ISTITUTO NAZIONALE DI FISICA NUCLEARE

Sezione di Napoli

INFN/TC-95/31
20 Novembre 1995

G. Miano, G. Panariello, V.G. Vaccaro, L. Verolino:

CAPACITANCE OF THE CIRCULAR PATCH RESONATOR

PACS: 41.10.Dq; 02.30.Rz

SIS-Pubblicazioni
dei Laboratori Nazionali di Frascati

INFN - Istituto Nazionale di Fisica Nucleare
Sezione di Napoli

INFN/TC–95/31
20 Novembre 1995

CAPACITANCE OF THE CIRCULAR PATCH RESONATOR

G. Miano(1), G. Panariello(2), V.G. Vaccaro(3), L. Verolino(1)
(1) Dipartimento di Ingegneria Elettrica and INFN Sezione di Napoli
Via Claudio 21, I-80125 Napoli - Italy
(2) Dipartimento di Ingegneria Elettronica Via Claudio 21, I-80125 Napoli - Italy
(3) Dipartimento di Scienze Fisiche and INFN Sezione di Napoli
Mostra d'Oltremare (Pad. 20), I-80125 Napoli - Italy

Abstract

In this paper the capacitance of the circular microstrip patch resonator is computed. It is shown that the electrostatic problem can be formulated as a system of dual integral equations, and the most interesting techniques of solutions of these systems are reviewed. Some useful approximated formulas for the capacitance are derived and plots of the capacitance are finally given in a wide range of dielectric constants.

1. Introduction

The practical advantages of microstrip structures have been discussed in many papers and [1,2] are now too well known to be repeated here. The relatively simple geometry of microstrip structures and the theoretical possibility to obtain very well suited mathematical models certainly contributed to their popularity. Indeed, every analytical technique commonly used in electromagnetism has been applied to microstrip, giving rise to a large number of different and apparently unrelated approaches.

The developments of numerical methods for solving integral equations in electromagnetic theory have been the subject of intensive research for more than twenty years. During these years, careful analysis has paved the way for the development of efficient and effective numerical methods and, of equal importance, has provided a solid foundation for a thorough understanding of the available techniques. Progress in understanding and procedures has been so great that a lot of integral equations can be solved handily by undergraduate engineering students.

In this article we mainly deal with the evaluation of the capacitance of a circular microstrip disk, a problem particularly interesting for applications in microwave integrated circuits [1-3]. The determination of this capacitance has been a subject of investigation from the middle of sixties. Itoh and Mittra [4] proposed a Galerkin technique in the spectral domain; their general theory gives numerical results only with a first-order approximation. Besides the coefficients of the Galerkin matrix involve infinite integral of Bessel functions, requiring a considerable computer time. The more accurate calculations of the present analysis confirm, however, that the one-term result for capacitance given in [4] are sometimes remarkably accurate. Borkar and Yang [5] present a method based on dual integral equations, adapting some classical results discussed by Tranter. Their numerical results exhibit a disparity of more than 10 percent compared with the ones given in [4]; therefore some authors doubted the validity of the application of the theory of such systems of integral equations. Coen and Gladwell [6] develop a solution expanding the charge density in terms of modified Legendre polynomials, taking into account an adequate factor to provide the correct singularity at the edge of the disk. Chew and Kong [7] considered the effects of fringing fields when the substrate thickness is small (compared with the radius of the patch); they obtained an asymptotic expansion for the capacitance and, using a seminumerical approach, a simple approximate formula to yield accurate results with the simple use of a calculator.

Our aim is to demonstrate that the electrostatics of the circular microstrip disk can be formulated as a system of dual integral equations, and that all the theoretical techniques developed can be obtained from the general theory of these systems. In particular we will show how such a system can be rewritten as a single Fredholm integral equation of the second kind with a continuous kernel; besides we will discuss a general procedure to reduce this new integral equation to an algebraic system of linear equation. Though a vast amount of literature has been published on the numerical computation of the capacitance of the circular patch resonator, a few have noted the importance of the edge singularity effect, and in some cases, the simplicity of the stratified media Green's function in spectral domain representation. We will obtain the charge distribution density in a factorized form with the edge singularity in evidence, and a plot of the capacitance of the system. In this way we will review all the classical method of solution for such kind of problem, emphasising advantages and disadvantages of each technique of solution, and the range of validity of obtained approximations.

