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BUNCHED BEAMS IN AXISYMMETRIC SYSTEMS*

M. De Magistris [(1)], V.G. Vaccaro [(2-3)], L. Verolino [(1-3)]
(1) Università di Napoli Federico II – Dipartimento di Ingegneria Elettrica, Via Claudio 21, I–80125 Napoli, Italy
(2) Università di Napoli Federico II – Dipartimento di Scienze Fisiche, Mostra d'Oltremare (Pad. 20), I–80125 Napoli, Italy
(3) Istituto Nazionale di Fisica Nucleare – Sezione di Napoli, Mostra d'Oltremare (Pad. 20), I-80125 Napoli, Italy

Abstract

We analyze, in this paper, the self field of a bunched beam in a cylindrical metallic pipe with arbitrary density. The field is calculated by superposition of the Green's functions of the single charge in the pipe. A general solution is given as a Bessel-Fourier expansion, which is characterised by very good convergence properties. This allows the evaluation of the dynamics of a large class of physical beams by using this kind of solution for the self field. As an example the field is calculated for the case of a parabolic radial and elliptic longitudinal density profile .

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1. Introduction

The growing interest on high intensity beams, which is related to applications like ICF (Inertial Confinement Fusion) [1], neutron sources [2] or energy amplifiers [3], is enforcing the analysis of some non linear problems in beam dynamics that have often been neglected in the past. Particles dynamics are generally evaluated by means of so called *tracking codes*, that perform trajectory evaluation once the force (self force and external force) is calculated. Moreover, the evaluation of the self force is also the basis for any stability analysis. This justifies the effort that recently has arisen in the scientific community about canonical solutions of classical electromagnetic problems in the time domain relevant to beam dynamics [4]. In fact, solutions to these problems in a computable form provide, firstly, benchmarks for numerical codes; besides, once the solution is fast converging, it can be used as a part of a more general effective semi-analytical code.

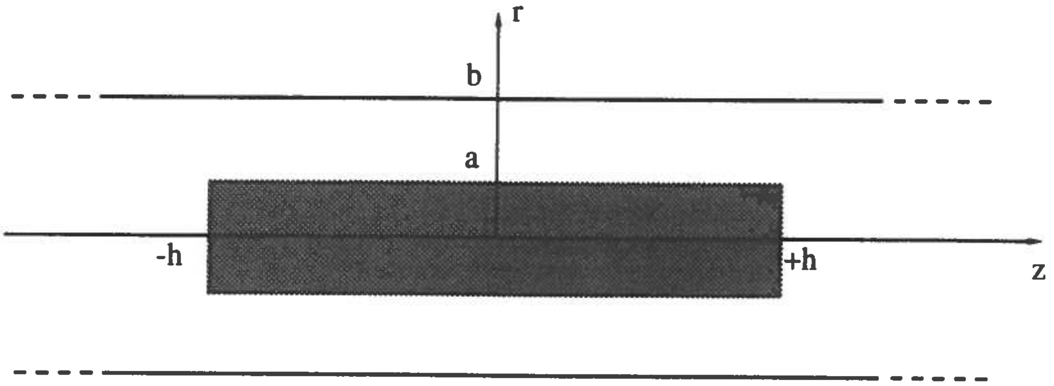

Figure 1: schematic representation of a bunch in a vacuum chamber.

We study, in this paper, such a kind of representation for the self field in the case of a bunch of charged particles travelling in vacuum with a constant velocity along the axis of an infinite perfectly conducting metallic tube (representing the vacuum chamber) of radius b, as shown in Figure 1. The analysis will be performed in the beam frame, so that an electrostatic problem has to be solved. Once the solution is given in the beam frame it is possible by means of a Lorentz transformation to obtain the fields in the laboratory frame [5]. Assuming the potential as unknown, we have to solve a Poisson problem with homogeneous Dirichlet conditions along the external cylinder. It is of primary importance the knowledge of the Green's function of the system to construct an effective procedure to compute the field. Green's functions can have an integral or a series representation, and both are useful in a certain range of the parameters. We use for our problem a series representation.

