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G. Miano, L. Verolino, V.G. Vaccaro

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SOME INTEGRAL FORMULATIONS OCCURRING IN ACCELERATOR PHYSICS

G. Miano, L. Verolino
Dipartimento di Ingegneria Elettrica and INFN Sezione di Napoli
Via Claudio 21 - I-80125 Napoli - Italy

V.G. Vaccaro
Dipartimento di Scienze Fisiche and INFN Sezione di Napoli
Mostra d'Oltremare (Pad. 20) I-80125 Napoli - Italy

Abstract

In this paper a powerful and robust analytical-numerical approach to study the electromagnetic interaction between a bunch of particles and the discontinuities of the vacuum chamber of a particle accelerator is discussed. In particular the diffraction of the electromagnetic field created by a bunch of charges travelling through an iris and a drift tube is considered. Choosing in both cases a spectral transform of the current density distribution on the scatterer as unknowns, an effective numerical model is obtained. These unknowns have to satisfy a system of dual integral equations. A general procedure to transform this system into only one Fredholm integral equation of the second kind (in the case of the iris) or to a system of linear algebraic equations by means of a Neumann series (in the case of the drift tube) is described. These models allow to compute the longitudinal coupling impedance with a good accuracy either in the low frequency limit or in the high frequency limit.

Les sciences sont comme un beau fleuve, dont le cours est facile à suivre, lorsqu'il a acquis une certaine regularité; mais si l'on veut remonter à la source, on ne la trouve nulle part, parsqu'elle est partout; elle est repandue en quelque sorte sur toute la surface de la terre: meme si l'on veut remonter a l'origine des sciences, on ne trouve que obscurité, idees vagues, cercles vicieux; et l'on se perd dans les idées primitives.

(L. Carnot 1783)

I. INTRODUCTION

A bunch of charged particles travelling in a linear or circular accelerator interacts with the surrounding structure producing electromagnetic wake fields which, reacting back on the bunch, influence its dynamics [1-2]. Many authors have devoted their efforts to the analysis of resonant (cavities, bellows) and non-resonant (discontinuities) structures, either in the time domain (wake field), or in the frequency domain (the machine impedance, namely the ω-transform of the wake field), adopting numerical or analytical procedures. Although the results of these studies agree on the behaviour of the long-range wake fields (low-frequency impedance), the results obtained so far for the short-range wake fields (high-frequency impedance) are somewhat contrasting. The development of new analytical and numerical procedures leading to a definitive solution of these problems would be clearly desirable.

In this paper we attempt to calculate the impedance of two structures relevant in accelerator physics, the iris and the drift tube, by using methods wich guarantee the correctness of the results either in the low frequency limit or in the high frequency limit.

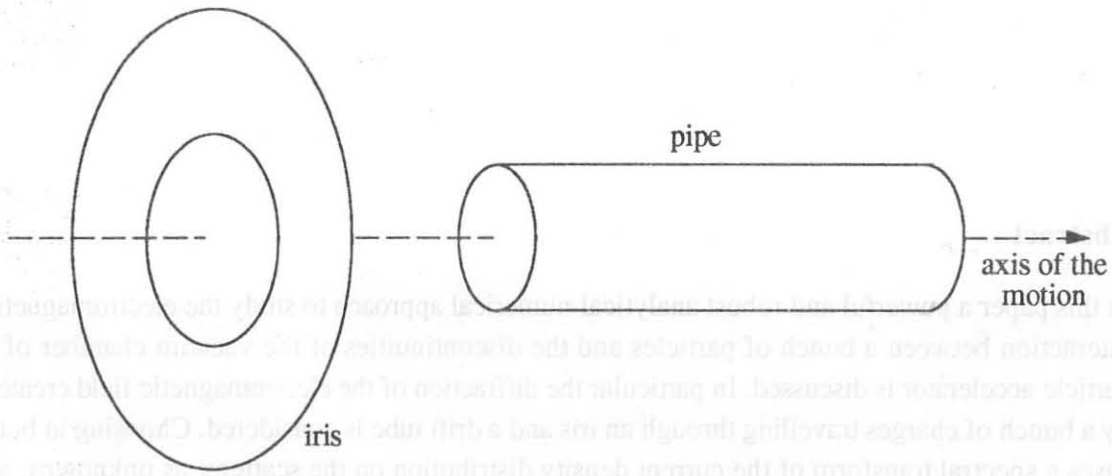

Fig. 1 – Two fundamental structures in accelerator physics.

An *iris* is a hole in an conducting screen, whereas a *drift tube* is a piece of a metallic pipe, Figure 1. We shall use cylindrical co-ordinates system whose $\hat{z}$ axis coincides with the symmetry axis of the structure; besides we shall assume that the charge moves in the positive $\hat{z}$ direction.

Let us consider a particle travelling on the axis of a metallic structure having a rotational symmetry. The field created by the charge in the presence of the scatterer will interact with the charge itself; thus we should find a change of the particle velocity. Such a radiation problem is very difficult to solve and therefore a simplifying assumption will be made: we shall suppose that the charge moves at *constant velocity* during its flight. The constant velocity, $\vec{v} = \beta c\,\hat{z}$, can be imagined as being maintained by an external source. The result will be a good approximation provided that the velocity of the charge does not change significantly during the interaction with the screen, namely this assumption is quite realistic when dealing with ultrarelativistic charges.

The aim of this study is to compute the *longitudinal coupling impedance* [3-4], that is one of the most important parameter which determines the performance of an accelerator. The coupling impedance allows for deriving the energy lost by the beam and the accelerator current thresholds set by the instability mechanism arising in the longitudinal beam dynamics. One can define the longitudinal impedance as

$$Z_{||}(k) = \frac{1}{q}\int_{-\infty}^{+\infty} E_z(r=0,z,k)\, \exp(jkz/\beta)\, dz\,, \qquad (I.1)$$

where q is the charge of the particle, k is the wavenumber, and $E_z(r,z,k)$ is the radiated field synchronous with the particle.

The theory of the diffraction of a plane wave by a circular aperture in an infinite screen or by a circular cylinder can easily be found in the literature [5-6]. The problem is usually analysed by modal expansion methods which give the solution as an infinite sum of eigenfunctions of the wave equation in a particular co-ordinate system. However, this solution has the shortcoming of poor convergence, especially for the case of short wavelengths. It is well known that a point charge crossing the hole will excite a continuous spectrum of frequencies, which extends to very high frequencies for ultrarelativistic charges; this general feature of the diffraction-radiation problem makes a modal expansion really impracticable for our problem, even to have an approximate solution [7].

Both problems will be treated as a boundary-value problems for Maxwell's equations: we have the radiation condition at infinity, the condition on the tangential component of the electric field on the screen or on the tube, and the Meixner (edge) condition [8] for the discontinuity at the edge of the holes. The last condition will ensure the uniqueness of the solution for our problems.

This diffraction problem is described by the field $(\vec{E}_0,\vec{H}_0)$ travelling with the charge itself and by the electromagnetic field $(\vec{E},\vec{H})$ radiated by the metallic structure, which has a travelling wave character. Accordingly, we can represent all the fields and/or potentials as the superposition of two terms

$$\begin{cases} \vec{E}_t = \vec{E}_0 + \vec{E}\,, \\ \vec{H}_t = \vec{H}_0 + \vec{H}\,. \end{cases}$$

The sum of both terms must satisfy the boundary conditions on the metallic structures.

