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POTENTIALS IN THE FRAMEWORK OF STOCHASTIC QUANTIZATION
APPROACH**

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Abstract. Based on the stochastic generalization of dynamical equations of motion we have shown that the equations of stochastic mechanics with a general but constant diffusion matrix can be transformed into a Schrodinger-like equation describing the motion of a particle in electromagnetic field. Using the gauge invariance of the Schrodinger equation an expression for the gauge potentials has been derived.

Introduction

Recently a thermal wave model for relativistic charged particle beam propagation building on remarkable analogies between particle beam optics and non relativistic quantum mechanics has been proposed [1]. The conjectured in Reference 1 Schrodinger-like equation for the transverse motion of azimuthally symmetric beams has been derived [2] in the framework of stochastic generalization of the Hamilton equations for a single particle, moving in the environment (thermal bath) provided by the rest of the beam under the action of external forces.

The basic idea of the approach presented here is to consider the beam a continuous medium, especially relevant for a collection of particles interacting directly or indirectly through the surroundings. We assume that a test particle constantly undergoes Brownian motion with zero friction, being in dynamical equilibrium between the random force (produced by the rest of the beam and the environment) and the external force providing the beam acceleration and focusing. The latter is included in the physical picture by means of the stochastic generalization of Newton's law, where the mean acceleration of the particle replaces its classical acceleration.

The beam consists of large number of particles, thus representing a complex physical system. Here we assume (without proof) that the full set of dynamic equations for all the degrees of freedom can be replaced with a much smaller set of stochastic differential equations, describing the motion of one representative of our original system. The latter corresponds to the physical observables of the beam, while the projected degrees of freedom constitute the environment whose influence is to generate random fluctuations. It is worth while to note that random fluctuations could be introduced into the system by the surroundings as well (RF noise, random imperfections in the magnetic elements etc.), which combined with the eliminated degrees of freedom comprise the complex environment of the test particle.

Following Nelson [3, 4] we represent the motion of an individual particle in terms of a stochastic process. Further two mean (forward and backward) velocities and four mean (forward-forward, forward-backward, backward-forward and backward-backward) accelerations of the stochastic motion can be defined [2]. The kinematical aspect (current velocity) of the stochastic process is expressed by the symmetric combination of the two mean velocities, while the diffusion aspect (osmotic velocity) is given by their antisymmetric combination. More difficult task however, is the choice among various combinations of the mean accelerations, leading to the model of Brownian motion. Two of the stochastic generalizations of acceleration deserve to be distinguished (for other possible choices see Ref. 5): the first one defined as the symmetric combination of the forward-backward and backward-forward mean accelerations is regarded by Nelson as relevant, and the second one being the fully symmetric stochastic generalization of acceleration [2]. The second definition of mean acceleration leads to the stochastic generalization of Newton's law as Euler's equation, representing the quasi-stochastic model of a moving classical particle. If the stochastic process is a diffusion process dynamics of the stochastic motion is governed by Nelson's stochastic generalization of Newton's equation of motion

and the forward and backward Fokker-Planck equations. The latter two in particular when combined yield the continuity and osmotic equation, thus completing the system of differential equations of stochastic mechanics. Here we assume that the beam evolution is governed by the equations of stochastic mechanics.

It is well known that equations of stochastic mechanics when the diffusion process is described by a diffusion coefficient that is the ratio of Planck's constant and the particle mass are (*formally*) equivalent to Schrodinger equation [3,4,9]. Objections against this one to one correspondence have been raised in [6], where it is pointed out that “not every nice solution of the Schrodinger equation yields a diffusion”. This seems to be critical for the dynamical principles of Nelson's stochastic mechanics (see also Ref. 7). However, in a subsequent work [8] it is demonstrated “how to recover a complete correspondence between quantum dynamics and the dynamical principles of stochastic mechanics”, showing that previous criticism is unfounded.

In the present paper we study the case when the diffusion process is characterized by an arbitrary (symmetric) diffusion matrix independent of coordinates and time. This model is suitable to cover the problem of propagation of asymmetric beams in accelerators and is of considerable interest by itself as well.