The solution we are going to present consist of an expansion for the electromagnetic field of a charged particle travelling in the vacuum chamber of an accelerator whose rate of convergence is exponential, what renders particularly attractive the proposed expansion. A comparison between the computing time of the present expansion and an integral representation is also given.

2. The bunch

We consider a bunch of charged particles travelling along the axis of an axisymmetric structure. Because we are in the bunch system, we can consider an electrostatic problem.

Undertaking a cylindrical coordinate system (r,φ,z) with the z-axis coaxial to the bunch and the pipe (Figure 1), the bunch can be modelled as a cylinder of charge of radius $a(<b)$ and length 2h, distributed in an axisymmetric way. We choose as fitting function for the bunch charge distribution the factorized functional form

$$\rho(r,z) = \rho_0\, \rho_r(r)\, \rho_z(z)\ , \tag{2.1}$$

where ρ_0 is a constant value of charge density.

This choice allows us to model quite in general realistic bunched beams from the knowledge of measured densities. Assuming that the charge distribution density vanishes outside the bunch, namely $\rho(r,z) = 0$ for $r \geq a$ and $|z| \geq h$, we require that the radial factor takes the form

$$\rho_r(r) = \left[1 - (r/a)^2\right]^s, \quad 0 \leq r \leq a\ , \tag{2.2}$$

or any linear combination of these polynomials. Besides we ask that the Fourier expansion of the longitudinal distribution $\rho_z(z)$ has coefficients which are easy to compute, analytically or numerically. We use the even periodical continuation with period 4h

$$\rho_z(z) = \sum_{m=1}^{\infty} b_m \cos(p_m z)\ , \quad |z| < h\ , \tag{2.3}$$

where $p_m = (m-1/2)\pi/h$.

3. The potential

A series expansion of the Green's function G(P,P') for a point charge located at the point P'(r',φ',z') inside a perfectly conducting cylinder of radius b, whose potential is zero, is well known [6], and takes the form

$$G(P,P') = \sum_{n=0}^{\infty} \frac{\delta_n}{2\pi\varepsilon_0 b} \sum_{k=1}^{\infty} \frac{J_n(x_{nk}r/b)J_n(x_{nk}r'/b)}{x_{nk}\, J_1^2(x_{nk})} \cos[n(\varphi-\varphi')] \exp(-x_k|z-z'|/b)\ , \quad (3.1)$$

where P(r,φ,z) is the observation point, and, as usual, $J_n(x)$ represents the Bessel function of order n, whose zeros are ordered as x_{nk} with $k \in \mathbf{N}$, and the Neumann's symbol δ_n is defined as

$$\delta_n = \begin{cases} 1 & n = 1\ , \\ 2 & n = 2, 3, 4 \cdots\ . \end{cases}$$

An integral representation of the same Green's function is discussed in some details in Appendix A. In order to evaluate the potential Φ(P) created by the charge distribution (2.1) of the metallic tube, we have to integrate Green's function (3.1) over all the bunch volume, weighting it to the charge density, namely

$$\Phi(P) = \iiint_{\text{BUNCH}} G(P,P')\, \rho(P')\, dV'\ , \quad (3.2)$$

with dV' = r'dr'dφ'dz'. Because we are able to perform the angular integration on φ', substituting expansion (3.1) into integral (3.2), and using the relevant result [7]

$$\int_0^1 x\,(1-x^2)^s\, J_0(bx)\, dx = 2^s\, \Gamma(s+1)\, \frac{J_{s+1}(b)}{b^{s+1}}\ ,$$

it is not difficult to show that the potential can be represented as

$$\Phi(r,z) = \frac{\rho_0}{\varepsilon_0} ha \left(\frac{2b}{a}\right)^s s! \sum_{k=1}^{\infty} \frac{J_0(x_k r/b)\, J_{s+1}(x_k a/b)}{x_k^{s+2}\, J_1^2(x_k)} B_k(z)\ , \quad (3.3)$$

where we have introduced the coefficients

$$B_k(z) = \int_{-1}^{+1} \rho_z(hx) \exp(-x_k|z-hx|/b)\, dx \; ; \qquad (3.4)$$

here instead of x_{0k} we employ the simpler notation x_k.

