + \varphi_0)\frac{kr}{2}\sum_{n,odd} a_n \cos\left(n\frac{kz}{\beta_S}\right)$$

By employing the Panofsky-Wenzel theorem, the radial momentum change Δp_r induced on a particle crossing the cell (extending between $z_i = -\frac{d}{2}$ and $z_e = \frac{d}{2}$) at constant speed β_S (so that $\omega t = \frac{k}{\beta_S} z$) is given by

$$\Delta p_r = q\left\{\left[A_r(z_i) - A_r(z_e)\right] + \int_{-d/2}^{d/2} \frac{\partial A_z}{\partial r} dz\right\} \qquad (2.7)$$

where the vector potential $\vec{\mathbf{A}}$ is such that $\vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{A}}}{\partial t}$. We find that

$$A_r = \frac{-r}{4c\beta_S}\sum_{n,odd} na_n\left[\cos\left((n-1)\frac{kz}{\beta_S}+\varphi_0\right)-\cos\left((n+1)\frac{kz}{\beta_S}+\varphi_0\right)\right];\text{ since } \frac{kz_i}{\beta_S}=-\frac{\pi}{2},\ \frac{kz_e}{\beta_S}=\frac{\pi}{2}$$

and $(n-1)$, $(n+1)$ are even numbers, the term in square brackets in Eq.1.7 vanishes, so that the momentum change arises only from the second order off-axis dependence of the longitudinal (accelerating) component E_z through the integral $\int_{-d/2}^{d/2}\frac{\partial A_z}{\partial r}dz$. This integral is null for a relativistic $(\beta_S \cong 1)$ particle crossing a $\lambda/2$ cell. In the present case, defining $a_1 = E_0$, $\bar{a}_3 = a_3/E_0$, $\bar{a}_5 = a_5/E_0$,...., we find

$$\frac{\Delta p_r}{m_p c} = -\alpha(kr)\frac{\pi}{2}\frac{\sin\varphi_0}{\beta_S\gamma_S^2}, \tag{2.8}$$

where the dimensionless quantity α is defined as $\alpha \equiv \frac{eE_0}{2m_p c\omega}$ (note that $\gamma_0' = \alpha k$). The "direct" momentum change Δp_r , which is a first order (in the field amplitude) effect, is therefore to be independent on the higher space harmonics. We may also apply Eq.1.7 to find the momentum change induced on a particle travelling at constant speed β which is slightly different from β_S , so that $\beta = \beta_S(1+\delta)$, with $\delta << 1$. Since we are interested in averaging the force over a cell, we calculate the average effective force $\bar{F}_{PW} \equiv \Delta p_r^{PW}/(T) = 2\beta c\Delta p_r^{PW}/(\beta_S\lambda_{RF})$ due to direct first order effects which are described by Panofsky-Wenzel theorem, to find

$$\bar{F}_{PW} = \frac{qE_0 kr}{4\beta_S}\sin\varphi_0\left[\left(1+3\beta_S^2\right)\frac{\delta}{2}-\frac{1}{\gamma_S^2}-\frac{2\delta}{\gamma_S^2}\Sigma\right], \tag{2.9}$$

where $\Sigma \equiv \sum_{n=3,odd}(-1)^{\frac{n-1}{2}}\frac{n\bar{a}_n}{n^2-1}$. The corresponding focusing gradient K_r^{PW} is then

$$K_r^{PW} = \frac{\gamma_0' k}{2\beta_S^3\gamma_S}\sin\varphi_0\left[-\left(1+3\beta_S^2\right)\frac{\delta}{2}+\frac{1}{\gamma_S^2}+\frac{2\delta}{\gamma_S^2}\Sigma\right]. \tag{2.10}$$

In order to calculate second order effects we must perform an averaging of the force over a cell of the RF structure. The net transverse force is given by

$$F_r = q\frac{kr}{2\beta_S}\sum_{n,odd}a_n\left\{n\cos(\omega t+\dot{\varphi}_0)\sin\left(n\frac{kz}{\beta_S}\right)+\beta\beta_s\sin(\omega t+\varphi_0)\cos\left(n\frac{kz}{\beta_S}\right)\right\} \tag{2.11}$$

We are mainly concerned about two contributions possible sources of second-order focusing — particle speed change across the cell and ponderomotive focusing. The first effect is treated assuming a small speed change, so that, since $d\beta = \frac{d\gamma}{\beta\gamma^3}$, to first order in $d\beta$ we can take a particle speed given by $\beta = \beta_S + \frac{\alpha kz}{\beta_S\gamma_S^3}$, so that $\omega t = \frac{1}{\beta_S c}\int_0^{kz}\frac{dx}{1+\frac{\alpha x}{\beta_S\gamma_S^3}} \cong \frac{kz}{\beta_S}$. The force then becomes

$$F_r^{\Delta\beta} = q\frac{kr}{2\beta_S}\times$$
$$\sum_{n,odd} a_n\left\{n\cos\left(\frac{kz}{\beta_S}+\varphi_0\right)\sin\left(n\frac{kz}{\beta_S}\right)+(1+\frac{\alpha kz}{\beta_S^2\gamma_S^3})\beta_S^2\sin\left(\frac{kz}{\beta_S}+\varphi_0\right)\cos\left(n\frac{kz}{\beta_S}\right)\right\}, \tag{2.12}$$

and the corresponding average focusing gradient

$$K_r^{\Delta\beta} = -\frac{\gamma_0'^2}{4\beta_S^2\gamma_S^4}\cos\varphi_0[1+4\Sigma]. \tag{2.13}$$