Dual integral equations system occur in general in the mixed boundary values problems where the metallic region is finite (or in the dual problem), such as holes in metallic screens, drift tube in accelerators [8], cylindrical antennas [9]. These coupled integral equations are obtained imposing the boundary conditions on the electric field and on the charge density distribution, as described in the following. An extensive survey of the historical developments, almost up to 1966, and of the methods of numerical treatment of the dual integral equations, such as to a Fredholm type of equation, or reduction to a system of algebraic equations, is well summarised by I.N. Sneddon [10]. Recently the solution of a general type of dual integral equations has been proposed by Eswaran [11]; he showed under what conditions a solution can exist, is unique and can be expressed as a Neumann series.

The basic idea to solve a system of dual integral equations is to find an adequate representation of the unknown satisfying automatically one of the two equations and transforming the other one into an expression, easy to manage and/or to treat numerically. What we define 'a numerical manageable expression' will be a well conditioned system of algebraic equations. There are many ways to conceive this transformation; one can transform the system into a single Fredholm integral equation of the second kind with a continuous kernel, and then sample the continuous variable in order to obtain a system of equations; one can expand the unknown as a series of Bessel functions, obtaining a Neumann series, and, by means of adequate projections in functional spaces, the problem is converted into a set of linear simultaneous equations with a symmetric coefficients matrix, whose solution can be obtained in a wide range of frequencies. Matrix inversion is easily accomplished with digital computer so that solutions of high order are feasible.

In our case the problem is formulated as a system of dual integral equations of Bessel type in section 2 and a first approximation (the patch very close to the ground plane) for the capacitance of the system is found. In section 3 a general way to transform Bessel functions is explained, and a first Fredholm integral equation of the second kind is derived; unfortunately this integral equation is defined on a non-compact support, and therefore can be used to deduce other approximate formulas. In section 4 the system is transformed into a single Fredholm integral equation with a continuous kernel; this equation is rewritten by means of a Neumann series as a system of linear algebraic equations in section 5. A more general Neumann expansion is proposed in section 6. Finally the numerical results are given in section 7, whereas conclusions and perspectives of the proposed methods are examined in section 8. Numerical details of each method are reported, if necessary, in appendix.

We hope that our various approaches to the derivation of approximate formulas for the microstrip disk resonator will be of interest to readers who are interested in the analysis of microwave integrated circuits.

2. Formulation of the problem

Consider the geometrical configuration shown in Figure 1 of a microstrip circular disk resonator of radius a, separated from a ground plane by a dielectric material of permittivity

$\varepsilon = \varepsilon_0\,\varepsilon_r$. The substrate and the ground plane have infinite transverse dimensions. Theoretical developments are given here for a single-layer substrate. Modifications needed to account for multiple layers will be mentioned in an other paper. The disk is charged to the potential V.