For the symmetry of the problem, the induced currents are directed radially for the iris and longitudinally for the drift tube, and the only components of the field are E_r, E_z, H_φ.

Moreover since in bith cases the induced currents are orthogonal to the edge, the Meixner or edge condition requires that the component of the electric field orthogonal to the edge diverges as $d^{-1/2}$, where d is the distance from the edge [8].

The choice of a suitable spatial transformation of the current density as unknown is the key to obtain a robust numerical model. One equation of this system is obtained by imposing that the tangential component of the electric field has to vanish on the metallic region; the other arises from the condition that the current has to be zero in the complementary region (the vacuum region). The result will be a system of coupled integral equation, known as dual integral equations because the equations have as range of definition two complementary regions.

These systems have been studied by many authors; a quite complete summary of all the methods of solution, almost up to 1966, can be found in Sneddon [9]. In particular, when the two domains are both semi-infinite, the problem can be solved by Wiener-Hopf techniques [10]. These techniques give the solution in a simple and elegant manner. Unfortunately they do not apply in our cases, where one of the domains is finite.

In order to illustrate two different methods of solution, the two problems quoted above will be tackled in different ways. An integral representation of the unknown is given for the scattering from circular iris, whereas the unknown for the drift tube problem is expanded as a Neumann series.

For the first problem, *circular iris*, we get a Fredholm integral equation of the second kind with a continuous kernel, solved by the method of moments. The way to reformulate the problem is not unique, as demonstrated in several papers [9]. This variety of methods can be used to find an 'ad hoc' Fredholm integral equation of the second kind, or still better a fast converging integral equation, namely an equation where the free term is already a good approximation of the complete solution; this is not an easy task. Lebedev-Skal'skaya's works (see for example [11]) are also very useful in order to understand the aim of these transformations; besides they contain very nice manipulations of some integral relations.

The solution for the expansion coefficients relevant to the second problem, *drift tube*, is obtained by expanding the unknown in a series of Bessel functions (Neumann series), in such a way that one of the integral equation of the system is automatically satisfied, whereas the other one can be transformed in a algebraic system of equation, by means of an adquate projection in an opportune functional space of orthogonal functions [12].

It is also worth noting that the method hereafter proposed can be applied to an infinite array of circular apertures or tubes and that the difficulty of the problem does not increase if one try to compute the longitudinal impedance for a particle travelling off of the symmetry axis [13].

In Section II we shall briefly recall the formulas to compute the electromagnetic field of a particle travelling at constant velocity in free space, representing the forcing term in both problems. The Sections III and IV treat respectively the problem of the circular iris and of the drift tube; approximate formulas and plots are given for the longitudinal coupling impedance.

Notations

The conventions for the transforms used in this paper are

• *time-dependent function*

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)\, \exp(-j\omega t)\, dt \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\, \exp(j\omega t)\, d\omega\,;$$

• *space-dependent function*

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(z)\, \exp(jkz)\, dz \qquad f(z) = \int_{-\infty}^{+\infty} F(k)\, \exp(-jkz)\, dk\,;$$

• *Hankel transform*

$$F(u) = \int_{0}^{\infty} r\, f(r)\, J_1(ur)\, du \qquad f(r) = \int_{0}^{\infty} u\, F(u)\, J_1(ur)\, du\,.$$

It will be assumed in the following that the order of integration in repeated integrals, and the orders of differentiation and integration, can be interchanged as necessary without explicit justification.

II. FIELD OF A UNIFORMLY MOVING CHARGE

Let us consider a cylindrical co-ordinate system (r,φ,z). The charge is located on the z-axis and moves with velocity $\vec{v} = \beta c\, \hat{z}$ in the positive direction. It can be shown [3] that the expressions of the fields in the free space have a TM structure in the ω-domain and they are given by

$$\vec{H}_0 = \hat{\varphi}\, \frac{q\kappa}{2\pi}\, K_1(\kappa r)\, \exp(-jzk/\beta)\,, \tag{II.1}$$

$$\vec{E}_0 = \zeta_0 \frac{q\kappa}{2\pi\beta\gamma}\left[\hat{r}\, \gamma K_1(\kappa r) + \hat{z}\, jK_0(\kappa r)\right] \exp(-jzk/\beta)\,, \tag{II.2}$$

where j is the imaginary unit, $\kappa = k/(\beta\gamma)$, $\zeta_0 = 120\pi\Omega$ is the characteristic impedance of the free space, γ is the relativistic factor defined as

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-\beta^2}}\,,$$

and c is the velocity of light in vacuo. The asymptotic behaviour of the modified Bessel functions of the second kind is given by [14]

$$K_n(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}\left[1+O\left(\frac{1}{x}\right)\right],$$

so it is easy to verify that the fields associated with the charge moving in free space are strongly attenuated in the radial direction.

III. CIRCULAR IRIS

In this Section we shall show, in some details, how the problem of the circular iris can be formulated as a system of dual integral equations [13], how this system can be rewritten as a single integral equations of the second kind, and how to solve by means of the method of moments this integral equation in order to obtain the longitudinal coupling impedance.

Statement of the problem

A point charge q is moving at constant velocity $v = \beta c$ on the axis of a hole in a perfectly conducting screen. We shall use cylindrical co-ordinates whose $\hat{z}$ axis passes through the centre of the aperture and is perpendicular to the plane of the screen; we shall assume that the charge moves in positive $\hat{z}$ direction, as shown in Figure 2.

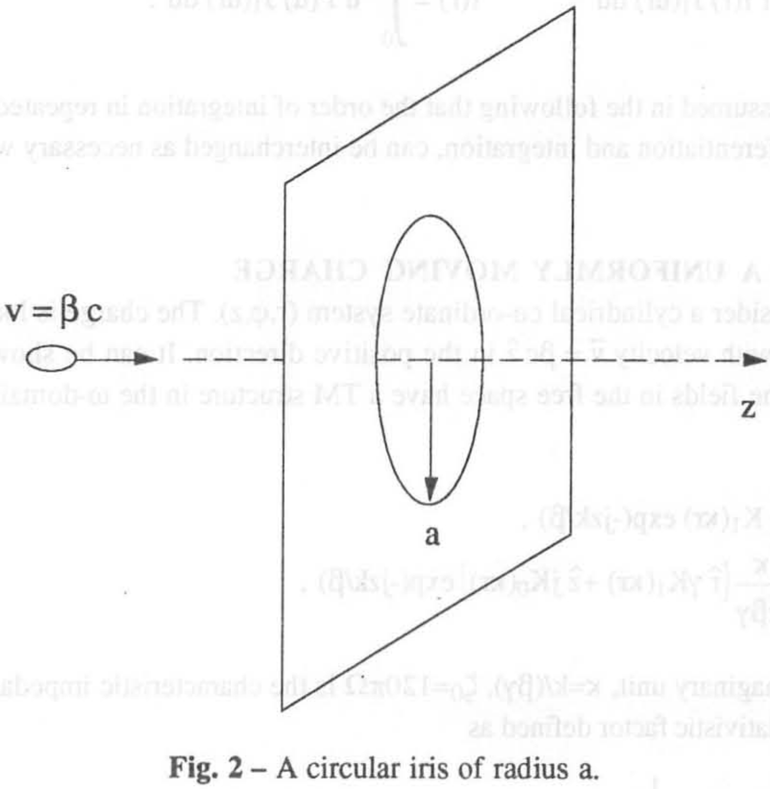

Fig. 2 – A circular iris of radius a.