The second effect is due to the ponderomotive focusing; we calculate it under the approximation of synchronous velocity, *i.e.* taking a force given by

$$F_r^{PO} = q\frac{kr}{2\beta_S}\sum_{n,odd} a_n\left\{n\cos\left(\frac{kz}{\beta_S}+\varphi_0\right)\sin\left(n\frac{kz}{\beta_S}\right)+\beta_S^2\sin\left(\frac{kz}{\beta_S}+\varphi_0\right)\cos\left(n\frac{kz}{\beta_S}\right)\right\} \tag{2.14}$$

Performing an average analogous to the treatment found in Ref.1 we obtain an effective focusing gradient

$$K_r^{PO} = \frac{1}{8}\frac{(\beta_S\gamma_S)'^2}{(\beta_S\gamma_S)^2\cos^2\varphi_0}\eta(a_n,\beta_S,\varphi_0), \tag{2.15}$$

with

$$\eta = \frac{(1+\beta_S^2)^2}{4}+2\sum_{n=3,odd}\frac{n^2/\gamma_S^4+(n^2-\beta_S)^2}{(n^2-1)^2}\bar{a}_n^2+\bar{a}_{n-2}\bar{a}_n\cos 2\varphi_0\left(1-\frac{1}{(n^2-1)^2\gamma_S^4}\right). \tag{2.16}$$

The envelope equation for transverse motion, Eq.1.1, then becomes

$$R''+R'\frac{(\beta\gamma)'}{(\beta\gamma)}+\left(K_r^{PO}+K_r^{\Delta\beta}+K_r^{PW}+\frac{1}{\beta_Q^2}+\frac{\Omega_L^2}{(\beta c\gamma)^2}\right)R=\frac{2I/I_0}{R\,(\beta\gamma)^3}+\frac{\varepsilon_{n,th}^2}{R^3\,(\beta\gamma)^2},$$

which can be cast in a more convenient form as (dropping the subscript $_S$ for β_S and γ_S)

$$R''+R'\frac{(\beta\gamma)'}{(\beta\gamma)}+\Omega^2 R\frac{(\beta\gamma)'^2}{(\beta\gamma)^2}=\frac{2I/I_0}{R\,(\beta\gamma)^3}+\frac{\varepsilon_{n,th}^2}{R^3\,(\beta\gamma)^2} \tag{2.17}$$

with

$$\Omega^2=\frac{1}{\cos^2\varphi_0}\left\{\begin{array}{l}\frac{\eta}{8}+\beta^2b^2-\frac{\beta^2(1+4\Sigma)\cos\varphi_0}{4\gamma^2}+\frac{\beta^4\gamma^2}{\beta_Q^2\gamma_0'^2}+\\ \frac{\sin\varphi_0}{2\alpha}\left[\frac{\beta}{\gamma}-\frac{\delta}{2}\left(\left(1+3\beta^2\right)\beta\gamma-4\frac{\beta}{\gamma}\Sigma\right)\right]\end{array}\right\}, \tag{2.18}$$

and $b\equiv cB_0/E_0$, which clearly reveals the symmetry of the equation once the laminarity assumption and the invariance of the normalized focusing frequency Ω is invoked. In fact, by dropping the emittance term (assuming the beam flow is in the laminar limit) we obtain

$$R''+R'\frac{p'}{p}+\Omega^2 R\frac{p'^2}{p^2}-\frac{2I/I_0}{R\,p^3}=0,\qquad (p\equiv\beta\gamma) \tag{2.19}$$

This result is a general, fully relativistically correct envelope equation for charged particle beams if thermal emittance effects and alternating gradient nature of the quadrupole focusing are ignored. These effects are trivial to recover, with the restoration of the emittance term from the right-hand-side of Eq. 2.17, and by using the exact local, rather than average, focusing effects of the quadrupole fields.

III. Relativistically Correct Invariant Envelope Solutions

Equation 2.19 is a nonlinear differential equation with no general analytical solution available. It has been noted in previous work[?], however, that an exact particular solution to this type of equation exists, which is a generalization of the equilibrium solution to the envelope equation of a coasting beam — the Brillioiun flow. In this case, this type of solution, which is termed the *invariant* envelope in Ref. 2, is given by

$$\hat{R} = \frac{2\beta_i}{\gamma_i' \cos\varphi_0} \sqrt{\frac{2\bar{I}}{\beta\gamma\left(1+4\Omega^2\right)}}. \tag{3.1}$$