1. The Rescaled Coordinate System

Let us consider the equations of stochastic mechanics [2-4, 9] with an arbitrary but constant in the coordinates $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ and time t diffusion matrix $\hat{\varepsilon}$ with components $\hat{\varepsilon}_{kl}$:

$$\frac{\partial \tilde{\rho}}{\partial t} + \tilde{\nabla} \cdot (\tilde{\mathbf{P}}\tilde{\rho}) = 0, \quad (1.1a)$$

$$\tilde{\rho}\tilde{\mathbf{U}}_k = -\frac{\hat{\varepsilon}_{kl}}{2}\tilde{\nabla}_l\tilde{\rho}, \quad (1.1b)$$

$$\frac{\partial \tilde{\mathbf{P}}}{\partial t} + (\tilde{\mathbf{P}} \cdot \tilde{\nabla})\tilde{\mathbf{P}} = -\tilde{\nabla}\mathcal{U} + (\tilde{\mathbf{U}} \cdot \tilde{\nabla})\tilde{\mathbf{U}} - \frac{\hat{\varepsilon}_{kl}}{2}\tilde{\nabla}_k\tilde{\nabla}_l\tilde{\mathbf{U}}, \quad (1.1c)$$

where $\tilde{\rho}(\tilde{\mathbf{x}};t)$ is the probability density of random coordinates, $\tilde{\mathbf{P}}(\tilde{\mathbf{x}};t)$ is the current velocity, $\tilde{\mathbf{U}}(\tilde{\mathbf{x}};t)$ is the osmotic velocity and $\mathcal{U}(\tilde{\mathbf{x}};t)$ is the mechanical potential (divided by the particle mass). In equations (1.1a-c) the symbol $\tilde{\nabla}$ denotes the well-known gradient operator acting with respect to $\tilde{\mathbf{x}}$ and summation over repeated indices is implied as well. The external force $\tilde{\mathcal{F}}(\tilde{\mathbf{x}};t) = -\tilde{\nabla}\mathcal{U}(\tilde{\mathbf{x}};t)$ is taken to be potential (non potential forces can be included in the formalism with slight modification) and has nothing to do with the interaction between the moving particle and the environment.

Our goal now is to cast the system (1.1) if possible into the form suggested by Nelson's mechanics with a constant and isotropic diffusion [see equations (1.7) below]. For that purpose we perform a coordinate transformation

$$\mathbf{x} = \hat{\mathcal{M}}\tilde{\mathbf{x}} \quad ; \quad x_n = \hat{\mathcal{M}}_{nk}\tilde{x}_k \quad (1.2)$$

such that the transformed diffusion matrix [10]

$$\hat{\varepsilon}'_{kl} = (\hat{\mathcal{M}}\hat{\varepsilon}\hat{\mathcal{M}}^T)_{kl} = \hat{\mathcal{M}}_{km}\hat{\mathcal{M}}_{ln}\hat{\varepsilon}_{mn} \quad (1.3a)$$

is proportional to the unit matrix by a diffusion scaling parameter ε , namely:

$$\hat{\varepsilon}' = \varepsilon\hat{\mathbf{I}} \quad ; \quad \hat{\varepsilon}'_{kl} = \varepsilon\delta_{kl}, \quad (1.3b)$$

where $\hat{\mathbf{I}}$ is the unit matrix and $(\dots)^T$ means matrix transposition. In fact, this is possible, provided the diffusion matrix $\hat{\varepsilon}$ is symmetric. Clearly the matrix $\hat{\mathcal{M}}$ has the following structure

$$\hat{\mathcal{M}} = \hat{\mathcal{A}}\hat{\mathcal{O}}, \quad (1.4)$$

where $\hat{\mathcal{O}}$ is an orthogonal matrix and $\hat{\mathcal{A}}$ is the diagonal matrix

$$\hat{\mathcal{A}}_{kl} = \sqrt{\frac{\varepsilon}{\varepsilon_k}}\delta_{kl}, \quad (1.5)$$

defining the equilateral diffusion scaling with ε_k ($k=1,2,3$) being the eigenvalues of the diffusion matrix $\hat{\varepsilon}$. Furthermore the transformed current and osmotic velocities are [10]

$$\mathbf{P} = \hat{\mathcal{M}}\tilde{\mathbf{P}} \quad ; \quad \mathbf{P}_n = \hat{\mathcal{M}}_{nk}\tilde{\mathbf{P}}_k \quad ; \quad \mathbf{U} = \hat{\mathcal{M}}\tilde{\mathbf{U}} \quad ; \quad \mathbf{U}_n = \hat{\mathcal{M}}_{nk}\tilde{\mathbf{U}}_k, \quad (1.6a)$$

while the probability density of new random coordinates is

$$p(\mathbf{x};t) = |\det(\hat{\mathcal{M}})|^{-1} \tilde{p}(\tilde{\mathbf{x}};t). \quad (1.6b)$$