A charge moving with uniform velocity in vacuo radiates only if of the material inhomogeneities are present close its path and the radiation is due to the diffraction of the field at the edges of the discontinuities. The fields of an ultrarelativistic charge are essentially confined within an angular region of aperture $\approx 1/\gamma$, where γ is the energy of the charge expressed in rest mass units (relativistic factor). As long as the charge is far from the hole, it barely perceives the presence of the scatterer. Its image charges are at a great distance so that there is only very weak interaction with them; moreover, they move at constant velocity towards the centre of the hole. In this situation, which persists up to quite small distances of the charge from the hole, little radiation is expected [15]. Only when the edge of the hole is seen by the charge within the narrow $1/\gamma$ cone, the image charges will experience a sudden change of their motion since they are released and start moving radially back to infinity: this process lasts for the time of passage of the charge through the hole and it is the main reason for radiation. The more relativistic is the charge, the shorter is the radiation time and the wider is the spectrum of radiation.

Let us consider the current flowing on the metallic plate; for each frequency component of $\vec{J} = J(r)\,\hat{r}$, we can find the expression of the vector potential $\vec{A} = A(r,z)\,\hat{r}$ by means of the equation

$$\nabla^2\vec{A} + k^2\,\vec{A} = -\,\mu_0\,\vec{J}\ ,$$

where the wave number $k=\omega/c$ and μ_0 is the permeability of vacuum. Firstly we solve for a δ-function source located at the point (r_0,z_0) which yields the Green's function $\vec{G}(r,z;r_0,z_0) = G(r,z;r_0,z_0)\,\hat{r}$ as solution of the equation

$$\nabla^2\vec{G} + k^2\,\vec{G} = -\,\mu_0\,\frac{\delta(r-r_0)\delta(z-z_0)}{2\pi r_0}\,\hat{r}\ , \tag{III.1}$$

which, in cylindrical co-ordinates, becomes [14]

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r}\frac{\partial G}{\partial r} - \frac{1}{r^2}\,G + k^2\,G = -\,\mu_0\,\frac{\delta(r-r_0)\delta(z-z_0)}{2\pi r_0}\ . \tag{III.2}$$

We can now transform partial differential equation (III.2) into an ordinary algebraic equation by means of the double (Hankel-Fourier) integral transformation [16]

$$A(k_r,k_z;r_0,z_0) = \int_0^{\infty} r\,J_1(rk_r)\left[\int_{-\infty}^{+\infty} A(r,z;r_0,z_0)\exp(jzk_z)\,dz\right]dr\ ,$$

which, from the properties of the δ-functions, enables us to write

$$\vec{A}(k_r,k_z;r_0,z_0) = \frac{\mu_0}{4\pi^2}\,\frac{J_1(r_0k_r)\exp(jz_0k_z)}{k_r^2 + k_z^2 - k^2}\,\hat{r}\ . \tag{III.3}$$

We can now integrate Green's function multiplied by the current over the whole space V_0, even if the actual current flows on the plates only for r>a (hole radius). Eventually we shall impose the condition that the current vanishes in the hole, which is a condition for the Hankel transform of the current. We obtain

$$A_r(r,z) = \frac{\mu_0}{2\pi}\int_0^{\infty} k_r J_1(rk_r)dk_r \left\{\int_{-\infty}^{+\infty} dk_z \frac{\exp(-jzk_z)}{k_r^2+k_z^2-k^2}\left[\int_0^{\infty} r_0 J_1(r_0k_r)J(r_0)dr_0\right]\right\} . \qquad \text{(III.4)}$$

The integration over r_0 is simply the Hankel transform of the current J(r), namely

$$F(u) = \int_0^{\infty} r_0\, J_1(ur_0)\, J(r_0)\, dr_0 \qquad \text{(III.5)}$$

where u represents the radial wavenumber k_r. The inverse transform reads

$$J(r) = \int_0^{\infty} u\, J_1(ur)\, F(u)\, du\ . \qquad \text{(III.6)}$$

From now on we choose the transform F(u) as the unknown of the problem. Equation (III.4) becomes

$$A_r(r,z) = \frac{\mu_0}{2\pi}\int_0^{\infty} u\, F(u)\, J_1(ur)\left[\int_{-\infty}^{+\infty} \frac{\exp(-jzk_z)}{k_r^2+k_z^2-k^2}dk_z\right] du\ . \qquad \text{(III.7)}$$

The integration in square brackets over k_z may be performed by means of the residue theorem; in fact, putting $U = \sqrt{k^2-u^2}$ the integrand function exhibits two simple poles at $k_z=\pm U$. In the evaluation of the residues it is necessary to take into account a small imaginary part for $k = \omega\sqrt{\varepsilon\mu}$ where $\varepsilon\mu$ is considered to be complex, so that Im(k)<0. For the two poles, we then have Im(U)<0 or $U = \sqrt{k^2-u^2} = -j\sqrt{u^2-k^2}$ when u>Re(k). By taking Im(U)<0, we implicitly satisfy the radiation condition at infinity. It is found that

$$\int_{-\infty}^{+\infty} \frac{\exp(-jzk_z)}{U^2 - k_z^2}dk_z = j\pi\frac{\exp(-jU|z|)}{U} .$$

Because the fields $\vec{E}$ and $\vec{H}$ are related to the vector potential $\vec{A}$ by [17]

$$\vec{E} = -j\omega\left(\vec{A} + \frac{\nabla\nabla\cdot\vec{A}}{k^2}\right), \quad \vec{H} = \frac{\nabla\times\vec{A}}{\mu_0},$$

we have the following expressions for the fields

$$\begin{cases} E_r(r,z) = \dfrac{j\zeta_0}{2k} \displaystyle\int_0^\infty u\, F(u)\, J_1(ur) \sqrt{u^2-k^2}\, \exp\!\left(-|z| \sqrt{u^2-k^2}\right) du , \\ E_z(r,z) = \dfrac{j\zeta_0}{2k} \operatorname{sgn}(z) \displaystyle\int_0^\infty u^2 F(u)\, J_0(ur) \exp\!\left(-|z| \sqrt{u^2-k^2}\right) du , \\ H_\varphi(r,z) = \dfrac{\operatorname{sgn}(z)}{2} \displaystyle\int_0^\infty u\, F(u)\, J_1(ur) \exp\!\left(-|z| \sqrt{u^2-k^2}\right) du \end{cases} \qquad \text{(III.8)}$$

As already mentioned, the solution can be found as a superposition of the solution of the inhomegeneous equations in free space and a solution of the homogeneous equations, chosen in such a way as to fulfil the boundary conditions on the plates. Accordingly the two conditions which are to be satisfied are

$$\begin{cases} J(r) = 0 , & 0 \le r < a \\ E_{0r}(r,z=0) + E_r(r,z=0) = 0 . & r > a \end{cases} \qquad \text{(III.9)}$$