The factorized form (3.3) is a direct consequence of assumption (2.1). In order to find a rapidly convergent expansion for the potential, and for the electric field, we have to evaluate longitudinal integrals (3.4); besides we have to try to accelerate the convergence of series (3.3).

4. Evaluation of potential integrals

We want to write potential integrals (3.4) in a computable form, in both regions, internal and external to the bunch. Substituting expansion (2.3) into the longitudinal integral of formula (3.4), we have for the external region

$$B_k(z) = B_k \exp[-x_k(|z|-h)/b]\,, \qquad |z| \geq h\,, \qquad (4.1)$$

where for sake of simplicity we called

$$B_k = 2 \exp(-x_k h/b) \int_0^1 \rho_z(hx)\, \mathrm{ch}(x_k hx/b)\, dx\,. \qquad (4.2)$$

First of all we may guess that the coefficients B_k are not difficult to derive according to the hypothesis of section 2 where we assumed that $\rho_z(z)$ may be easily expanded as the Fourier series (2.3). As a matter of fact for all the longitudinal profiles suggested by the experience, we were able to give simple analytical expressions to the coefficients B_k. An example will be given in the next section. Equation (3.3) exhibits in this region a very good behaviour from the computational point of view. Indeed for $|z| > h$ the decrease of each addendum is exponential as a function of k and z.

In the internal region the functions $B_k(z)$ take the form

$$B_k(z) = \frac{\mathrm{ch}(zx_k/b)}{\mathrm{ch}(hx_k/b)} B_k + 2\frac{bx_k}{h} \sum_{m=1}^{\infty} \frac{b_m \cos(p_m z)}{x_k^2 + (p_m b)^2}\,, \qquad |z| \leq h\,. \qquad (4.3)$$

We note that in addition to a term similar to equation (4.1), which has been already discussed, there is an extra term which is expanded as a function of the Fourier coefficients b_m. This expansion has a weak decrease as a function of the indices m and k, and is not adequate for a fast numerical treatment, even more if it should be a part of an extended numerical code. Therefore our effort will be to find a better expansion by inverting the order of summations, after introducing expansion (4.3) into equation (3.3). We get the following expression for the potential in the internal region

$$\Phi(r,z) = \frac{\rho_0}{\varepsilon_0} ha \left(\frac{2b}{a}\right)^s s! \sum_{k=1}^{\infty} \frac{J_0(x_k r/b)\, J_{s+1}(x_k a/b)}{x_k^{s+2}\, J_1^2(x_k)} \frac{ch(x_k z/b)}{ch(x_k h/b)} B_k + \tag{4.4}$$

$$+ \frac{\rho_0}{\varepsilon_0} ab \left(\frac{2b}{a}\right)^s s! \sum_{m=1}^{\infty} b_m\, C_{s,m}(r) \cos(p_m z)\,, \qquad |z| \leq h\,,$$

where the function $C_{s,m}(r)$ represents the following series

$$C_{s,m}(r) = 2 \sum_{k=1}^{\infty} \frac{J_0(x_k r/b)\, J_{s+1}(x_k a/b)}{x_k^{s+1}\, J_1^2(x_k) \left[x_k^2 + (p_m b)^2\right]}\,, \tag{4.5}$$