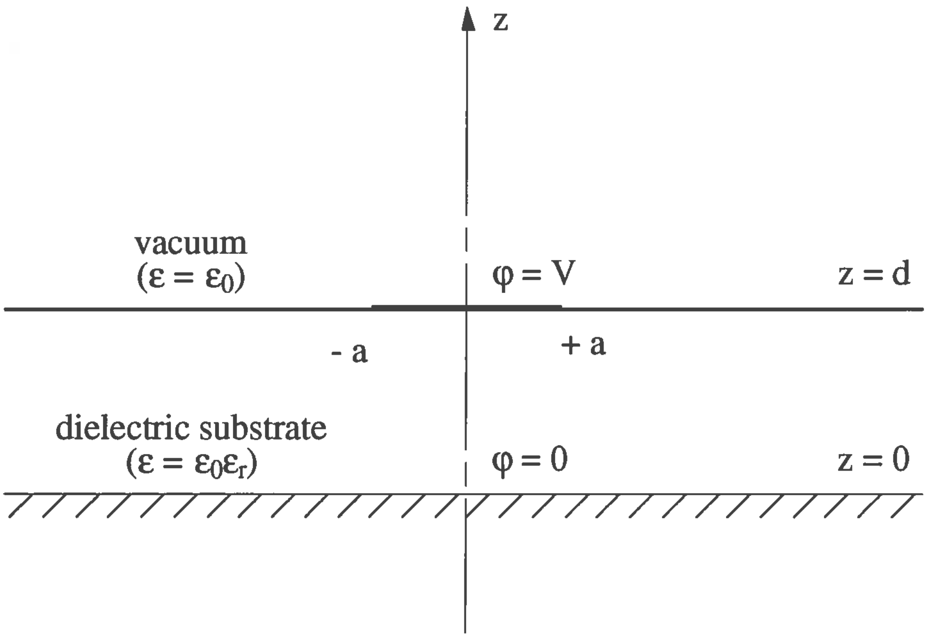

Figure 1 – schematic representation of the electromagnetic system.

The potential functions are considered to be defined in the two different regions

$$\varphi(r,z) = \begin{cases} \varphi_1(r,z)\,, & z \in [0,d]\,, \\ \varphi_2(r,z)\,, & z \in [d,\infty[\,. \end{cases} \tag{2.1}$$

The first step is to write Laplace equation for the potentials in each region (k=1,2)

$$\nabla^2\varphi_k = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi_k}{\partial r}\right) + \frac{\partial^2\varphi_k}{\partial z^2} = 0\,, \tag{2.2}$$

in which the azimuthal terms vanished because of the circular symmetry.
Let us now introduce the Hankel transform of the order zero for the potential

$$\begin{cases} \Phi_k(w,z) = \displaystyle\int_0^\infty r\,J_0(wr)\,\varphi_k(r,z)\,dr & \text{(Hankel transform)}\,, \\ \varphi_k(r,z) = \displaystyle\int_0^\infty w\,J_0(wr)\,\Phi_k(w,z)\,dw & \text{(Hankel inverse-transform)}\,. \end{cases} \tag{2.3}$$

Applying Hankel inverse-transform to equation (2.2), we obtain

$$\nabla^2 \varphi_k = \int_0^\infty w\, J_0(wr) \left[\frac{\partial^2 \Phi_k(w,z)}{\partial z^2} - w^2 \Phi_k(w,z) \right] dw = 0\,, \tag{2.4}$$

or equivalently the partial differential equations (k=1,2)

$$\frac{\partial^2 \Phi_k(w,z)}{\partial z^2} - w^2 \Phi_k(w,z) = 0\,. \tag{2.5}$$

The general solution of equations (2.5) is

$$\begin{cases} \Phi_1(w,z) = A(w)\,\mathrm{sh}(wz) + C(w)\,\mathrm{ch}(wz)\,, & z \in [0,d]\,, \\ \Phi_2(w,z) = B(w)\,\exp(-wz) + D(w)\,\exp(wz)\,, & z \in [d,\infty[\,, \end{cases} \tag{2.6}$$

and satisfying the boundary conditions

$$\begin{cases} \varphi_1(r,z=0) = 0 \\ \varphi_2(r,z=\infty) = 0 \end{cases} \quad \forall r \tag{2.7}$$

we have the simplified forms

$$\begin{aligned} &\Phi_1(w,z) = A(w)\,\mathrm{sh}(wz)\,, && z \in [0,d]\,, \\ &\Phi_2(w,z) = B(w)\,\exp(-wz)\,, && z \in [d,\infty[\,. \end{aligned} \tag{2.8}$$