Therefore the transformed equations of stochastic mechanics read as

$$\frac{\partial p}{\partial t} + \nabla \cdot (\mathbf{P}p) = 0, \quad (1.7a)$$

$$p\mathbf{U} = -\frac{\varepsilon}{2}\nabla p, \quad (1.7b)$$

$$\frac{\partial \mathbf{P}}{\partial t} + (\mathbf{P} \cdot \nabla)\mathbf{P} = -\varepsilon\nabla_\varepsilon \mathcal{U} + (\mathbf{U} \cdot \nabla)\mathbf{U} - \frac{\varepsilon}{2}\nabla^2\mathbf{U}, \quad (1.7c)$$

where

$$(\nabla_\varepsilon)_n = \frac{\nabla_n}{\varepsilon_n}. \quad (1.8)$$

We here note that the transformed force $\mathcal{F}(\mathbf{x};t) = -\varepsilon\nabla_\varepsilon \mathcal{U}(\mathbf{x};t)$ is non potential in general. One possible physical field producing the same effect on particle's motion as the transformed force could be a suitably introduced "electromagnetic-like" field. The latter bounds up the interaction between the particle and the environment with the external force through the dynamical characteristics of the moving object.

2. Derivation of the Schrodinger-like Equation

We are looking now for a Schrodinger-like equation of the type

$$i\varepsilon \frac{\partial \psi}{\partial t} = \hat{\mathcal{H}}\psi \quad (2.1)$$

equivalent to the system (1.7) through the well-known quantum mechanical ansatz

$$\psi(\mathbf{x};t) = \sqrt{p(\mathbf{x};t)} \exp\left[\frac{i}{\varepsilon} S(\mathbf{x};t)\right], \quad (2.2)$$

where $\hat{\mathcal{H}}$ is a second order differential operator with known (constant) coefficients in front of the second order derivatives. The basic requirement the operator $\hat{\mathcal{H}}$ should satisfy is to be Hermitian

$$\int d^3\mathbf{x} \psi_1^* \hat{\mathcal{H}} \psi_2 = \int d^3\mathbf{x} \psi_2 \hat{\mathcal{H}}^* \psi_1^* \quad (2.3)$$

which defines it (as can be easily shown) up to a scalar and a vector function. Without loss of generality one can write

$$\hat{\mathcal{H}} = \frac{1}{2} [i\varepsilon \nabla + \mathbf{A}(\mathbf{x}; t)]^2 + \Phi(\mathbf{x}; t), \quad (2.4)$$

where the vector potential $\mathbf{A}(\mathbf{x}; t)$ and the scalar potential $\Phi(\mathbf{x}; t)$ describing the electromagnetic field mentioned above are some yet unknown functions of the coordinates and time. Substitution of the ansatz (2.2) into equation (2.1) followed by separation of terms by real and imaginary part gives

$$\frac{\partial \mathcal{P}}{\partial t} = -p \nabla \cdot (\nabla S - \mathbf{A}) - (\nabla S - \mathbf{A}) \cdot \nabla p, \quad (2.5a)$$

$$\frac{\partial \mathcal{S}}{\partial t} = \frac{\varepsilon^2}{4} \frac{\nabla^2 p}{p} - \frac{\varepsilon^2}{8} \frac{(\nabla p)^2}{p^2} - \frac{1}{2} (\nabla S - \mathbf{A})^2 - \Phi. \quad (2.5b)$$

From the continuity equation (1.7a) and equation (2.5a) it follows:

$$\mathbf{P} = \nabla S - \mathbf{A}. \quad (2.6)$$

Using the above definition of the current velocity vector and the osmotic equation (1.7b) it is straightforward to cast the equation (2.5b) into the form

$$\frac{\partial \mathcal{S}}{\partial t} = -\frac{\varepsilon}{2} \nabla \cdot \mathbf{U} + \frac{\mathbf{U}^2 - \mathbf{P}^2}{2} - \Phi. \quad (2.7)$$