The first equation of the system (III.9) states that there is no current on the aperture ($r<a$), whereas the second one is the boundary condition for the radial component of the electric field on the metallic surface ($r>a$)

Dual integral equations system

Boundary conditions (III.9) allow us to find the expression of the current transform $F(w)$ and the system of coupled integral equations is easily obtained

$$\int_0^\infty w\, F(w)\, J_1(wr)\, dw = 0 , \qquad 0 \le r < a ; \qquad \text{(III.10)}$$

$$\int_0^\infty w \sqrt{w^2-k^2}\, F(w)\, J_1(wr)\, dw = \frac{jq\gamma\kappa^2}{\pi} K_1(\kappa r) , \qquad r > a , \qquad \text{(III.11)}$$

where for brevity we called $\kappa = k/(\beta\gamma)$. Equation (III.10) states that there is no current on the aperture ($r<a$) or the field H_φ is zero on the hole, whereas equation (III.11) is the boundary condition for the radial component of the electric field which has to vanish on the metallic surface ($r>a$).

Such pairs of integral equations, with one equation of the pair holding over one part of the range of the independent variable and the other over the other part of the range, are known as dual integral equations. These types of integral equations seem to have been first encountered in potential theory. Though in this paper we are interested in the application of dual integral equations to the solution of the wave equation for diffraction problems (where the integrals are normally singular), it will not be inappropriate to give a brief summary of the developments of dual integral equations in potential theory.

A quite exhaustive survey of the historical developments of dual integral equations up to 1966 is given by Sneddon [9] in his book. One of the earliest (1873) encounters with dual integral equations was made by Weber [18], who solved the axisymmetric potential problem of a uniformly charged disk by noticing the similarity with certain discontinuous integrals. In 1881 Beltrami [19] could provide a logical basis and generalise the solution for any given axisymmetric potential distribution on the disk. This development was further generalised by Copson [20] in 1947 for any given potential distribution and his method has provided the basis for the solution of many types of dual integral equations. Solutions have been also obtained through the use of integral operators such as Erdelyi and Kober operators of fractional integration and modified Hankel operators [9]. The solution of a class of dual integral equations involving Bessel function kernels has been studied by Nicholson [21], by Tichmarsh [22], Noble [23], Peters [24], Gordon [25], Erdelyi and Sneddon [26], Lebedev in the solution of an electrostatic problem [27], among others. The application of dual integral equations to diffraction problems has been extensively studied by Lur'e [28], Lebedev and Skal'skaya [29-32]. Recently solutions involving generalised functions have been found by Estrada and Kanwal [33-34]. Various methods of numerical treatment of dual integral equations, namely, reduction to a system of algebraic equations, or to a Fredholm type of equation, the multiplying factor method, the integral representation method, the reduction to a single integral equation of the second kind have been very well reviewed by Sneddon [9] and also by Williams [35].

Once we know the spectrum F(w), we can, in principle, compute the longitudinal component of the electric field on the z-axis which is necessary to calculate the coupling impedance we are searching for; so we get (III.8)

$$E_z(r=0,z) = \frac{j\zeta_0}{2k}\,\mathrm{sgn}(z)\int_0^{\infty} u^2\,F(u)\;\exp\!\left(-|z|\sqrt{u^2-k^2}\right)du$$

where ζ_0 is the impedance of free space. Thus, making use of the integral [14]

$$\int_{-\infty}^{+\infty} \exp(jkz/\beta)\,\exp\!\left(-|z|\sqrt{u^2-k^2}\right)\mathrm{sgn}(z)\,dz = \frac{2jk}{\beta(u^2+\kappa^2)}\,,$$

it can be shown that longitudinal coupling impedance (I.1) becomes

$$Z_{||}(k) = \frac{\zeta_0}{q\beta}\int_0^{\infty} F(u)\,\frac{u^2}{u^2+\kappa^2}\,du\,. \qquad \text{(III.12)}$$

This equation links directly the problem unknown F(w) to the longitudinal coupling impedance.

Reduction to a single Fredholm integral equation

We can reduce the problem of solving *dual integral equations* (III.10) and (III.11) to that of solving only one Fredholm integral equation by using a method extensively described by Sneddon [9]; a shorter description of this method, which is at the same time synthetic but also complete for our propose, can be found in reference [11]. The method proposed enables us to represent the solution of the problem by means of an auxiliary function, which satisfies a Fredholm integral equation of second kind with a continuous kernel. As a first step we have to find an *ad hoc* representation of the unknown function F(w), so that one of the two equations of the system (the second in our case)is automatically satisfied. The other equation will become apparently a Fredholm equation of first kind, but a careful study of its kernel will reveal the presence of a δ-function that will transform it into an equation of the second kind. An integral equation of the second kind is quite easy to treat numerically because it does not present all the problems of instability of the equation of first kind [36]. This is possible if the spectrum F(u) is sought in the form [9-10]

$$\sqrt{w^2-k^2}\,F(w) = \frac{jqk}{\pi\beta}\left[\frac{w}{w^2+\kappa^2} - a\int_0^1 p(t)\,\sin(uat)\,dt\right], \qquad \text{(III.13)}$$

where p(t) is some unknown auxiliary[1] function, continuous, together with its first derivative, in the closed interval (0,1). One can verify that equation (III.11) is satisfied identically. It is interesting to note that Meixner's condition (requiring that the electric field near an infinitely thin edge behaves as the -1/2 power of the distance from the edge) [8] is automatically satisfied with the particular choice of equation (III.13), what one can verify, for example, on the radial component of the electric field in the aperture.

Substituting (III.13) in this last relation, after some algebraic manipulations, one can finally write[2]

$$p(x) = T(kax) + \frac{ka}{2}\int_0^1 p(t)\,\{G(ka|x-t|) - G[ka(x+t)]\}\,dt \qquad \text{(III.14)}$$

with $0 \le x \le 1$. The free term T(x) of equation (III.14) defined by the integral relation

$$T(x) = \frac{2}{\pi}\int_0^{\infty} \frac{u^2 \sin(ux)}{u^2 + (\beta\gamma)^{-2}}\,\frac{du}{\sqrt{u^2-1}}$$

---

1 A different transformation could also be used, where the function F(u) would be an integral from 1 to ∞ of another auxiliary function [9, 12].

2 It is interesting to note that one can also substitute (III.13) in equation (III.10) and then operate the Abel's transformation, inverting the order proposed here.

has been carefully studied in [14] and some interesting approximations have been found. In particular if $\beta = 1$, it is easily verified that

$$T(x) = J_0(x) - j\, \mathbf{H}_0(x)\,,$$

namely a complex combination of Bessel and Struve functions.

The kernel G(x) can be written in a closed form by means of the relation [11]

$$G(x) = \frac{2}{\pi}\int_0^{\infty}\left(1 - \frac{u}{\sqrt{u^2-1}}\right)\cos(ux)\,du = J_1(x) - j\mathbf{H}_1(x) + \frac{2j}{\pi}\,,$$

where $J_1(x)$ and $H_1(x)$ are respectively Bessel and Struve functions [37-38] of the first kind. It follows from the definition of the function G(x) that the kernel of equation (III.14) is a continuous and symmetrical function of the two variables x and t; this implies that this equation can be solved by use of numerical methods by converting the integral equation into a linear system of algebraic equations. Besides for small values of the parameter ka, the solution of the integral equation can be expressed in the form of a converging power series of this parameter.