The continuity of the potential on the separation surface z=d, namely

$$\varphi_1(r,z=d^-) = \varphi_2(r,z=d^+) \qquad \forall r\,, \tag{2.9}$$

implies the following relation between the two unknowns A(w) and B(w)

$$B(w) = A(w)\,\exp(wd)\,\mathrm{sh}(wd)\,. \tag{2.10}$$

We have to specify the boundary conditions on the plane z=d. The metallic patch is charged to the potential V, and therefore

$$\varphi_1(r,z=d) = \varphi_2(r,z=d) = V\,, \qquad r \in [0,a]\,, \tag{2.11}$$

namely the following integral equation

$$\int_0^\infty A(w)\, J_0(wr)\, w\, \mathrm{sh}(wd)\, dw = V\,, \qquad r \in [0,a]\,. \tag{2.12}$$

Besides the continuity of normal component of the field **D** implies

$$\varepsilon_0\, \varepsilon_r \left[\frac{\partial \varphi_1}{\partial z}\right]_{z=d^-} = \varepsilon_0 \left[\frac{\partial \varphi_2}{\partial z}\right]_{z=d^+} , \qquad (2.13)$$

that is equivalent to the integral equation

$$\int_0^\infty A(w)\, J_0(wr)\, w^2 \left[\varepsilon_r\, ch(wd) + sh(wd)\right] dw = 0 , \qquad r \in [a,\infty[\; . \qquad (2.14)$$

Summarising equations (2.12) and (2.14), we obtain the following system of integral equations

$$\begin{cases} \displaystyle\int_0^\infty A(w)\, J_0(wr)\, w\, sh(wd)\, dw = V , & r \in [0,a] , \\[2ex] \displaystyle\int_0^\infty A(w)\, J_0(wr)\, w^2 \left[\varepsilon_r\, ch(wd) + sh(wd)\right] dw = 0 , & r \in [a,\infty[\; . \end{cases} \qquad (2.15)$$

System (2.15) represents a system of dual integral equations and it describes a boundary-value problem in which the boundary conditions are mixed in the sense that the unknown function satisfies different types of boundary conditions over distinct portions of the same boundary. The first equation states that the metallic circular plate is to the potential V, whereas the second one represents the continuity of the normal component of the field **D** outside the patch.

Finally we can evaluate the relations linking the unknown A(w) and the charge distribution density on the metallic patch, and, in order to compute the capacitance of the system, the total charge. We have for the density

$$\sigma(r) = \varepsilon_0\, \varepsilon_r \left[\frac{\partial \varphi_1}{\partial z}\right]_{z=d^-} - \varepsilon_0 \left[\frac{\partial \varphi_2}{\partial z}\right]_{z=d^+} = \varepsilon_0 \int_0^\infty A(w)\, w^2 \left[\varepsilon_r\, ch(wd) + sh(wd)\right] J_0(wr)\, dw , \qquad (2.16)$$

and consequently, because [12]

$$\int_0^a r\, J_0(wr)\, dw = \frac{a}{w}\, J_1(wa) ,$$

the capacitance is

$$C = \frac{Q}{V} = \frac{2\pi}{V} \int_0^a r\, \sigma(r)\, dr = \frac{2\pi}{V}\, \varepsilon_0\, a \int_0^\infty A(w)\, w \left[\varepsilon_r\, ch(wd) + sh(wd)\right] J_1(wa)\, dw . \qquad (2.17)$$

It is helpful to find an approximate solution of the system (2.15); this solution can be useful to test the numerical solutions we shall find in the next sections. A solution in a closed form can be obtained for small values of d (d«a); in this case we can say that the system (2.15) can be approximated as

References

[1] Mosig J.R., *Arbitrarily shaped microstrip structures and their analysis with a mixed potential integral equation*, IEEE Transactions on Microwave Theory and Techniques **36**(2), pp. 314-323, February 1988.

[2] Davidovitz M., Lo Y.T., *Rigorous analysis of a circular patch antenna excited by a microstrip transmission line*, IEEE Transactions on Antennas and Propagation **37**(8), pp. 949-958, August 1989.