Taking into account the fact that $\nabla \times \mathbf{U} = 0$ [following from the definition of the osmotic velocity by (1.7b)] and the well-known vector relations

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}, \quad (2.8a)$$

$$\mathbf{F} \times (\nabla \times \mathbf{F}) = \frac{1}{2} \nabla F^2 - (\mathbf{F} \cdot \nabla) \mathbf{F}, \quad (2.8b)$$

holding for arbitrary vector function $\mathbf{F}(\mathbf{x})$, we finally arrive at

$$\frac{\partial \mathcal{P}}{\partial t} + (\mathbf{P} \cdot \nabla) \mathcal{P} = \mathbf{E} + \mathbf{P} \times \mathbf{B} + (\mathbf{U} \cdot \nabla) \mathcal{U} - \frac{\varepsilon}{2} \nabla^2 \mathcal{U}. \quad (2.9)$$

The electric field $\mathbf{E}(\mathbf{x}; t)$ and the magnetic field $\mathbf{B}(\mathbf{x}; t)$ are given by [11]

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} \quad ; \quad \mathbf{B} = \nabla \times \mathbf{A} \quad (2.10)$$

and satisfy the Maxwell equations

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = p \mathbf{P}, \quad (2.11a)$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (2.11b)$$

$$\nabla \cdot \mathbf{E} = p, \quad (2.11c)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (2.11d)$$

Note that the effective charge distribution takes after the probability density $p(\mathbf{x}; t)$, provided the continuity equation must hold. Comparing equation (2.9) with equation (1.7c) we conclude that the transformed external force equals the force produced by the electromagnetic field per unit charge, namely:

$$-\varepsilon \nabla \cdot \mathcal{U} = p(\mathbf{E} + \mathbf{P} \times \mathbf{B}). \quad (2.12)$$

At this stage we would like to point out that if the diffusion process is isotropic, that is $\varepsilon_k = \varepsilon = \hbar / m$ ($k = 1, 2, 3$) where \hbar and m being the Planck's constant and the particle mass respectively, the magnetic field vanishes and the electric potential is equal to the mechanical potential. Moreover one obtains the Schrodinger equation

$$i\hbar \frac{\partial \psi(\mathbf{x}; t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + m\mathcal{U}(\mathbf{x}; t) \right] \psi(\mathbf{x}; t). \quad (2.13)$$

3. Gauge Electromagnetic Potentials

In order to find the electromagnetic potentials $\mathbf{A}(\mathbf{x}; t)$ and $\Phi(\mathbf{x}; t)$ we note that the Schrodinger equation (2.1) is gauge invariant under the *local* phase transformation

$$\psi_1(\mathbf{x}; t) = \psi(\mathbf{x}; t) \exp\left[\frac{i}{\varepsilon} S_1(\mathbf{x}; t) \right]. \quad (3.1)$$

This means that the equation for the new wave function $\psi_1(\mathbf{x}; t)$ has the same structure as (2.1) with

$$\mathbf{A}(\mathbf{x}; t) \longrightarrow \mathbf{A}'(\mathbf{x}; t) = \mathbf{A}(\mathbf{x}; t) + \nabla S_1(\mathbf{x}; t), \quad (3.2a)$$

$$\Phi(\mathbf{x}; t) \longrightarrow \Phi'(\mathbf{x}; t) = \Phi(\mathbf{x}; t) - \frac{\partial S_1(\mathbf{x}; t)}{\partial t}. \quad (3.2b)$$

Moreover equation (2.12) written in the form

$$-\varepsilon \nabla_{\varepsilon} \mathcal{U} + \nabla \Phi = -\frac{\partial \mathbf{A}}{\partial t} + (\nabla S - \mathbf{A}) \times (\nabla \times \mathbf{A}) \quad (3.3)$$

is gauge invariant under (3.1) with

$$S(\mathbf{x}; t) \longrightarrow S'(\mathbf{x}; t) = S(\mathbf{x}; t) + S_1(\mathbf{x}; t). \quad (3.2c)$$

Choosing $S_1(\mathbf{x}; t) = -S(\mathbf{x}; t)$ we obtain the Euler equation

$$-\varepsilon \nabla_{\varepsilon} \mathcal{U} + \nabla \Phi' = -\frac{\partial \mathbf{A}'}{\partial t} - \mathbf{A}' \times (\nabla \times \mathbf{A}') \quad (3.4)$$

for the gauge electromagnetic potentials $-\mathbf{A}'(\mathbf{x}; t)$ and $\Phi'(\mathbf{x}; t)$.