The longitudinal coupling impedance defined by equation (III.12), can now be computed by means of equation (III.13) as (k>0)

$$\frac{Z_{||}(k)}{\zeta_0} = \frac{1}{2\pi\beta}\left\{\left(1 - \frac{1}{2\gamma^2}\right)\left[\ln\left(\frac{1+\beta}{1-\beta}\right) + j\pi\right] - \beta\right\} - \frac{jka}{2\beta^2}\int_0^1 T(kax)\,p(x)\,dx\,. \qquad \text{(III.15)}$$

It is interesting to note that this impedance can be directly expressed in quadrature by means of the function p(x), without the intermediate formula (III.12). Equation (III.15), besides, can be used for a numerical study of the impedance.

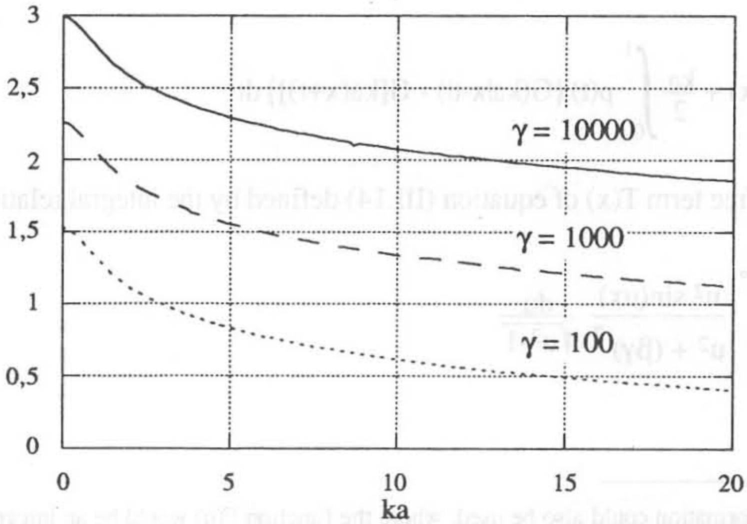

Fig. 3 – Normalized values of the real part of the longitudinal impedance (III.15).

As for the auxiliary function p(x), a series expansion for the impedance can be easily found; but this formula useful for the approximation of the low frequencies (ka<1), does not exhibit a wide range of convergence, becoming impracticable at ka=1 for large values of γ.

Numerical resolution

In the solution procedure the auxiliary function p(x) of equation (III.14) is approximated by a linear combination of known, linearly independent basis functions [39]. In our case, pulses are employed which means that p(x) is represented in (0,1) by a piecewise-constant approximation as

$$p(x) \approx \sum_{n=1}^{N} P_n \, \Pi_n(x)$$

in which each P_n is an unknown constants and the pulse function Π_n is defined by.

$$\Pi_n(x) = \begin{cases} 1 & x \in (x_n - \Delta/2, x_n + \Delta/2)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

The approximation is illustrated in Figure 4. The interval (0,1) is divided into N segments of equal length $\Delta = 1/N$ with their centres at

$$x_n = \left(n - \frac{1}{2}\right)\Delta\,, \qquad\qquad n = 1, 2, ..., N.$$

Subject to the above approximation of p(x), equation (III.14) becomes

$$\sum_{n=1}^{N} P_n \, \Pi_n(x) = T(kax) + \sum_{n=1}^{N} P_n \, S_n(x)\,, \qquad x \in (0,1) \qquad\qquad (III.16)$$

where

$$S_n(x) = \frac{ka}{2} \int_{x_n - \Delta/2}^{x_n + \Delta/2} \{G(ka|x-t|) - G[ka(x+t)]\}\, dt\,.$$

These integrals can be computed readily using the definition of the kernel G(x). Equation (III.16) is enforced exactly at N points x_m, m = 1, 2, ... N, in the interval (0,1), as illustrated in Figure 4, to obtain the following set of N equations in N unknowns P_n

$$P_m = T_m + \sum_{n=1}^{N} P_n \, S_{nm}\,. \qquad\qquad \text{for } m = 1, 2, ..., N. \qquad\qquad (III.17)$$

In equation (III.17), $T_m = T(kax_m)$ and $S_{nm} = S(x_m)$.

The matrix $S = \{S_{nm}\}$ is a symmetric matrix and the matrix (I-S) is diagonally dominant, that is, the magnitude of the main diagonal elements is greater than that of any of the off-diagonal elements. For this reason the numerical inversion of this matrix is very easy and the solution of the linear system (III.17) is a very stable process.

An approximate solution of the system (III.17) is presented in Figure 5.

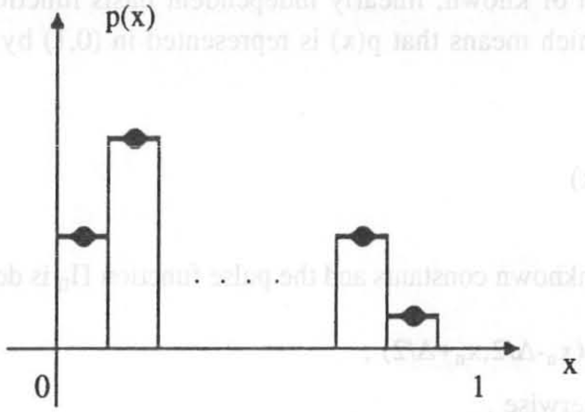

Fig. 4 – Weighted pulse-functions representation of p(x).

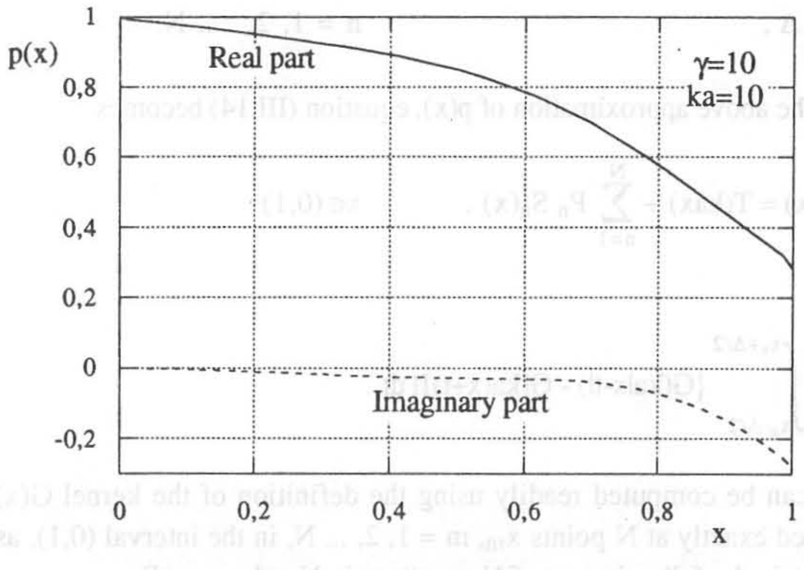

Fig. 5 – An example of the auxiliary function p(x).