According to Watson [8] a simplified series similar to equation (4.5) can be closed by means of a procedure originated by works of Kneser and Sommerfeld. Unfortunately in all the literature consulted [8, 9, 10], in addition to several misprints, there are also conceptual errors in the demonstration. In order to get the exact form, one has to resort to the original papers [11,12]. A deep analysis of this misquoting, which propagated and scattered in many authors, can be found in [13], where the more general form (4.5) is also given. Also in this case we get two expansions, one for the external region, with respect to the radial position, and another one for the internal region. This general form is then

$$C_{s,m}(r) = \frac{I_{s+1}(p_m a)}{(p_m b)^{s+1}} \left[K_0(p_m r) - \frac{K_0(p_m b) I_0(p_m r)}{I_0(p_m b)}\right], \qquad a \leq r \leq b\,, \tag{4.6}$$

$$C_{s,m}(r) = \left(\frac{b}{2a}\right)^{s+1} \sum_{k=1}^{s+1} \left(\frac{2r}{p_m b^2}\right)^k \frac{W_k(p_m r)}{(s+1-k)!} \left(\frac{a^2}{b^2} - \frac{r^2}{b^2}\right)^{s+1-k} \tag{4.7}$$

$$- \frac{I_0(p_m r)}{(p_m b)^{s+1}} \left[(-1)^s K_{s+1}(p_m a) - \frac{K_0(p_m b) I_{s+1}(p_m a)}{I_0(p_m b)}\right],$$

where $W_n(z)$ is a generalized wronskian, defined as

$$W_n(z) = K_0(z)\, I_n(z) - (-1)^n\, I_0(z)\, K_n(z)\,, \qquad n \in \mathbb{N}_0\,; \tag{4.8}$$

here $I_n(z)$ and $K_n(z)$ are modified Bessel functions [8] of the first and second kind, respectively. It is worth noting that this Wronskian is an inverse power polynome, which can be easily computed by means of a recursive relation [13].

It is apparent from equations (4.6) and (4.7) the improvement that we get. In the external region (4.6) we have an exponential decrease as a function of m; in the internal region (4.7), in addition to a term similar to the one previously discussed, there is a second term which decreases as the inverse powers of m. This latter term, at least in principle, and in practice for all cases taken as examples, can be summed over m. This is due to the fact that the generalized Wronskian (4.8) is a combination of inverse powers of p_m.

In order to show in a specific case how it is possible to apply this formula, we shall illustrate in the next section a particular case. There will be also a discussion on the computing time in comparison with the integral representation of the Appendix A.

5. An example

In order to give an example of the numerical computations discussed in the previous section, we refer to a special case, particularly interesting in heavy ion accelerators. We select s = 1 in relation (2.2), and

$$\rho_z(z) = \sqrt{1 - (z/h)^2}\ , \tag{5.1}$$

namely a bunch carrying the total charge $Q = \pi^2 a^2 h \rho_0 / 4$, whose expansion coefficients, according to Fourier series (2.3), are [7]

$$b_m = \pi \frac{J_1(p_m h)}{p_m h}\ . \tag{5.2}$$

Let us compute expansion integrals (4.2) for this special case. The coefficients B_k can be easily computed as [7]

$$B_k = \frac{\pi b}{h x_k} \exp(-x_k h/b)\, I_1(h x_k/b)\ . \tag{5.3}$$

We can conclude that

$$\Phi(r,z) = \frac{2\pi\rho_0 b^2}{\varepsilon_0} \sum_{k=1}^{\infty} \frac{J_0(x_k r/b)\, J_2(x_k a/b)}{x_k^4\, J_1^2(x_k)} I_1(x_k h/b)\, e^{-x_k |z|/b} \qquad |z| \geq h\ , \tag{5.4}$$

and similarly

$$\Phi(r,z) = \frac{2\pi\rho_0 b^2}{\varepsilon_0} \sum_{k=1}^{\infty} \frac{J_0(x_k r/b)\, J_2(x_k a/b)}{x_k^4\, J_1^2(x_k)} \frac{ch(x_k z/b)}{ch(x_k h/b)} I_1(x_k h/b)\, e^{-x_k h/b} +$$

$$+ \frac{2\rho_0 b^2}{\varepsilon_0} \sum_{m=1}^{\infty} b_m\, C_{1,m}(r) \cos(p_m z)\,, \qquad |z| \le h\,. \tag{5.5}$$