It should be noted that this solution is valid under the assumption, to be justified of constant peak current I (here $\bar{I} \equiv I/I_0$) and invariance of the *momentum* growth rate, *i.e.* $p' = (\beta\gamma)' = \gamma'/\beta = \text{const}$. This second condition implies that the accelerating gradient must scale as β, *i.e.* $\gamma' = \gamma_i'\beta/\beta_i$, where $\gamma_i' = \alpha_i k \cos\varphi_0$ and β_i are the corresponding values of γ' and β at injection. It is important to note at this point that Eqs. 1.18-19 and 2.1 are generalizations of the previous results obtained for ultra-relativistic beam envelopes. In fact, the solution of Eq.1.20 for $\hat{R}$ becomes the invariant envelope $\hat{\sigma}$ found in Ref. 2

in the limit that $\beta \to 1$. Also the beam divergence $\hat{R}'$ comes out to be negative (convergent beam) and proportional to the beam spot $\hat{R}$, as given by $\hat{R}' = -\frac{\hat{R}}{2}\frac{(\beta\gamma)'}{(\beta\gamma)}$, in full analogy with the ultra-relativistic case ($\hat{\sigma}' = -\frac{\hat{\sigma}}{2}\frac{\gamma'}{\gamma}$). The normalized frequency Ω also tends to the value $\Omega^2 = \frac{\eta/8 + b^2}{\cos^2\varphi_0}$ in this limit, a value which is invariant along the acceleration (*i.e.* no longer dependent on γ). This is due to the vanishing as $1/\gamma$ of the first order RF effects in the expression for Ω, thus removing the strong velocity dependence of the transverse defocusing effects of the synchronous component of the accelerating wave.

In order to keep Ω constant during the acceleration in the general case we are now considering, one can vary the focusing of the quadrupole lattice as a function of γ and the phase φ_0. For the simple case of an ideally synchronous linac (*i.e.* $\delta = 0$) with a pure first harmonic accelerating field (*i.e.* $a_3 = a_5 = ... = 0$, so that $\Sigma = 0$, $\eta = (1+\beta^2)^2/4$), and one obtains

$$\beta_Q = \frac{\beta_i \beta\gamma}{\gamma_i' \sqrt{\Omega^2\cos^2\varphi_0 - \frac{(1+\beta^2)^2}{32} + \frac{\beta^2\cos\varphi_0}{4\gamma^2} - \frac{\beta_i\sin\varphi_0}{2\alpha_i\gamma}}}. \qquad (3.2)$$

Since α is of the order of 10^{-3} for proton linacs, and assuming a negative accelerating phase φ_0 as required for longitudinal stability, Eq. 3.2 can be well approximated by

$$\lambda_{\beta Q} = \lambda_{RF}\beta\sqrt{\frac{2\beta_i\gamma^3}{\alpha_i \sin|\varphi_0|}}, \tag{3.3}$$

which gives the betatron wavelength $\lambda_{\beta Q}$ which must be provided by the quadrupoles in order to cancel the first order RF defocusing effects.

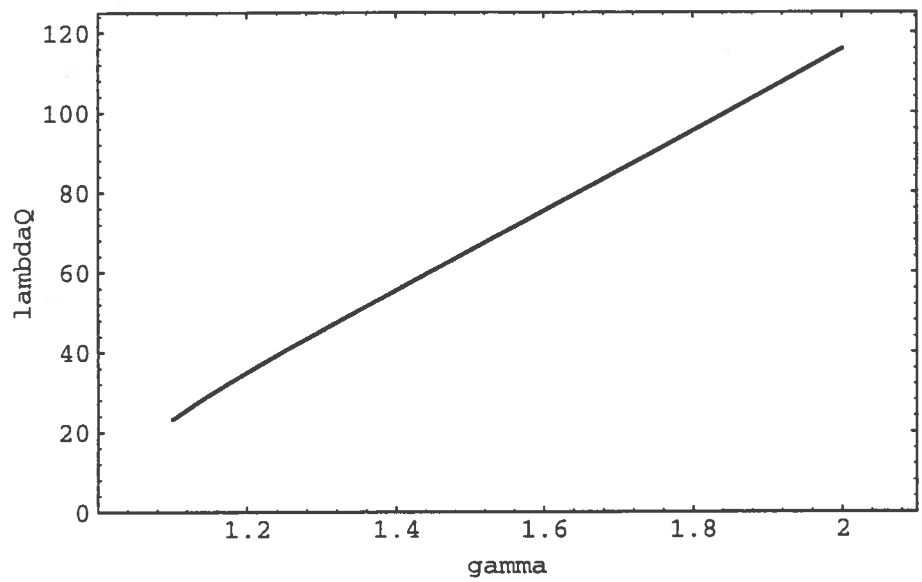

Fig.1 - Betatron wavelength from quadrupoles needed to cancel the first order RF defocusing, for $\lambda_{RF} = 0.85$ m $E_{acc} = 5$ MV/m and $\varphi_0 = -20^\circ$.

For a typical value $\alpha_i = 0.7 \cdot 10^{-3}$ (corresponding, *e.g.*, to $E_0 = 10$ MV/m and $\lambda_{RF} = 0.85$ m, and hence $E_{acc} = 5$ MV/m) and $\varphi_0 = -20^\circ$, the corresponding $\lambda_{\beta Q}$ is shown in Fig.1.