Asymptotic approximation

The solution of equation (III.17) enables us to compute the longitudinal coupling impedance up to an upper bound of ka, because of the numerical discretization. To increase ka we have to increase N in the approximation of the unknown p(r). As it is easy to imagine, N cannot be large as we want because of the large dimensions of the matrices describing the approximate version of equation (III.14). Thus it seems interesting to have (at least for the real part of the impedance), an asymptotic formula that can help us for a better understanding of the high frequency behaviour. For this reason we can rewrite the fundamental system (III.10) and (III.11), in the limit $ka \to \infty$, as the single integral equation

$$\int_0^\infty u\, F(u)\, J_1(u)\, du \cong \frac{q\kappa}{\pi\beta} K_1(\kappa r)\, u(r-a) \qquad \text{(III.18)}$$

where u(r) represents the unit step function (Heaviside step function). This equation can be interpreted as the Hankel transform of the unknown F(u); making use of the inversion formula [14], we obtain

$$F(u) \cong \frac{qa}{\pi\beta}\frac{\kappa}{u^2+\kappa^2}\left[\kappa J_1(ua)K_2(\kappa a) + uJ_2(ua)K_1(\kappa a)\right] . \qquad \text{(III.19)}$$

If we put (III.19) in the definition (III.12), after some manipulations [14, 37], we finally have the following real part of the impedance

$$\mathrm{Re}[Z_{\parallel}(k)] \cong \frac{\zeta_0}{2\pi\beta^2}(\kappa a)^2\left[K_0(\kappa a)K_2(\kappa a) - K_1^2(\kappa a)\right] \qquad \text{(III.20)}$$

This formula gives a good asymptotic approximation of the real part of the impedance, in principle, for very large values of the dimensionless product ka. But as it is shown in Table 1, when γ is sufficiently large (let us say greater then 10), one can assume that equation (III.20) represents the impedance also for ka>1. Figure 3 has been obtained by making use of equation (III.15) in the range $0 \le ka \le 20$ and it could be completed almost for the real part using formula (III.20) in the rest of the range.

Table 1 – Ratio between the real part of the longitudinal coupling impedance computed by means of the approximate formula (III.20) and the actual value given by (III.15), on increasing γ, for ka=1.

γ	Ratio
1.1	8.65499
10^1	1.04545
10^2	1.01702
10^3	1.01090
10^4	1.00803
10^5	1.00669

Finally, in order to have an easier formula to manage the asymptotic real part of the impedance, we can substitute in equation (III.20) Bessel functions with their asymptotic expansions [14], simplifying the expression as follows

$$\mathrm{Re}[Z_{||}(k)] \cong \frac{\zeta_0}{4\beta^2} \exp(-2\kappa a)$$

when $\kappa a \gg 1$. The problem to find a more accurate approximation for the longitudinal impedance, involving also the imaginary part, is still an open question and has to be investigated carefully.

IV. DRIFT TUBE

In this Section we shall show how the problem of the pipe of finite length can be formulated as a system of dual integral equations, how this system can be rewritten as a system of linear algebraic equations by means of a Neumann series, and how to solve it in order to obtain the longitudinal coupling impedance.

Statement of the problem

As in the previous case, all the electromagnetic quantities, fields and/or potentials, can be imagined as the superposition of two terms: the free space solution, discussed in Section II, and the solution sustained by the current induced on the surface of the metallic tube (r=a)

$$\vec{J} = \hat{z}\, J(z) \,. \qquad \text{(IV.1)}$$

This density current flows (Figure 6) along the z-axis and produces a TM propagation (like in the case of the free space solution).

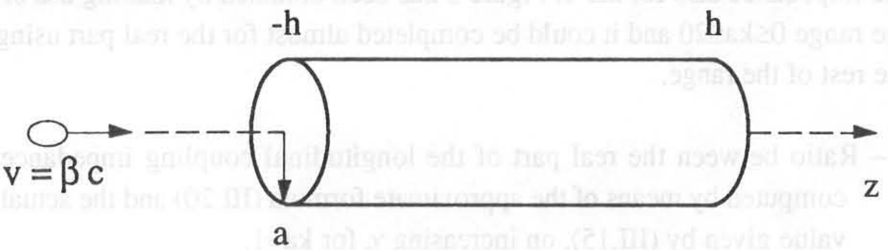

Fig. 6 – Drift tube.

We choose as unknown of the problem the transform of J(z) in the wavenumber domain, namely (we used the same symbol used above for the circular iris, but the physical meaning is completely different)

$$F(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} J(z) \exp(juz)\, dz \,. \qquad \text{(IV.2)}$$

Let us start to write the vector potential as a function of the selected unknown. The apparent azimuthal symmetry enables us to affirm [40]

$$\vec{A} = \hat{z}\, A_z(r,z) = \hat{z}\,\frac{\mu_0 a}{2}\int_{-\infty}^{+\infty} J(z_0)\left[\int_0^{2\pi} \frac{\exp(-jkR)}{R}\, d\varphi_0\right] dz_0\,, \tag{IV.3}$$

R is the distance between the generic observation point $P(r,\varphi,z)$ and the source point $P_0(r_0,\varphi_0,z_0)$, namely

$$R^2 = r^2 + a^2 - 2ra\cos(\varphi-\varphi_0) + (z-z_0)^2 = D^2 + (z-z_0)^2$$

and D represents the distance in the transverse plane. Using a relevant result of the theory of Bessel function, useful to express the Green's functions in cylindrical co-ordinates, one can write [40]

$$\frac{\exp(-jkR)}{R} = \frac{1}{\pi}\int_{-\infty}^{+\infty} K_0\!\left(D\sqrt{u^2-k^2}\right)\exp[-ju(z-z_0)]\, du\,,$$

where $K_0(x)$ the modified Bessel function of zero order.

It is worth noting that, as for the case of the iris, the previous integral converges if we take into account a small imaginary part for k, and the branch cut has to be chosen such that $\mathrm{Im}\sqrt{k^2-u^2} \le 0$.

The above result enables us to factorize the spatial dependence on r and z, that is

$$\int_0^{2\pi} \frac{\exp(-jkR)}{R}\, d\varphi_0 = 2\int_{-\infty}^{+\infty} G(u,r)\exp[-ju(z-z_0)]\, du\,, \tag{IV.4}$$

and the function G(u,r), because[1] of the addition theorem of the Bessel functions [38]

$$K_0\!\left(m\sqrt{r^2+\rho^2-2r\rho\cos x}\right) = \begin{cases} \sum_{n=0}^{\infty} \varepsilon_n I_n(m\rho)\, K_n(mr)\cos(nx), & \rho \le r; \\ \sum_{n=0}^{\infty} \varepsilon_n I_n(mr)\, K_n(m\rho)\cos(nx), & \rho \ge r. \end{cases}$$

can be simply written as

1 Neumann's symbol ε_n is defined as

$$\varepsilon_n = \begin{cases} 1 & \text{if } n=0\,, \\ 2 & \text{if } n=1,2,3,\dots \end{cases}$$

$$G(u,r) = \begin{cases} I_0\left(r\sqrt{u^2-k^2}\right) K_0\left(a\sqrt{u^2-k^2}\right), & r\leq a; \\ I_0\left(a\sqrt{u^2-k^2}\right) K_0\left(r\sqrt{u^2-k^2}\right), & r\geq a. \end{cases} \qquad \text{(IV.5)}$$