The evaluation of the function $C_{1,m}(r)$ can be done by means of relations (4.6) and (4.7)

$$C_{1,m}(r) = \begin{cases} \dfrac{I_0(p_m r)}{(p_m b)^2}\left[K_2(p_m a) - \dfrac{K_0(p_m b) I_2(p_m a)}{I_0(p_m b)}\right] + \dfrac{p_m^2(a^2 - r^2) - 4}{2 p_m^4 a^2 b^2} & 0 \le r \le a\,, \\ \dfrac{I_2(p_m a)}{(p_m b)^2}\left[K_0(p_m r) - \dfrac{K_0(p_m b) I_0(p_m r)}{I_0(p_m b)}\right] & a \le r \le b\,. \end{cases} \tag{5.6}$$

It is worth noting that the mathematical structure of formula (5.6) is identical to the definition (A.7) of the function F(u,r) of Appendix A. The main advantage of the formulation (5.5) with respect to integral formulation (A.5) is that the series can be summed using a few terms (depending upon the desired accuracy), while the integrand (A.5) can exhibit oscillations, and therefore it can be numerically uncontrollable.

Finally Figures 2 and 3 show the electric field according to the potential (5.4) and (5.5), whereas Figures 4 and 5 the electric field sustained by the bunch in absence of the metallic screen. All the numerical values of the field are normalized to the reference field

$$E_{ref} = \frac{\pi^2}{4} \frac{\rho_0 h}{\varepsilon_0}\,.$$

Particular attention is also payed to the computing time of this expansion for a given mean accuracy ε; this is represented in Figure 6 where various examples are given. In particular the calculation was performed for thousand points ranging between zero and $z/h = 1, 2, 3$ for the three lines a, b, and c respectively; the computing time of the integral representation τ_i is normalized to the one of the series expansion τ_s.

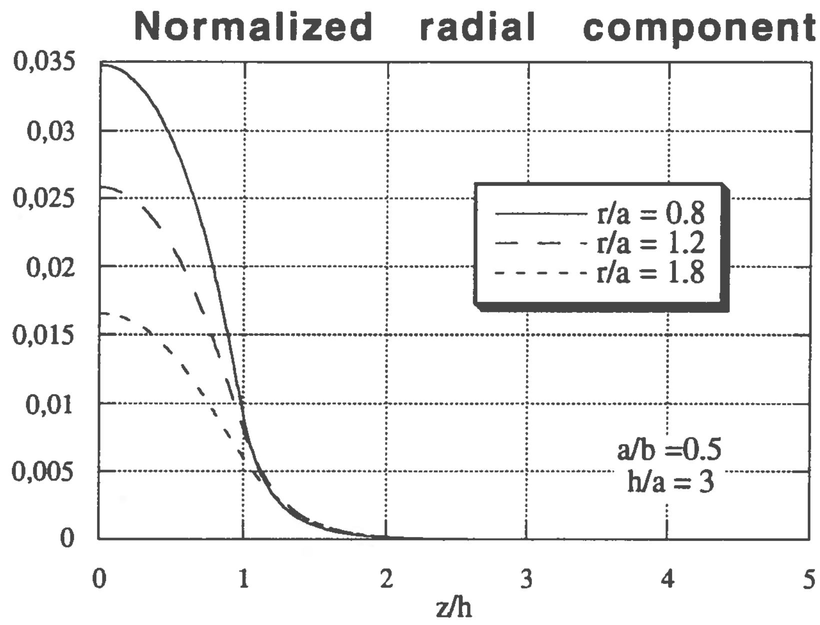

Figure 2: electrostatic field in presence of the metallic tube.