One could also notice that such a focusing regime (*i.e.* an ideal second order focusing with constant normalized gradient $\Omega^2 \cos^2 \varphi_0$) can be achieved by applying a solenoid field, which is represented by the parameter b in Eq. 2.18. We find that, taking again $\delta = 0$, $\Sigma = 0$ (in this case $E_0 = 2E_{acc}$), the magnetic field produced by the external solenoid should be given by

$$B_0 = \frac{2E_{acc}}{\beta c}\sqrt{\Omega^2 \cos^2 \varphi_0 - \frac{\left(1+\beta^2\right)^2}{32} + \frac{\beta^2 \cos \varphi_0}{4\gamma^2} + \frac{\beta_i \sin|\varphi_0|}{2\alpha_i \gamma}} \qquad (3.4)$$

The maximum amplitude of the solenoid field B_0 occurs at injection (where the first order defocusing effect by the cavities is stronger): for the same parameter set listed above we find $B_0[kG] = 0.8\sqrt{92.5 + 0.88\Omega^2}$, giving $B_0 = 1\text{T}$ at $\Omega^2 = 70$.

In order to check the validity of our assumption of beam laminarity, we must evaluate the ratio between the space-charge term and the emittance term in Eq. 2.17. This ratio is defined as the parameter $\rho \equiv \dfrac{2R^2\, I/I_0}{\varepsilon_{n,th}^2 \;\; \beta\gamma}$, where the laminar flow assumption holds whenever $\rho >> 1$. When the beam is transported under the equilibrium condition specified by $\hat{R}$ (Eq. 3.1) , we have $\hat{\rho} \equiv \dfrac{1}{1+4\Omega^2}\left[\dfrac{4\beta_i \bar{I}}{\gamma_i' \cos\varphi_0 \varepsilon_{n,th} \beta\gamma}\right]^2$. The behavior of $\hat{R}$ and $\hat{\rho}$ versus γ for a linac boosting 100 MeV protons (γ=1.1) up to 1 GeV (γ=2) is shown in Fig. 2, for the cases of a 10 A peak current beam with $\varepsilon_{n,th} = 1$ mm·mrad and $E_{acc} = 5$ MV/m , at different values of Ω^2 (20 and 80 for the solid and dashed lines respectively).

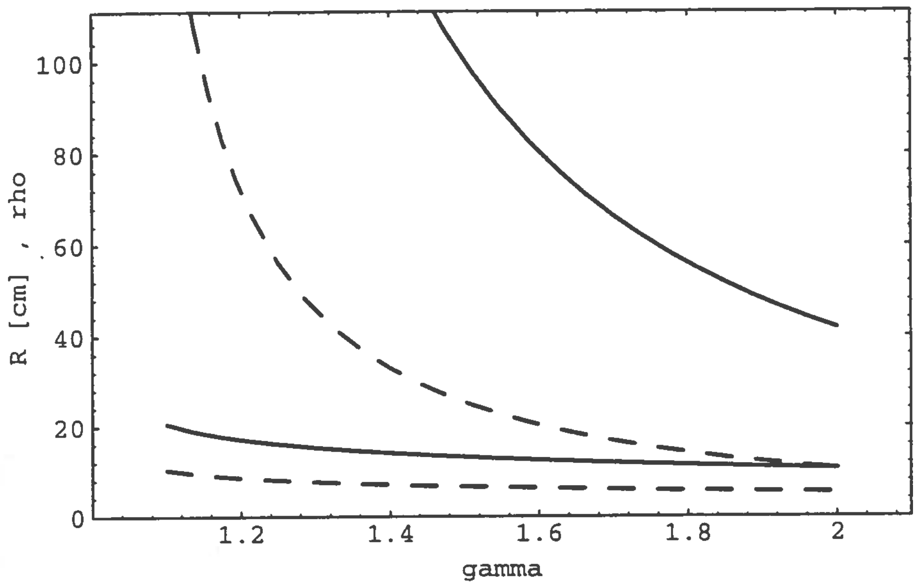

Fig.2 - Behavior of $\hat{R}$ (lower curve in mm) and $\hat{\rho}$ (upper curve) versus γ .

As clearly shown in the chosen, the flow is actually laminar since $\hat{\rho}$ is large, but in order to keep the beam radius $\hat{R}$ as small as possible one should provide a stronger second order focusing than that one produced by the ponderomotive potential, $\Omega^2 = 1/8$. An invariant envelope can be defined for any value of Ω^2, thus this parameter can be chosen to guarantee that the beam is laminar, while still minimizing the beam size under the constraint of available transverse aperture.