The knowledge of the function G(u,r) allows us to find intelligible integral relations linking the potential and the electromagnetic filed to the unknown F(u) and, as we shall see, it will be rather simple to write an integral equation describing the phenomenon; in other words, G(u,r) is the function which renders algebraic the links between the spectra of the fields and the potential. Coming back to the vector potential (IV.3), it is (ζ_0 is the characteristic impedance of the medium filling the whole space)

$$A_z(r,z) = \frac{a\zeta_0}{c} \int_{-\infty}^{+\infty} G(u,r)\, F(u) \exp(-juz)\, du ,$$

that is the electromagnetic field

$$\begin{cases} H_\varphi(r,z) = -a \displaystyle\int_{-\infty}^{+\infty} F(u) \frac{\partial G(u,r)}{\partial r} \exp(-juz)\, du , \\ E_r(r,z) = -\dfrac{a\zeta_0}{k} \displaystyle\int_{-\infty}^{+\infty} u \frac{\partial G(u,r)}{\partial r} F(u) \exp(-juz)\, du , \\ E_z(r,z) = j\, \dfrac{a\zeta_0}{k} \displaystyle\int_{-\infty}^{+\infty} \left(u^2-k^2\right) G(u,r)\, F(u) \exp(-juz)\, du , \end{cases} \qquad \text{(IV.6)}$$

and in particular we have the longitudinal component of the electric field that has to satisfies the opportune boundary condition on the metallic surface of the pipe. So if we suppose that *the waveguide is infinite,* we can write a single integral equation by imposing that this component vanishes on the lateral surface (r=a) of the pipe, namely

$$\int_{-\infty}^{+\infty} F(u)\, T(u) \exp(-juz)\, du = -\frac{q\kappa^2}{\pi} K_0(\kappa a) \exp(-jzk/\beta) , \qquad \forall z , \qquad \text{(IV.7)}$$

where the kernel T(u) is given by

$$T(u) = 2a \left(u^2-k^2\right) I_0\left(a\sqrt{u^2-k^2}\right) K_0\left(a\sqrt{u^2-k^2}\right) .$$

Integral equation (IV.7) can be interpreted as a Fourier transform [41] because it is homogeneous on z. Thus we have immediately

$$F(u) = -\frac{q\delta(u-k/\beta)}{2\pi a I_0(\kappa a)}, \qquad \text{(IV.8)}$$

where $\delta(x)$ is the usual Dirac's function. The function F(u) is the key function to compute the whole electromagnetic field; but here we fix our attention only on the current induced on the pipe $I(z)=2\pi aJ(z)$, given by the inverse transform of equation (IV.2)

$$I(z) = 2\pi a \int_{-\infty}^{+\infty} F(u)\exp(-juz)\,du = -q\frac{\exp(-jzk/\beta)}{I_0(\kappa a)}. \qquad \text{(IV.9)}$$

The current I(z) is in fact the Fourier transform of the current i(z,t) flowing along the metallic tube. It is not easy task to come back to the time domain; but this is possible [42-43]. Only for shortness we do not give here some details of this interesting computation.

Dual integral equations system

We have considered an infinite waveguide only to introduce formulas and relations; we are going now to use for the case of *a cylinder of finite length.*

Let us consider a drift tube (Figure 6) and, as usual, the charged particle moves at constant velocity along the symmetry axis of the system. This is an electromagnetic system whose metallic region is finite.

The system of integral equations governing the unknown F(u), space-time transform of the current flowing on the pipe, can be easily written using the previous results, and in particular equation (IV.6) representing the electric field sustained by the induced current and the field in the free space given in Section II. We have the following system of dual integral equations

$$\int_{-\infty}^{+\infty} F(u)\exp(-juz)\,du = 0 \qquad |z|\geq h\,; \qquad \text{(IV.10)}$$

$$\int_{-\infty}^{+\infty} F(u)\,T(u)\exp(-juz)\,du = -\frac{q\kappa^2}{\pi}K_0(\kappa a)\exp(-jkz/\beta) \qquad |z|<h\,. \qquad \text{(IV.11)}$$

Equation (IV.10) states that the current must be zero outside the pipe, whereas equation (IV.11) represents the boundary condition on the tangential component of the electric field. The kernel T(u) is the same as before.

Solution of the system of integral equations

The basic idea to solve this system of dual integral equations is to find an adequate representation of the unknown satisfying automatically one of the two equations and transforming the other one into an expression, easy to manage and/or to treat numerically. What we defined 'a numerical manageable expression' will be a system of algebraic equations.

There are many ways to conceive this transformation; we shall select the shorter one, namely the same we have recently used to solve a classical problem of antennas theory, Hallén's equation [44]. The method we are going to apply is also well described by Eswaran [12].

In order to find a solution of the system of integral equations (IV.10) and (IV.11) we adopt an expansion of the complex unknown in series of analytical functions whose generic term has one or more Bessel functions, or other functions related to these ones. Any series of the type

$$f(u) = \sum_{n=0}^{\infty} a_n \frac{J_{n+\nu}(\sigma u)}{u^{\nu}}$$

is called Neumann series, although in fact Neumann considered only the special type of series for which ν is an integer (σ is a given real constant). The investigation of the more general series is due to Gegenbauer. The possibility of expanding an arbitrary function into a Neumann series is discussed in the Watson's monumental treatise on Bessel functions [38]; the theory is not so important as it appears to be at first sight, because, as the reader will presently realise, it has to deal with functions which must not only behave in a prescribed manner as the variable tends to $\pm\infty$, but must also satisfy an intricate integral equation. Recently Eswaran solved the question of the expansion; he demonstrated that any function whose Fourier transform is of a compact support can be developed in a Neumann series [12].

Equation (IV.10) states that the Fourier transform of the unknown F(u), namely the current density J(z), is a function of a compact support (the function J(z)=0 for |z|>h). Thus F(u) can be expanded in a Neumann series defined as [12, 44]

$$F(u) = -\frac{q\beta^s}{\pi(ka)^{2s}} (\kappa a)^2 K_0(\kappa a) \sum_{n=1}^{\infty} b_n \frac{J_{n-1+s}(uh)}{u^s}, \qquad \text{(IV.12)}$$

where b_n is an expansion coefficient. It is immediate to verify that equation (IV.10) is automatically verified if we use expansion (IV.12) because [14]

$$\int_{-\infty}^{+\infty} \exp(-jux) \frac{J_{r+p}(u)}{u^p} du = 0, \quad |x|>1. \qquad \text{(IV.13)}$$

Equation (IV.11) becomes

$$\frac{\beta^s a^2}{(ka)^{2s}} \sum_{n=1}^{\infty} b_n \int_{-\infty}^{+\infty} T(u) \frac{J_{n-1+s}(uh)}{u^s} \exp(-juz)\, du = \exp(-jkz/\beta), \quad |z|<h. \qquad \text{(IV.14)}$$