[3] Kuester E.F., Martinson T.M., *Accurate analysis of arbitrarily shaped patch resonators on thin substrates*, IEEE Transactions on Microwave Theory and Techniques **36**(2), pp. 324-331, February 1988.

[4] Itoh T., Mittra R., *A new method for calculating the capacitance of a circular disk for microwave integrated circuits*, IEEE Transactions on Microwave Theory and Techniques , pp. 431-432, June 1973.

[5] Borkar S.R., Yang R.F.H., *Capacitance of a circular disk for applications in microwave integrated circuits*, IEEE Transactions on Microwave Theory and Techniques, pp. 588-591, July 1975.

[6] Coen S., Gladwell G.M.L., *A Legendre approximation method for the circular microstrip disk problem*, IEEE Transactions on Microwave Theory and Techniques **25**(1), pp. 1-6, January 1977.

[7] Chew W.C., Kong J.A., Effects of fringing fields on the capacitance of circular microstrip disk, IEEE Transactions on Microwave Theory and Techniques **28**(2), pp. 98-104, February 1980.

[8] Dôme G., E. Gianfelice, Palumbo L., Vaccaro V.G., Verolino L., *Longitudinal coupling impedance of a circular iris*, Il Nuovo Cimento **104A**, pp. 1241-1255, 1991.

[9] Miano G., Vaccaro V.G., Verolino L., *A new numerical method of solution of Hallén's problem*, Journal of Mathematical Physics **36**(8), pp. 4087-4099, 1995.

[10] Sneddon I.N., *Mixed boundary value problems in potential theory*, North-Holland, Amsterdam, 1966.

[11] Eswaran K., *On the solution of a class of dual integral equations occurring in diffraction problems*, Proc. R. Soc. London A **429**, pp. 399-427, 1990.

[12] Gradshteyn I.S., Ryzhik I.M., *Table of Integrals, Series, and Products*, Academic Press, 1980.

[13] Lewin M., Schmeltzer R.A., *Function-theoretic solution of a class of dual integral equations and an application to diffraction theory*, Quarterly of applied mathematics **21**(4) pp. 269-283, 1963.

[14] Sneddon I.N., *The use of the integral transform*, McGraw-Hill, New York, 1972.

[15] Meixner J., *The behavior of electromagnetic fields at edges*, IEEE Transactions on Antennas and Propagation **20**, pp. 442-446, 1972.

[16] Watson G.N., *A treatise on the theory of Bessel functions*, second edition, Cambridge University Press, Cambridge, England ,1966.

[17] Abramowitz M., Stegun I.A., *Handbook of mathematical functions*, Dover, New York, ninth edition, 1972.

[18] Flannery B.P., Press W.H., Teukolsky S.A., Vetterling W.T., *Numerical Recipes*, Cambridge University Press, 1986.

[19] Oldham K.B., Spanier J., *An atlas of functions*, Hemisphere Publishing Corporation, New York, 1987.

Appendix A

A way to accelerate the convergence is the use of the polylogarithm functions, defined as [19]

$$\mathrm{poln}_\nu(x) = - \sum_{j=1}^{\infty} \frac{(1 - x)^j}{j^\nu} \tag{A.1}$$

where ν is not necessarily an integer; we need only integer index for our purposes. If $\nu=1$, definition (A.1) coincides with the usual logarithmic function, whereas we have the so called dilogarithmic and trilogarithmic function for $\nu=2,3$. Formally we indicate

$$\mathrm{poln}_1(x) = \ln(x)\,, \qquad \mathrm{poln}_2(x) = \mathrm{diln}(x)\,, \qquad \mathrm{poln}_3(x) = \mathrm{triln}(x)\,.$$

The recurrence relation linking two polylogarithmic functions can be easily deduced by definition (A.1), being

$$\frac{d}{dx}\,\mathrm{poln}_j(x) = \frac{\mathrm{poln}_{j-1}(x)}{x - 1}, \qquad j = 2, 3, \cdots . \tag{A.2}$$

The first polylogarithmic functions are represented in Figure 9.