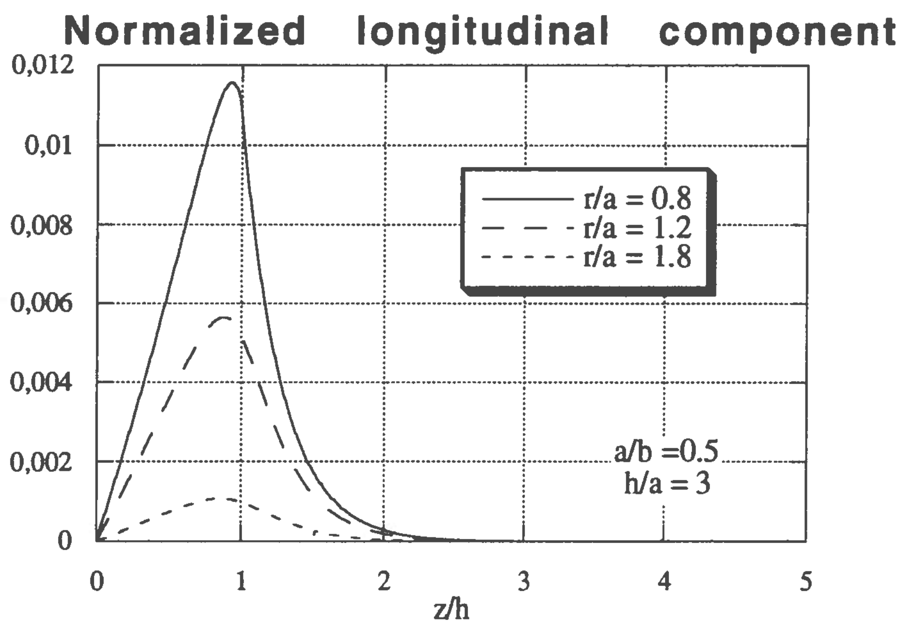

Figure 3: electrostatic field in presence of the metallic tube.

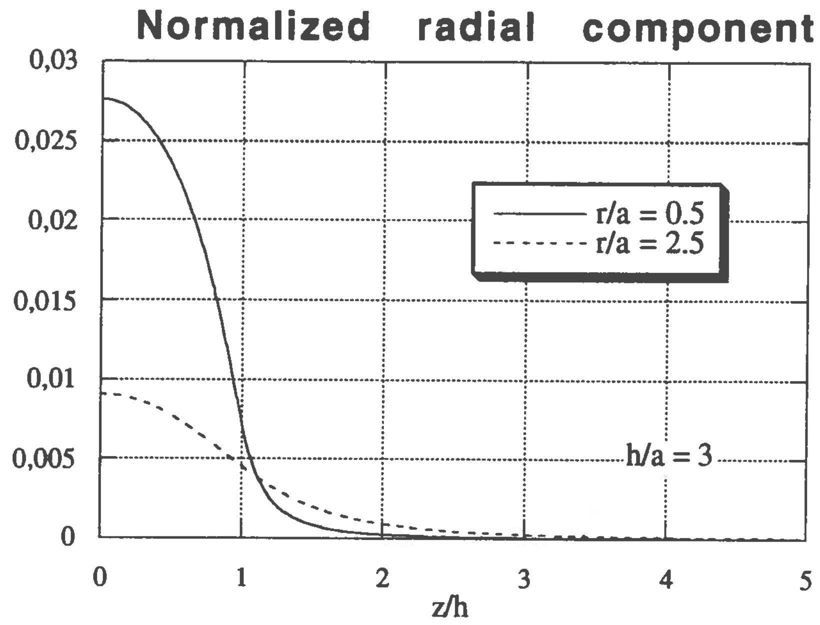

Figure 4: electrostatic field in the free space.