The vector $\mathbf{A}'(\mathbf{x}; t)$ describes the macroscopic behavior of the system considered a fluid, as though its structure is continuous and all physical quantities are continuously distributed in the volume, where the motion is confined. It determines the velocity vector (actually in the opposite direction) of the fluid and represents the basic parameter of the flow. The scalar potential $\Phi'(\mathbf{x}; t)$ plays the role of static pressure in Euler's representation of the local velocity aggregate, while

$$\hat{\sigma}_{kl} = \left(\frac{\varepsilon}{\varepsilon_k} \mathcal{U} - \Phi' \right) \delta_{kl} \quad (3.5)$$

is the hydrodynamic-stress tensor [12].

Let us now specify the mechanical potential $\mathcal{U}(\tilde{\mathbf{x}}; t)$ assuming that

$$\mathcal{U}(\tilde{\mathbf{x}}; t) = \mathcal{U}_0(\tilde{\mathbf{x}}; t) + \mathcal{V}(\tilde{\mathbf{x}}; t), \quad (3.6)$$

where $\mathcal{U}_0(\tilde{\mathbf{x}}; t)$ governs the linear motion and is represented by

$$\mathcal{U}_0(\tilde{\mathbf{x}}; t) = \frac{1}{2} \tilde{\mathbf{x}}^T \hat{\mathbf{G}}(t) \tilde{\mathbf{x}}, \quad (3.7)$$

while $\mathcal{V}(\tilde{\mathbf{x}}; t)$ is a sum of all nonlinear terms. The matrix $\hat{\mathbf{G}}(t)$ entering the right hand side of (3.7) is considered to be symmetric. Further we split the electric potential $\Phi'(\mathbf{x}; t)$ into two parts according to the relation

$$\Phi'(\mathbf{x}; t) = \Phi_0(\mathbf{x}; t) + \Phi_1'(\mathbf{x}; t), \quad (3.8)$$

where

$$\Phi_0(\mathbf{x}; t) = \frac{1}{2} \mathbf{x}^T \hat{\mathbf{G}}'(t) \mathbf{x} \quad ; \quad \hat{\mathbf{G}}'(t) = \hat{\mathcal{M}} \hat{\mathbf{G}}(t) \hat{\mathcal{M}}^{-1}, \quad (3.9)$$

$$\Phi_1'(\mathbf{x}; t) = -\frac{1}{2} \mathbf{A}'^2(\mathbf{x}; t). \quad (3.10)$$

Equation (3.4) takes now the form

$$-\varepsilon \nabla_\varepsilon \psi = -\frac{\partial \mathbf{A}'}{\partial t} + (\mathbf{A}' \cdot \nabla) \mathbf{A}', \quad (3.11)$$

which transformed back to the original coordinates reads as

$$-\tilde{\nabla} \psi = -\frac{\partial \tilde{\mathbf{A}}'}{\partial t} + (\tilde{\mathbf{A}}' \cdot \tilde{\nabla}) \tilde{\mathbf{A}}'. \quad (3.12)$$

If the velocity field $\tilde{\mathbf{A}}'(\tilde{\mathbf{x}}; t)$ is non vortex one can introduce the velocity potential [12, 13]

$$\tilde{\mathbf{A}}'(\tilde{\mathbf{x}}; t) = -\tilde{\nabla} \varphi(\tilde{\mathbf{x}}; t) \quad (3.13)$$

and obtain the first integral of motion

$$\frac{\partial \varphi(\tilde{\mathbf{x}}; t)}{\partial t} + \frac{1}{2} \tilde{\mathbf{A}}'^2(\tilde{\mathbf{x}}; t) + \mathcal{V}(\tilde{\mathbf{x}}; t) = f(t). \quad (3.14)$$

Without loss of generality the function $f(t)$ in equation (3.14) may be set equal to zero as a result of the uncertainty in the definition of the velocity potential (3.13). As a matter of fact, equation (3.14) is the Hamilton-Jacobi equation for the velocity potential, defining the velocity field $\tilde{\mathbf{A}}'(\tilde{\mathbf{x}}; t)$. This is a direct corollary of Bohr's correspondence principle and is a well-known result in quantum mechanics.