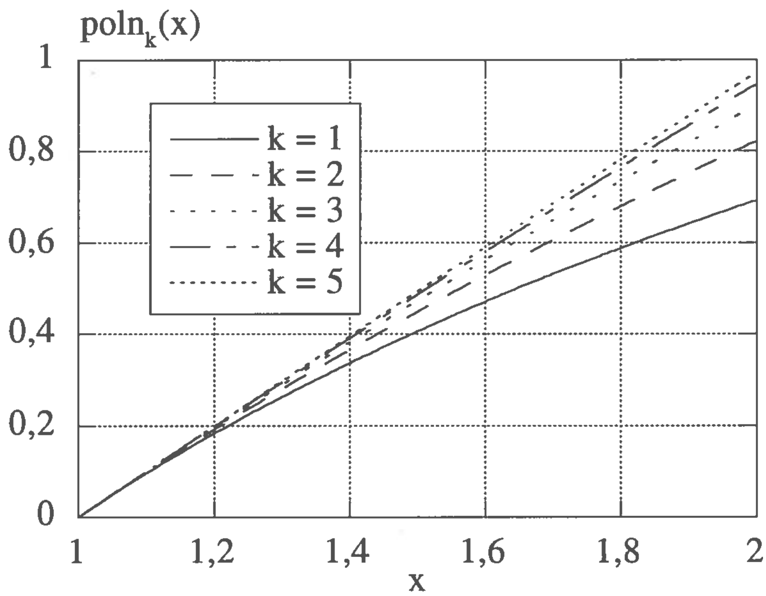

Figure 9 – : polylogarithmic functions of integer order.

In order to show how to employ these functions, we start by the relevant identity

$$\frac{1}{1+\left(\frac{x}{2dm}\right)^2} = \sum_{k=0}^{N} (-1)^k \left(\frac{x}{2dm}\right)^{2k} + \frac{1-(-1)^{N+1}\left(\frac{x}{2dm}\right)^{2N+2}}{1+\left(\frac{x}{2dm}\right)^2} \,. \tag{A.3}$$

Substituting expansion (A.3) into series (4.7), and using definitions (A.1), it is not difficult to conclude that

$$N(x) = \frac{\varepsilon_r}{\pi\, d\,(\varepsilon_r - 1)} \sum_{k=0}^{N} (-1)^k \left(\frac{x}{2d}\right)^{2k} \mathrm{poln}_{2k+1}\left(\frac{2\,\varepsilon_r}{1+\varepsilon_r}\right) + R_N(x) \,, \tag{A.4}$$

where we called for brevity with $R_N(x)$ the following series

$$R_N(x) = \frac{\varepsilon_r}{\pi\, d\,(1 - \varepsilon_r)} (-1)^{N+1} \left(\frac{x}{2d}\right)^{2N+2} \sum_{m=1}^{\infty} \left(\frac{1-\varepsilon_r}{1+\varepsilon_r}\right)^m \frac{1}{m^{2N+1}\left[m^2 + \left(\frac{x}{2d}\right)^2\right]} \,. \tag{A.5}$$

It is evident that the series in expansion (A.5) converges more rapidly that the one in (4.7). For numerical calculations we selected N=3. But, if $d/a > 1/2$, identity (A.3) can be used also in the limit $N = \infty$, and therefore we can write the Taylor expansion of the function N(x)

$$N(x) = \frac{\varepsilon_r}{\pi\, d\,(\varepsilon_r - 1)} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2d}\right)^{2k} \mathrm{poln}_{2k+1}\left(\frac{2\,\varepsilon_r}{1+\varepsilon_r}\right), \qquad d/a > 1/2 \,. \tag{A.6}$$

Appendix B

Aim of this appendix is to find an expansion of the capacitance as a function of the aspect ratio a/d, whose first two terms are derived in section 4 and are given by formula (4.14).