The last equation could be already used to compute the expansion coefficients; but an opportune projection will transform it in a system of algebraic equations with a symmetric matrix of the coefficients. In order to realise this, it is necessary to transform the complex exponential exp(-juz) in a Bessel function. Gegenbauer polynomials [14] can help us and, because (m=1,2,3,...)

$$\int_{-h}^{+h} (h^2-z^2)^{s-1/2}\, C^s_{m-1}(z/h)\, \exp(-j\alpha z)\, dz = h^s \frac{\pi 2^{1-s}(-j)^{m-1}\Gamma(2s+m-1)}{(m-1)!\Gamma(s)} \frac{J_{m-1+s}(\alpha h)}{\alpha^s},$$

equation (IV.14) can be formally rewritten as the following algebraic (complex) system of linear equations

$$\sum_{n=1}^{\infty} A_{m,n}\, b_n = J_{m-1+s}(kh/\beta)\,, \qquad \text{(IV.15)}$$

whose coefficients matrix is defined by

$$A_{m,n} = A_{n,m} = \frac{1}{a^{2(s-1)}} \int_{-\infty}^{+\infty} T(u)\, J_{n-1+s}(uh)\, J_{m-1+s}(uh) \frac{du}{u^{2s}}\,.$$

The kernel T(u) is an even function; therefore the expansion coefficients $A_{m,n}$ are non vanishing if n and m are both even or odd. In this situation, it is

$$A_{n,m} = 2a \int_0^{+\infty} T\!\left(\frac{x}{a}\right) J_{n-1+s}\!\left(\frac{xh}{a}\right) J_{m-1+s}\!\left(\frac{xh}{a}\right) \frac{dx}{x^{2s}}\,. \qquad \text{(IV.16)}$$

The numerical evaluation and the strategy of the acceleration the convergence of the matrix coefficients (IV.16) are discussed in [44]. Accordingly the matrix of the coefficients is of the following form

$$A = \begin{bmatrix} A_{1,1} & 0 & A_{1,3} & 0 & A_{1,5} & \cdots \\ 0 & A_{2,2} & 0 & A_{2,4} & 0 & \cdots \\ A_{3,1} & 0 & A_{3,3} & 0 & A_{3,5} & \cdots \\ 0 & A_{4,2} & 0 & A_{4,4} & 0 & \cdots \\ A_{5,1} & 0 & A_{5,2} & 0 & A_{5,5} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

The structure of the this matrix renders simpler the numerical evaluation of the expansion coefficients b_n. The problem, in fact, can be considered as the superposition of two independent problems: the subsystem for the even, and the one for the odd coefficients. In detail it is (r = 1, 2, 3, ...)

$$\sum_{p=1}^{\infty} A_{2r-1,2p-1} b_{2p-1} = J_{2r-2+s}(kh/\beta) \qquad \text{(IV.17)}$$

$$\sum_{p=1}^{\infty} A_{2r,2p} b_{2p} = J_{2r-1+s}(kh/\beta) \qquad \text{(IV.18)}$$

System (IV.15) is now decoupled into two systems (IV.17) and (IV.18) for the evaluation of the odd and even coefficients, respectively.

Longitudinal coupling impedance

We are now ready to compute the longitudinal coupling impedance defined in Section I. From relations (IV.6) we are able to evaluate the electromagnetic field, and in particular the longitudinal component of the electric field evaluated at r=0

$$E_z(r=0,z) = j \frac{a\zeta_0}{k} \int_{-\infty}^{+\infty} (u^2-k^2)\, K_0\!\left(a\sqrt{u^2-k^2}\right) F(u) \exp(-juz)\, du \,. \qquad \text{(IV.19)}$$

Substituting (IV.19) in the definition of the impedance (I.1), we have [14]

$$Z_{||}(k) = -\, 2j\zeta_0 \frac{\pi k a}{q(\beta\gamma)^2} K_0(\kappa a)\, F(k/\beta) \,,$$

or, using the Neumann series (IV.12) defining F(u), it is

$$Z_{||}(k) = 2j\zeta_0 \frac{(ka)^{3-2s}}{\gamma^4 \beta^{2(2-s)}} K_0^2(\kappa a) \sum_{n=1}^{\infty} b_n\, J_{n-1+s}(kh/\beta) \,. \qquad \text{(IV.20)}$$

As an example, a plot of this impedance is given in Figure 7; the normalization factor is the impedance of the free space ζ_0. For numerical calculations the maximum value of s has been chosen (s=2). It is worth noting that the real and the imaginary part of the impedance assume the same value in the asymptotic region (large values of ka). We did not discover any asymptotic formula at the moment, but some approximation proposed for similar cases are under investigation.

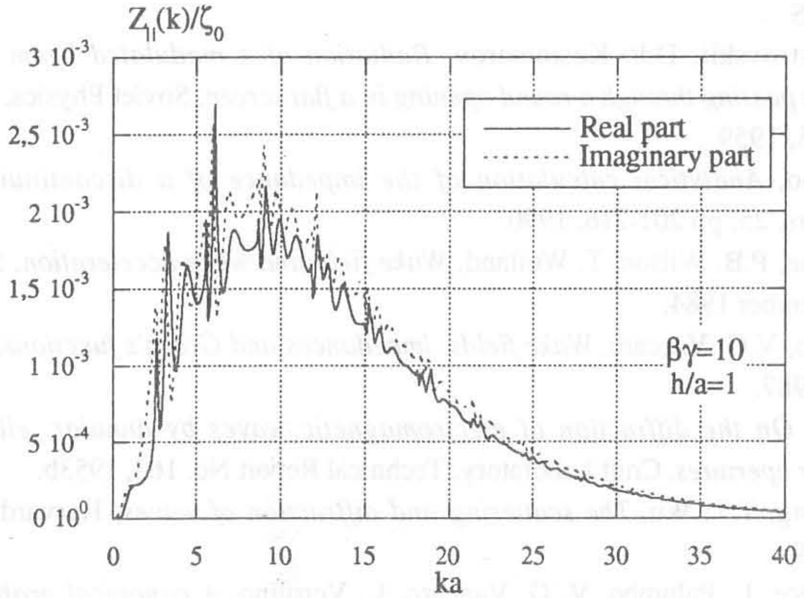

Fig. 7– Normalized values of the longitudinal impedance (IV.20).

V. CONCLUSIONS

We have presented a method for computing the longitudinal coupling impedance of a circular iris in an infinite, perfectly conducting plane, and of a metallic tube of finite length. Both problems have been formulated as systems of dual integral equations.

From a *mathematical* point of view the proposed methods of solution represent the two fundamental approaches to the study of such kinds of coupled system of integral equations: the transformation into a single Fredholm integral equation of the second kind with continuous kernel is the strategy of solution used for the case of the circular iris, whereas the reduction to a linear system of algebraic equations is the one adopted for the case of the drift tube.

From a *numerical* point of view the method of moments can be used to find an approximate solution of the Fredholm integral equation; we have discussed an ingenuous application in order to show the wide range of possibilities and improvements that did not still find an adequate arrangement.

From a *physical* point of view two relevant problems in accelerator physics have been discussed. The problem of the asymptotic behaviour of global parameters (longitudinal coupling impedance), useful in the machine project, is still an open problem and the scientific community of the accelerator is performing a large effort to solve it.

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