Performing the phase transformation (3.1) once again according to

$$\psi_2(\mathbf{x}; t) = \psi_1(\mathbf{x}; t) \exp\left[\frac{i}{\varepsilon} \varphi(\hat{\mathcal{M}}^{-1} \mathbf{x}; t)\right] \quad (3.15)$$

we obtain the gauge potentials $\mathbf{A}_2(\mathbf{x}; t)$ and $\Phi_2(\mathbf{x}; t)$ entering the Schrodinger equation for the wave function $\psi_2(\mathbf{x}; t)$. They are

$$\mathbf{A}_2(\mathbf{x}; t) = (\hat{\mathbf{I}} - \hat{\mathcal{A}}^2) \nabla \varphi(\hat{\mathcal{M}}^{-1} \mathbf{x}; t), \quad (3.16a)$$

$$\Phi_2(\mathbf{x}; t) = \Phi_0(\mathbf{x}; t) + \mathcal{V}(\hat{\mathcal{M}}^{-1} \mathbf{x}; t) + \frac{1}{2} [\nabla \varphi(\hat{\mathcal{M}}^{-1} \mathbf{x}; t)]^T \hat{\mathcal{A}}^2 (\hat{\mathbf{I}} - \hat{\mathcal{A}}^2) \nabla \varphi(\hat{\mathcal{M}}^{-1} \mathbf{x}; t). \quad (3.16b)$$

The Schrodinger equation we are looking for, called upon to replace equation (2.1) reads as

$$i\varepsilon \frac{\partial \psi_2(\mathbf{x}; t)}{\partial t} = \frac{1}{2} [i\varepsilon \nabla + \mathbf{A}_2(\mathbf{x}; t)]^2 \psi_2(\mathbf{x}; t) + \Phi_2(\mathbf{x}; t) \psi_2(\mathbf{x}; t). \quad (3.17)$$

Retracing the sequence of phase transformations (3.1) and (3.15) we find

$$\psi(\mathbf{x}; t) = \psi_2(\mathbf{x}; t) \exp\left\{\frac{i}{\varepsilon} [S(\mathbf{x}; t) - \varphi(\hat{\mathcal{M}}^{-1} \mathbf{x}; t)]\right\}, \quad (3.18a)$$

$$\mathbf{A}_2(\mathbf{x}; t) = \mathbf{A}(\mathbf{x}; t) + \nabla [\varphi(\hat{\mathcal{M}}^{-1} \mathbf{x}; t) - S(\mathbf{x}; t)], \quad (3.18b)$$

$$\Phi_2(\mathbf{x}; t) = \Phi(\mathbf{x}; t) - \frac{\partial}{\partial t} [\varphi(\hat{\mathcal{M}}^{-1} \mathbf{x}; t) - S(\mathbf{x}; t)]. \quad (3.18c)$$

The relations (3.18a-c) indicate the equivalence of equations (2.1) and (3.17) up to a *global* phase transformation defined by the constant in the coordinates and time phase

$$S(\mathbf{x}; t) - \varphi(\hat{\mathcal{M}}^{-1}\mathbf{x}; t) = \mathcal{C} = \text{const}. \quad (3.19)$$

At the end of this Section it is worth to note that the anisotropy of the diffusion matrix does not change really the structure of the equations of stochastic mechanics (1.7a-c) and therefore the structure of the wave equation (2.1). Its effect however, is embedded in the electromagnetic potentials. From the expressions (3.16) it follows that the Schrodinger equation (2.13) is a particular case of equation (3.17) when the diffusion process characterized by the diffusion matrix $\hat{\varepsilon}$ is isotropic. In this case one simply has $\hat{\mathcal{A}} = \hat{\mathbf{I}}$ and as a consequence the vector potential vanishes, while the scalar potential is equal to the mechanical potential defining the external force.