In order to write integral equation (4.3) in a dimensionless form, we start by introduce the new variables

$$\rho = \frac{r}{a}\,, \quad \tau = \frac{t}{a}\,, \quad \xi = \frac{a}{2\,d}\,,$$

and the new functions

$$p(\rho a) \rightarrow \omega(\rho,\xi)\,, \qquad\qquad aN(a\rho) \rightarrow N(\rho,\xi) = \frac{2\,\varepsilon_r\,\xi}{\pi\,(\varepsilon_r - 1)} \sum_{k=0}^{\infty} (-1)^k \mathrm{poln}_{2k+1}\!\left(\frac{2\,\varepsilon_r}{1+\varepsilon_r}\right) \xi^{2k} \rho^{2k}\,.$$

Therefore integral equation (4.3) can be written as

$$\begin{cases} \omega(\rho,\xi) = 1 + \displaystyle\int_0^1 \omega(\tau,\xi)\left[N(\rho+\tau,\xi) + N(\rho-\tau,\xi)\right] d\tau\,, \\ \rho \in [0,1],\ \xi \in\,]1/2,\infty[\,. \end{cases} \tag{B.1}$$

Let us assume a Taylor expansion for the unknown

$$\omega(\rho,\xi) = 1 + \sum_{n=1}^{\infty} \omega_n(\rho)\,\xi^n\,, \tag{B.2}$$

and for the kernel

$$N(\rho,\xi) = \sum_{n=1}^{\infty} N_n(\rho)\,\xi^n\,, \tag{B.3}$$

where obviously we called

$$\omega_n(\rho) = \frac{1}{n!}\left[\frac{\partial^n \omega(\rho,\xi)}{\partial \xi^n}\right]_{\xi=0}\,, \qquad N_n(\rho) = \frac{1}{n!}\left[\frac{\partial^n N(\rho,\xi)}{\partial \xi^n}\right]_{\xi=0}\,. \tag{B.4}$$

Expansion coefficient (B.4) for the kernel derives directly from (A.6) and is defined by

$$N_n(\rho) = \frac{2\,\varepsilon_r}{\pi\,(\varepsilon_r - 1)}\,\frac{1-(-1)^n}{2}\,(-1)^{(n-1)/2}\,\mathrm{poln}_n\!\left(\frac{2\,\varepsilon_r}{1+\varepsilon_r}\right)\rho^{n-1}, \quad n \in N\,.$$

Thus integral equation (B.1) becomes

$$\begin{cases} \omega_n(\rho) = \displaystyle\int_0^1 \omega_{n-k}(\tau)\left[N_k(\rho+\tau) + N_k(\rho-\tau)\right] d\tau\,, \\ \rho \in [0,1]\,. \end{cases} \tag{B.5}$$

Because $\omega_0(\rho) = 1$, we can easily obtain the expansion functions $\omega_n(\rho)$ and discover that they are polynomials, defined by

$$\omega_n(\rho) = \sum_{i=0}^{n} \alpha_i(n)\, \rho^i \qquad (n \geq 1)\,, \tag{B.6}$$

where, as usual, the generic $\alpha_i(n)$ represents a Taylor expansion coefficient

$$\alpha_i(n) = \frac{1}{i!}\left[\frac{d^i\omega_n(\rho)}{d\rho^i}\right]_{\rho=0},$$

and the recurrence equation among them is

$$\alpha_i(n) = \sum_{k=i+1}^{n} \binom{k-1}{i}\left[1 - (-1)^{k-i-1}\right] N_k(1) \left[\sum_{s=0}^{n-k-1} \frac{\alpha_s(n-k)}{k-i+s}\right]. \tag{B.7}$$

Finally it is possible to derive directly an expansion formula for the capacitance. Substituting, in fact, relations (B.1) and (B.6) into formula (4.10), we have

$$\frac{C}{C_0} = \frac{4\,d}{a}\,\frac{1+\varepsilon_r}{\varepsilon_r}\left\{1 + \sum_{n=1}^{\infty}\left[\sum_{i=0}^{n-1}\frac{\alpha_i(n)}{i+1}\right]\left(\frac{a}{2\,d}\right)^n\right\}. \tag{B.8}$$