IV. The Longitudinal Envelope Equation

At this point we investigate the longitudinal envelope equation in order to find under what condition the peak current can be kept invariant along the linac, which implies a quasi-free relativistic bunch expansion, so that $L = \frac{L_i}{\beta_i}\beta$, *i.e*. the bunch length must scale like β (L_i is the bunch length at injection). This condition removes our previous approximation of adiabatic variation of β. Substitution of R with $\hat{R}$ in Eq. 1.2 and assuming *longitudinal* laminarity we find that

$$L'' + \frac{2\alpha_i k^2 \sin|\varphi_0|}{\beta_i \beta^2 \gamma^3} L = \frac{2\alpha_i k \cos\varphi_0}{3\beta_i \beta^3 \gamma^4} \sqrt{\frac{\beta\gamma(1+4\Omega^2)\bar{I}}{2}} \quad . \tag{4.1}$$

Assuming $L = \frac{L_i}{\beta_i}\beta$ Eq. 4.1 becomes

$$-\frac{3\gamma_i'\beta L_i}{\beta_i^2 \gamma} + \frac{2kL_i \tan|\varphi_0|}{\beta_i \beta} = \frac{2}{3\beta^3\gamma} \sqrt{\frac{\beta\gamma(1+4\Omega^2)\bar{I}}{2}}, \tag{4.2}$$

where we have used the relations $\gamma' = \frac{\gamma_i'}{\beta_i}\beta$, $\beta' = \frac{\gamma_i'}{\beta_i \gamma^3}$, $\gamma'' = \left(\frac{\gamma_i'}{\beta_i}\right)^2 \frac{1}{\gamma^3}$, and $\beta'' = -3\left(\frac{\gamma_i'}{\beta_i}\right)^2 \frac{\beta}{\gamma^4}$. With some additional algebra, and recalling that $\gamma_i' = \alpha_i k \cos\varphi_0$, we obtain

$$\tan|\varphi_0| = \frac{1}{3\phi} \sqrt{\frac{\bar{I}(1+4\Omega^2)}{2\beta^3\gamma}} + \frac{3\beta^2 \alpha_i \cos\varphi_0}{2\beta_i \gamma}. \tag{4.3}$$

where the constant phase length $\phi \equiv kL_i / \beta_i$ has been introduced. Assuming again that α_i

is small and the bunch is not too long, *i.e.* taking $\frac{\alpha_i \phi}{\Omega}\sqrt{\frac{1}{\bar{I}\gamma}} << 1$ (which is the case for a 10 A peak current beam at $\alpha_i = 10^{-3}$, where $1.8\phi/\Omega << 1$ is easily satisfied), the second term on the R.H.S. of Eq.1.25 can be neglected, so that the accelerating phase φ_0 must follow, along the acceleration, the relation

$$\varphi_0 = -\arctan\left(\frac{1}{3\phi}\sqrt{\frac{\bar{I}(1+4\Omega^2)}{2\beta^3\gamma}}\right) \tag{4.4}$$

in order to obtain a constant peak current. It is interesting to note that this expression is independent of the accelerating gradient, because the focusing transverse focusing strength is also tied to this quantity. The variation of the accelerating phase variation φ_0 along the linac is shown in Fig. 3 for a 16 A beam at Ω^2=120 and ϕ=1.7 RF degrees.

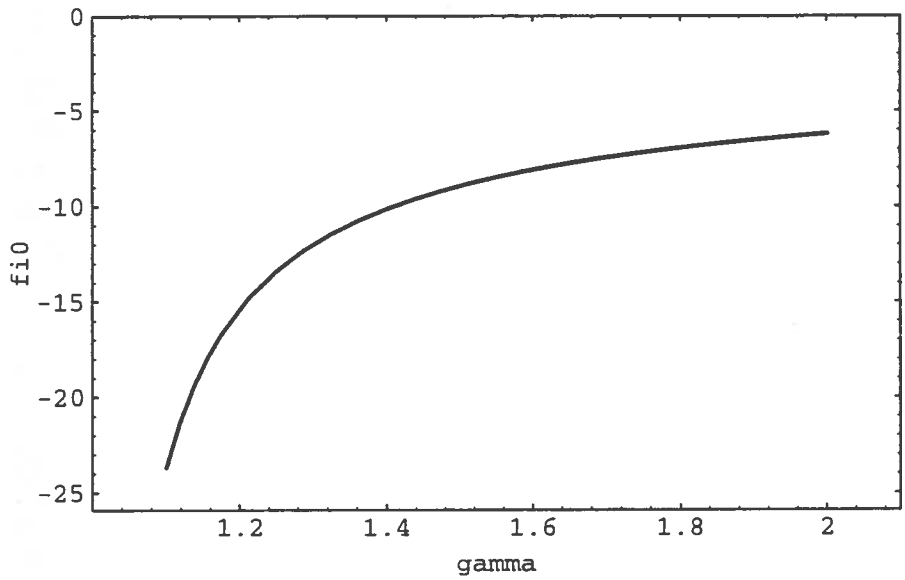

Fig.3 - Accelerating phase variation along the linac

Since it is necessary that all the particles in the bunch are accelerated at negative phases to avoid loss of the bunch tail, the bunch phase length ϕ must be smaller than the minimum φ_0 (at the linac exit $\beta_f\gamma_f$), implying

$$\phi \le \phi_{max} \equiv \left[\frac{\bar{I}(1+4\Omega^2)}{18\beta_f^3\gamma_f} \right]^{1/4} \tag{4.5}$$

This is plotted in Fig.4 as a function of Ω^2, for a 1 A (dashed line) and 10 A (solid line) peak current.