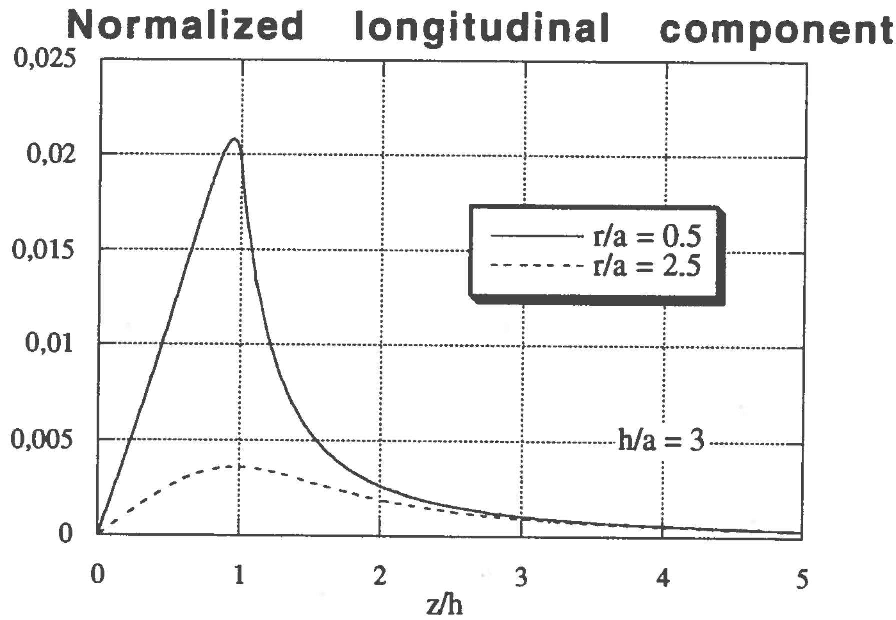

Figure 5: electrostatic field in the free space.

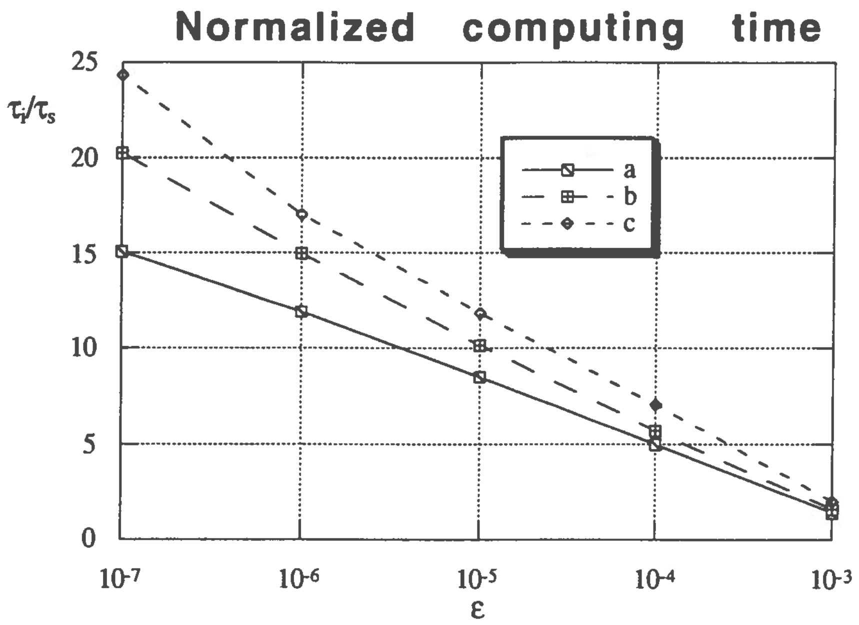

Figure 6: ratio of the computing time as a function of the accuracy ε

6. Conclusions and perspectives

An effective technique to compute the electrostatic field created by a bunch of charged particles has been discussed. The main hypothesis is that the charge distribution density does not depend by the angular position, and can be factorized according to the radial and the longitudinal coordinate.