4. The 2D Harmonic Oscillator

In order to demonstrate how the formalism developed in the preceding sections works let us consider the simple but very useful example taken from accelerator physics. We shall analyze the linear betatron motion of particles in an accelerator governed by the two dimensional harmonic oscillator potential

$$\mathcal{U}_0(\tilde{x}_1, \tilde{x}_2; \theta) = \frac{1}{2} [G_1(\theta)\tilde{x}_1^2 + G_2(\theta)\tilde{x}_2^2]. \quad (4.1)$$

The azimuth θ widely used as an independent variable in accelerator theory stands for the time variable here. The two functions $G_{1,2}(\theta)$ are periodic functions of the azimuth and characterize the linear stability properties of the system. The diffusion matrix $\hat{\varepsilon}$ is taken to be diagonal

$$\hat{\varepsilon}_{kl} = R\varepsilon_k\delta_{kl} \quad (4.2)$$

and is proportional to the average beam emittance tensor $\varepsilon_k\delta_{kl}$ [2] by a factor R that is the mean machine radius. In fact, the average emittance is the area in phase space occupied by the beam averaged over the ensemble realization of the stochastic process. The matrix $\hat{\delta}$ is trivially the unit matrix, while as a scaling parameter ε one can choose the product of one of the emittances (for example ε_1) and the mean radius. Clearly, the vector potential $\mathbf{A}_2(\mathbf{x}; \theta)$ vanishes (as a result of $\mathcal{V} = 0$) and since $\hat{\mathcal{G}}'(\theta) = \hat{\mathcal{G}}(\theta)$ the scalar potential $\Phi_2(\mathbf{x}; \theta)$ is given by

$$\Phi_2(\mathbf{x}; \theta) = \Phi_0(\mathbf{x}; \theta) = \frac{1}{2} [G_1(\theta)x_1^2 + G_2(\theta)x_2^2]. \quad (4.3)$$

Therefore equation (3.17) reads as

$$i\varepsilon \frac{\partial \psi(\mathbf{x}; \theta)}{\partial \theta} = \left[-\frac{\varepsilon^2}{2} \nabla^2 + \Phi_0(\mathbf{x}; \theta) \right] \psi(\mathbf{x}; \theta). \quad (4.4)$$

We are seeking the solution to equation (4.4) in the form $\psi(\mathbf{x}; \theta) = \xi(\mathbf{x}; \theta) \exp\left[\frac{i}{\varepsilon} \mathcal{S}(\mathbf{x}; \theta)\right]$. Separation of variables by real and imaginary part gives

$$\frac{\partial \xi}{\partial \theta} + \nabla \mathcal{S} \cdot \nabla \xi = -\frac{\xi}{2} \nabla^2 \mathcal{S}, \quad (4.5a)$$

$$\frac{\partial \mathcal{S}}{\partial \theta} + \frac{1}{2} (\nabla \mathcal{S})^2 + \Phi_0 = \frac{\varepsilon^2}{2} \frac{\nabla^2 \xi}{\xi}. \quad (4.5b)$$

The orthogonality condition for the eigenfunctions of equation (4.4) leads to the appearance of a discrete spectrum in its solution. Moreover the eikonal function $\mathcal{S}(\mathbf{x}; \theta)$ for the harmonic oscillator is quadratic in the coordinates

$$\mathcal{S}_{n_1 n_2}(\mathbf{x}; \theta) = \frac{1}{2} \left[\frac{x_1^2}{\rho_1(\theta)} + \frac{x_2^2}{\rho_2(\theta)} \right] + \varphi_{n_1 n_2}(\theta), \quad (4.6)$$

where

$$\frac{1}{\rho_k} = \frac{1}{\mathcal{R}_k} \frac{d\mathcal{R}_k}{d\theta} \quad (k=1,2) \quad ; \quad \frac{d\varphi_{n_1 n_2}}{d\theta} = -\left(n_1 + \frac{1}{2}\right) \frac{\varepsilon^2}{\mathcal{R}_1^2} - \left(n_2 + \frac{1}{2}\right) \frac{\varepsilon^2}{\mathcal{R}_2^2}. \quad (4.7)$$

Here \mathcal{R}_k are the beam envelope functions satisfying the well-known differential equations

$$\frac{d^2 \mathcal{R}_k}{d\theta^2} + G_k(\theta) \mathcal{R}_k = \frac{\varepsilon^2}{\mathcal{R}_k^3}. \quad (4.8)$$