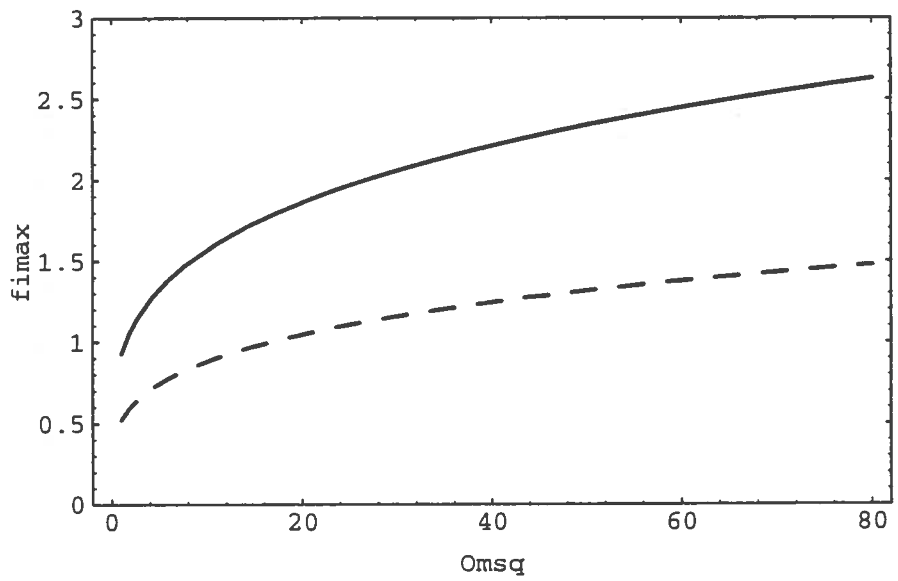

Fig.4 - Maximum phase bunch length (in RF degrees) as given by Eq.1.27, for examples with 1 A (dashed line) and 10 A (solid line) .

The hypothesis of longitudinal laminarity holds whenever the parameter

$$\rho_z \equiv \frac{2\alpha_i \cos\varphi_0}{3\beta_i (k\varepsilon_{n,z})^2} \sqrt{\frac{\beta\gamma(1+4\Omega^2)\bar{I}}{2}} \beta^2\gamma^2\phi^3 \qquad (4.6)$$

is much larger than unity . The minimum value for ρ_z is achieved at injection, where, substituting ϕ_{max} for ϕ, we obtain

$$\rho_z^{min} = \sqrt{\frac{1}{3\sqrt{2}}} \frac{\alpha_i \cos\varphi_0 \beta_i^{3/2}\gamma_i^{5/2}}{9(k\varepsilon_{n,z})^2 (\beta_f^3\gamma_f)^{3/4}} \left[(1+4\Omega^2)\bar{I}\right]^{5/4} \qquad (4.7)$$

Taking again as an example $\gamma_0 = 1.1$, $\gamma_f = 2$, $\alpha_i = 10^{-3}$, $\lambda_{RF} = 0.85$ m and a normalized longitudinal emittance $\varepsilon_{n,z} = 1$ mm·mrad , we have $\rho_z^{min} = 1.2 \cdot 10^{-4}\left[(1+4\Omega^2)I\right]^{5/4}$, which is plotted in Fig.5 at $I = 10$ A as a function of Ω^2.

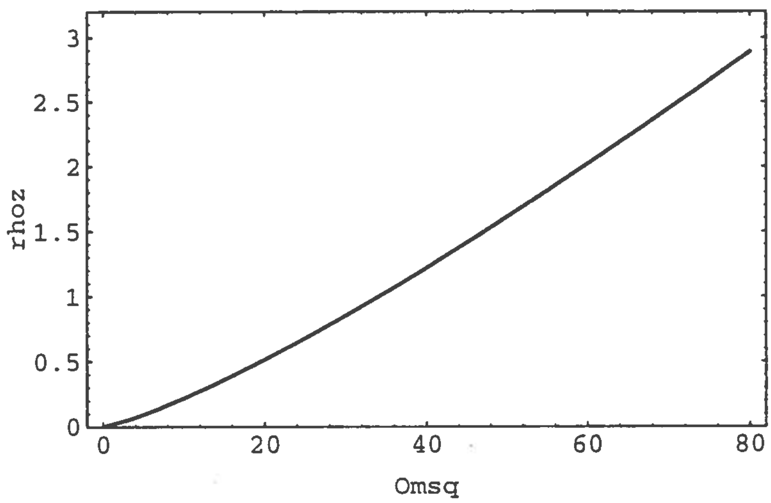

Fig.5 - Minimum value ρ_z^{min} at injection for the longitudinal laminarity parameter ρ_z , plotted versus Ω^2 at $I = 10$ A .

Since ρ_z grows along the acceleration as $(\beta\gamma)^{5/2}$, one can achieve longitudinal laminar flow by applying normalized large focusing frequencies Ω^2, making the beam denser and thus boosting the relative strength of the longitudinal space-charge in comparison with the longitudinal emittance effects. It should be emphasized that this must focusing is eventually limited by the need to preserve transverse laminarity.