The problem has been solved by means of a series expansion of the Green's function that can put as a computable series using a generalization of the Kneser-Sommerfeld formula. The comparison between the computing time of series expansion and integral representation is quite in favor of the former one.

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Appendix A

We shall give in this appendix formulas to compute the potential created by the bunch using an integral approach.

We start considering the bunch in the free space without the metallic screen ($b = \infty$). Because the charge density does not depend on the angular coordinate φ, we manage the angular integration according to the formula [7]

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\varphi}{|P-P'|} = \frac{2}{\pi}\int_0^{\infty} \cos[k(z-z')]\ H(kr,kr')\ dk\ , \qquad (A.1)$$

where

$$H(kr,kr') = \begin{cases} K_0(kr)\ I_0(kr') & r \geq r'\ , \\ I_0(kr)\ K_0(kr') & r \leq r'\ . \end{cases}$$

Relation (A.1) enables us to obtain a factorized form of the potential, and as a consequence the electrostatic field. For sake of simplicity, we shall give result only in the particular case (5.1). As for the series expansion proposed in section 4, also in this case we have to evaluate the radial and longitudinal contributions.

• *Radial factor*

We distinguish two cases: the observation point is inside or outside the bunch.
a) If it is inside the bunch, namely $0 \leq r \leq a$, using the generalized Wronskian (4.8) between modified Bessel functions

$$I_0(kr)\ K_2(kr) - I_2(kr)\ K_0(kr) = \frac{2}{(kr)^2}\ ,$$

it is easy to show that

$$\int_0^a r'\ \rho_r(r')\ H(kr,kr')\ dr' = \frac{1}{k^2}\left[1 - \frac{r^2}{a^2} - \frac{4}{k^2a^2} + 2\ I_0(kr)\ K_2(ka)\right]. \qquad (A.2)$$

b) If it is outside the bunch, namely $r > a$, a trivial integration reveals that [7]

$$\int_0^a r'\,\rho_r(r')\,H(kr,kr')\,dr' = \frac{2\,K_0(kr)\,I_2(ka)}{k^2}\,. \tag{A.3}$$

• *Longitudinal factor*

From relation (A.1), we conclude immediately that [7]

$$\int_{-h}^{+h} \cos[k(z-z')]\,\rho_z(z')\,dz' = \frac{\pi}{k}\cos(kz)\,J_1(kh)\,. \tag{A.4}$$

The previous results (A.2-4) enables us to write that the potential can be written in the functional factorized form

$$\Phi(r,z) = \frac{2\rho_0 h^2}{\varepsilon_0}\int_0^\infty \cos(uz/h)\,\frac{J_1(u)}{u^3}\,F(u,r)\,du\,, \tag{A.5}$$

where the function F(u,r) has been introduced to simplify the notations, and is defined by

$$F(u,r) = \begin{cases} I_0(ur/h)\,K_2(ua/h) + \frac{1}{2} - \frac{r^2}{2a^2} - \frac{2h^2}{u^2a^2}\,, & 0 \le r \le a\,, \\ K_0(ur/h)\,I_2(ua/h)\,, & r \ge a\,. \end{cases} \tag{A.6}$$

To take into account the presence of the metallic tube, if we repeat the previous procedure, it is not difficult to state that formula (A.5) formally does not change. We have only to change the function F(u,r) as follows

$$F(u,r) = \begin{cases} I_0(ur/h)\left[K_2(ua/h) - \frac{I_2(ua/h)K_0(ub/h)}{I_0(ub/h)}\right] + \frac{1}{2} - \frac{r^2}{2a^2} - \frac{2h^2}{u^2a^2} & 0 \le r \le a\,, \\ I_2(ua/h)\left[K_0(ur/h) - \frac{I_0(ur/h)K_0(ub/h)}{I_0(ub/h)}\right] & a \le r \le b\,. \end{cases} \tag{A.7}$$