From equation (4.5a) with (4.6) and (4.7) in hand one readily obtains:

$$\xi_{n_1 n_2}(\mathbf{x}; \theta) = \frac{g_{n_1 n_2}(z_1, z_2)}{\sqrt{\mathcal{R}_1 \mathcal{R}_2}} \quad ; \quad z_k = \frac{x_k}{\mathcal{R}_k},$$

where the function $g_{n_1 n_2}(z_1, z_2)$ satisfies the equation

$$\left(\frac{1}{\mathcal{R}_1^2} \frac{\partial^2}{\partial z_1^2} + \frac{1}{\mathcal{R}_2^2} \frac{\partial^2}{\partial z_2^2} \right) g_{n_1 n_2} = \left[\frac{1}{\mathcal{R}_1^2} (z_1^2 - 2n_1 - 1) + \frac{1}{\mathcal{R}_2^2} (z_2^2 - 2n_2 - 1) \right] g_{n_1 n_2}. \quad (4.9)$$

The solution of the latter is represented in terms of Hermite polynomials as follows

$$g_{n_1 n_2}(\mathbf{x}; \theta) = g_{n_1}^{(1)}(x_1; \theta) g_{n_2}^{(2)}(x_2; \theta) \quad ; \quad g_{n_k}^{(k)}(x_k; \theta) = \frac{\exp\left(-\frac{x_k^2}{2\mathcal{R}_k^2}\right)}{\sqrt{2^{n_k} n_k! \sqrt{\pi}}} \mathbf{H}_{n_k}\left(\frac{x_k}{\mathcal{R}_k}\right), \quad (4.10)$$

so that for the \mathbf{n} -th mode solution $[\mathbf{n} = (n_1, n_2)]$ of the wave equation (4.4) we obtain

$$\psi_{n_1 n_2}(\mathbf{x}; \theta) = \frac{\exp\left(-\frac{x_1^2}{2\mathcal{R}_1^2} - \frac{x_2^2}{2\mathcal{R}_2^2}\right)}{\sqrt{\pi \mathcal{R}_1 \mathcal{R}_2} \sqrt{2^{n_1+n_2} n_1! n_2!}} \mathbf{H}_{n_1}\left(\frac{x_1}{\mathcal{R}_1}\right) \mathbf{H}_{n_2}\left(\frac{x_2}{\mathcal{R}_2}\right) \exp\left[\frac{i}{\varepsilon} \mathcal{S}_{n_1 n_2}(\mathbf{x}; \theta)\right]. \quad (4.11)$$

Unfortunately there are relatively few systems of interest in quantum mechanics and accelerator theory for which the wave equation (3.17) can be solved exactly. Approximation methods are therefore expected to play an outstanding role in virtually all important applications of the Schrodinger equation with electro-magnetic potentials in the case of anisotropic diffusion. This is for example the case when anharmonic terms of higher order (sextupoles, octupoles, etc.) represent a relevant contribution and should be included in the real physical picture. The problems for which exact solutions can be found are of great importance, since they often provide a good starting point for approximate calculations.

Concluding remarks

In the present paper we have confined ourselves to the case when the stochastic process describing the motion of a charged particle in an accelerator is a diffusion process with a general, independent of coordinates and time diffusion matrix. As a result of the investigation performed we have shown that the continuity equation (1.1a), the osmotic equation (1.1b) and Nelson's generalization of Newton's equation of motion (1.1c) can be transformed by a change of coordinates and dependent variables into a Schrodinger-like equation (2.1). Regardless of the type of the external forces one need to introduce a gauge electromagnetic field that represents something of a relation between the local characteristics (trajectory) of the moving test particle and the way it interacts with the environment. If the environment is an isotropic medium (holding in the case of symmetric beams) the gauge vector potential vanishes and as a consequence the scalar potential is equal to the potential that accounts for the external force.

The gauge transformation (3.2) is the well-known transformation in classical electromagnetic theory [11] introduced by Weyl, indicating transition to alternative electromagnetic potentials, which sometimes are easier to find compared to the original ones. Besides that, the transformed potentials define the same electro-magnetic field tensor. Taking into account this fact we have found the gauge electromagnetic potentials explicitly, depending on the solution of Hamilton-Jacobi equation for the classical motion of the particle in the external potential.

We presume that the procedure outlined in the present paper could be generalized to cover the case when the diffusion matrix is a function of coordinates and time and the case of non zero friction as well. The latter is very important for lepton machines where the effect of synchrotron radiation on particle dynamics is considerable.

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