V. Transverse Envelope Oscillations

Assuming the mode of operation for the linac described in the previous sections, let us recall that the beam spot will be given by the invariant envelope expression $\hat{R} = \frac{2}{p'}\sqrt{\frac{2\bar{I}}{p(1+4\Omega^2)}}$, where $p = \beta\gamma$ and $p' = \gamma_i'/\beta_i$. In case of a weak transverse mismatching of the beam at injection, the beam spot will be $R = \hat{R} + \delta R$ (with $\delta R/\hat{R} << 1$). By a perturbative linear analysis of Eq.1.19 we can easily derive an equation for the mismatch δR, which is

$$\delta R'' + \delta R' \frac{p'}{p} + \xi^2 \delta R \frac{p'^2}{p^2} = 0 \qquad (5.1)$$

where $\xi \equiv \sqrt{2\Omega^2 + 1/4}$. The general solution of this linear equation can be expressed as

$$\begin{cases} \delta R = \delta R_i \cos\left[\xi \ln\left(p/p_i\right)\right] + \dfrac{p_i}{p'\xi}\,\delta R_i' \sin\left[\xi \ln\left(p/p_i\right)\right] \\ \delta R' = -\dfrac{p'\xi}{p}\,\delta R_i \sin\left[\xi \ln\left(p/p_i\right)\right] + \dfrac{p_i}{p}\,\delta R_i' \cos\left[\xi \ln\left(p/p_i\right)\right] \end{cases}, \tag{5.2}$$

displaying oscillations stable in amplitude, and adiabatically damped derivative of the mismatch trajectory $\delta R'$. The total phase advance of the mismatch oscillation $\Delta\Phi$ along the linac is given by $\Delta\Phi = \xi \ln\left(p_f / p_i\right)$. This is plotted in Fig.6, as a function of the normalized focusing frequency Ω^2 , for the case of a linac boosting protons from 100 MeV up to 1 GeV. At Ω^2=40 the total phase advance corresponds to nearly to two full transverse envelope oscillations during acceleration.

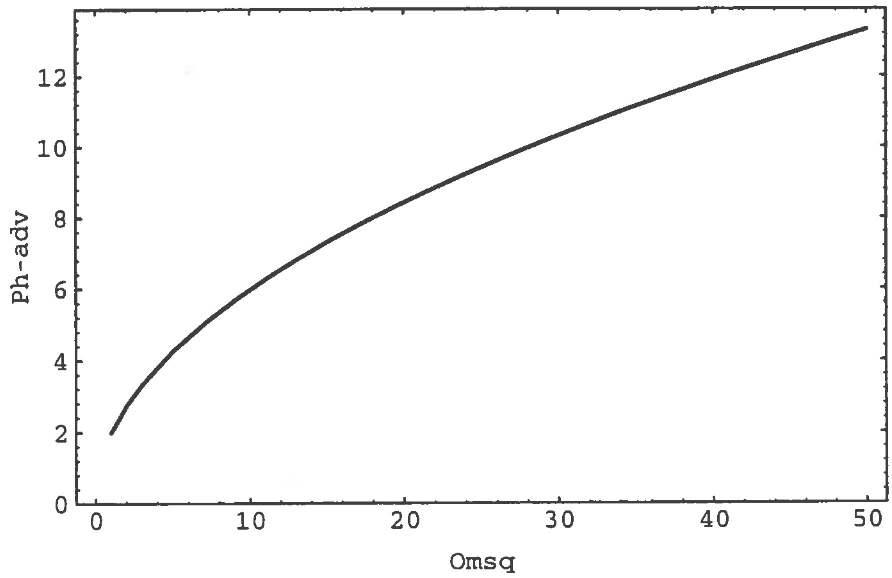

Fig.6 - Total phase advance on a linac boosting protons from 100 MeV up to 1 GeV.

The bunch eccentricity in the beam rest frame, defined as $\varepsilon \equiv \dfrac{R}{\gamma L}$, is thus

$$\varepsilon = \frac{2\beta_i}{\alpha_i \phi \cos\varphi_0}\sqrt{\frac{2\bar{I}}{\beta^3\gamma^3\left(1+4\Omega^2\right)}} \tag{5.3}$$

Typically we have $1 \le \varepsilon \le 5$ for the parameters considered above, and the beam is essentially round in its reference frame.

VI. Mass and Frequency Scaling of The Envelope Equations

In this section we present a possible scaling of the envelope equations based on a coupled scaling of the particle mass and the RF frequency which assures that a laminar non-relativistic electron beam accelerated in a high frequency (typically X-band) linac undergoes the same envelope propagation as the proton beam under discussion so far. This scaling resembles the one discussed elsewhere [3], but includes here a generalization to take into account also different species (masses).

We start by re-writing Eq. 2.1 and 2.2 in terms of the dimensionless $\bar{R} \equiv kR$ and $\bar{L} \equiv kL$ functions of the independent dimensionless variable $\bar{z} \equiv kz$:

$$\frac{d^2\bar{R}}{d\bar{z}^2} + \frac{d\bar{R}}{d\bar{z}}\frac{(\beta\gamma)'}{(\beta\gamma)k} + \frac{K_r}{k^2}\bar{R} = \frac{2I/I_0}{\bar{R}\,(\beta\gamma)^3} + \frac{\varepsilon_{n,th}^2 k^2}{\bar{R}^3\,(\beta\gamma)^2} \tag{6.1}$$

$$\frac{d^2\bar{L}}{d\bar{z}^2} + \frac{K_z}{k^2}\bar{L} = \frac{4I/I_0}{3\beta^3\gamma^4\bar{R}} + \frac{\varepsilon_{n,z}^2 k^2}{\beta^2\gamma^6\bar{L}^3} \tag{6.2}$$

These equations scale invariant as far as the following scaling are respected: $\gamma' \propto k$, which implies (since $\gamma' \propto \alpha k$) $\alpha = const.$; $K_r \propto k^2$; $K_z \propto k^2$; $I \propto I_0 \propto m_0$

(m_0 is the particle rest mass) . Such scaling laws imply that the accelerating gradient $E_{acc} \propto m_0\gamma' \propto m_0 k$ should scale like the product of the rest mass times the frequency, the beam current I must scale like the rest mass and the focusing gradients (both the transverse and the longitudinal one) should scale like the square of the frequency. In case of an ideal second order focusing we have (from Eqs. 2.16 and 2.17) $K_r = \Omega^2 (\beta\gamma)'^2 / (\beta\gamma)^2$, therefore the required scaling for K_r is achieved by scaling the normalized focusing gradient unchanged, *i.e.* $\Omega^2 = const.$. Consequently, the amplitude of the magnetic field, provided by an external solenoid to compensate the RF defocusing (see Eq. 3.4), will scale as the accelerating gradient, *i.e.* $B_0 \propto m_0 k$. From Eq. 2.4 we also see that the longitudinal focusing gradient shows a natural scaling $K_z \propto k\gamma' \propto k^2$, as required, provided that the accelerating phase scales unchanged, *i.e.* $\varphi_0 = const.$.

The laminar regime is attained whenever the two parameters ρ and ρ_z are much larger than 1: since we want to scale these parameters unchanged, in order to be in the same regime, this requires to scale the transverse emittance like $\varepsilon_{n,th} \propto 1/k\sqrt{m_0}$, as well as the longitudinal emittance, namely $\varepsilon_{n,z} \propto 1/k\sqrt{m_0}$.

The geometrical sizes of the beam will scale down by the inverse of the frequency (i.e. $R \propto 1/k$, $L \propto 1/k$), so that the bunch charge will scale like $Q \propto m_0/k$. The active length of the linac L_{act} will also scale down by a factor $1/k$ as far as the same change in γ is desired from injection to final acceleration.

As a practical example we consider the scaling of the proton linac used as an example in sect. IV down to an electron linac whose beam is expected to behave in the same way as the proton beam from the point of view of the envelope behavior. The mass scaling ratio is $m_e/m_p = 5.45\cdot 10^{-4}$ and we choose a frequency for the electron linac $\nu_e = 11.4$ GHz, so that the frequency scaling ratio is $\nu_e/\nu_p = 32.4$.

The accelerating gradient scales down from $E^p_{acc} = 5$ MV/m ($\gamma_i'^p = 0.0054$ m^{-1}) for protons to $E^e_{acc} = 89\,\mathrm{kV/m}$ for electrons (values at injection): the initial value for α is taken $\alpha_i = 7.3\cdot 10^{-3}$ at injection, where $\beta_i = 0.42$, which corresponds to 100 MeV protons and 51 keV electrons (the final energies are 1 GeV and 545 keV, respectively). The active length of the proton linac is given by $L_{act} = \frac{\beta_i}{\gamma_i'}\int_{\gamma_i}^{\gamma_f}\frac{d\gamma}{\beta} = \frac{\beta_i}{\gamma_i'}\left(\sqrt{\gamma_f^2-1}-\sqrt{\gamma_i^2-1}\right)$,

which comes out to be 105 m, which turns into 3.2 m for the electron linac. Due to the combined mass and frequency scaling of the solenoid field amplitude B_0, the prohibitive $B_0 = 1\mathrm{T}$ (at $\Omega^2 = 70$) solenoid field required by protons (see Eq. 3.4) becomes an easy $B_0 = 177\,\mathrm{Gauss}$ (maximum amplitude at injection) for electrons.

Also the demands on the longitudinal and transverse emittance are somewhat relaxed: the 1 mm·mrad value taken as a reference for protons (both for the transverse and the longitudinal one) becomes a factor $100/\left(32.4\cdot\sqrt{5.45}\right) = 1.32$ times larger for electrons. The bunch population, which is $1.8\cdot 10^9$ for the proton beam at 16 A of peak current and $L = 4$ mm of bunch length (1.7 °RF), goes down by a huge factor for electrons, namely at $N_e = 3\cdot 10^4$ (the electron bunch length becomes $L_e = 0.4$ ps).

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