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THE SPIN-SPLITTER CONCEPT

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SUMMARY

We consider a method, based on the coherent repetition of Stern-Gerlach kicks, which aims at separating the opposite spin states of the (anti)protons revolving in a storage ring. Drawbacks associate with damping and technical accuracies are analysed together with possible cures. Quantum-mechanical limitations are discussed and proven to be harmless.

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1. Introduction

The polarization mechanism presented here is based on the Stern-Gerlach effect: in an inhomogeneous magnetic field particles with spin aligned parallel to the field are deflected in opposite direction. The spin-orbit coupling term of the Hamiltonian is:

$$H = -\mu \vec{S} \cdot \vec{B} \quad (1.1)$$

with

$$\mu = g \frac{e\hbar}{4m} \quad (1.2)$$

which in the (anti)protons case yields $\mu = 1.41 \times 10^{-26} \text{ JT}^{-1}$. Crossing a quadrupole lens of field $\vec{B} = G(z\hat{x} + x\hat{z})$ and length l_Q , the particle experiences a variation of its transverse momentum:

$$\delta p_x = \mu S_z \frac{Gl_Q}{\beta c} \quad (1.3)$$

where the transverse location of the particle during the transit in the magnet have been considered constant (kick approximation). The angular divergence of that particle is therefore increased by the amount

$$\eta = \frac{\delta p_x}{p} = \frac{\mu S_z Gl_Q}{\beta^2 \gamma m c^2} \quad (1.4)$$

In the original experiment a neutral beam, at thermal energy, crossed a magnetic field with a transverse gradient and the effect was detected in a single pass. For a beam of antiprotons or protons the energy cannot be reasonably lower than 100 or 50 MeV. Therefore it is necessary to implement a constructive mechanism that allows the kicks to add up turn after turn for particles accumulated in a storage ring. In order to explain this process some notions of spin dynamics in an accumulation ring are required; for sake of completeness we shall recall them in the next section.

2. Spin Dynamics

The time evolution of the spin vector \vec{S} along a ring is described by the B.M.T. equation:

$$\frac{d\vec{S}}{dt} = \frac{e}{m\gamma} \vec{S} \times [\vec{B} + a(\vec{B}_{\parallel} + \gamma \vec{B}_{\perp})] \quad (2.1)$$

with $a=1.793$ proton anomaly and with the usual meaning for the other symbols. Here \vec{S} is in the particle frame and \vec{B} is in the laboratory reference system. This equation has the same form as the cyclotron precession equation

$$\frac{d\vec{v}}{dt} = \frac{e}{m\gamma} \vec{v} \times \vec{B} = \vec{\omega} \times \vec{v} \quad (2.2)$$

where \vec{v} is the particle velocity and $\vec{\omega} = -\frac{e\vec{B}}{\gamma m}$ is the instantaneous cyclotron frequency.

The reference orbit solves equation (2.2) and defines a curvilinear coordinates system $(\hat{x}, \hat{y}, \hat{z})$, where \hat{y} is always oriented as the reference particle velocity and \hat{z} is vertical at $s = 0$, being s the length of the arc.

If there is no coupling between the motions in the two planes, we have that $\hat{z}(s+L) = \hat{z}(s)$, with L the length of the ring. We may now express equation (2.1) in the coordinate system $(\hat{x}, \hat{y}, \hat{z})$; we have:

$$\frac{d\vec{S}}{dt} = \frac{e}{m\gamma} (a\gamma)\vec{S} \times \vec{B}_{\perp} = a\gamma\vec{\omega}_{\perp} \times \vec{S} \quad (2.3)$$

$$\frac{d\vec{S}}{dt} = \frac{e}{m\gamma} (1+a)\vec{S} \times \vec{B}_{\parallel} \quad (2.4)$$

Both equations preserve the modulus of \vec{S} ; therefore the spin transfer matrix for one revolution, solution of eqs. (2.3) and (2.4) between s and $s+L$, is a rotation:

$$\vec{S}(s+L) = R_0 \vec{S}(s) \quad (2.5)$$

$$R_0 R_0^T = R_0^T R_0 = I$$

being I the unity matrix.

The spectrum of eigenvalues of a real tridimensional rotation is given by 1 and two complex conjugate values. Therefore the spectrum of R_0 can be written as:

$$(1, e^{2\pi i\nu_s}, e^{-2\pi i\nu_s}) \quad (2.6)$$

where the quantity ν_s , number of spin precessions in one revolution, is called spin tune. If ν_s is integer, $R_0 = I$ and we are in a degenerated case. Otherwise the eigendirection $\vec{n}(s)$ associated to the eigenvalue 1 is uniquely defined:

$$R_0(s)\vec{n}(s) = \vec{n}(s) \quad (2.7)$$

The direction $\vec{n}(s)$ is called the equilibrium direction of polarization and has some interesting features. If we inject polarized particles with their polarization parallel to \vec{n} , such polarization keeps its direction turn after turn (and for example it can easily be measured by a polarimeter).

Moreover the polarization injected along $\vec{n}(s)$ is stable with respect to perturbations due to focussing and error fields. In fact if we consider for example the focussing field, each particle will experience a slightly different spin transfer matrix $R(\epsilon)$, where ϵ is the perturbation parameter proportional to the amplitude of betatron oscillations (emittance). It is assumed $R(\epsilon=0) = R_0$ and the dependence upon the parameter to be smooth.

The direction of injection of the polarization along \vec{n} secures the conservation of the polarization under more general conditions than the other two. Let's justify this sentence: \vec{P} is defined as

$$\vec{P} = \langle \vec{S} \rangle \quad (2.8)$$

where $\langle \rangle$ indicates the average over the particle distribution. More precisely we here consider the polarization vector \vec{P} as the macroscopic average over the particle distribution in the beam, joint with a quantum average evaluated by the quantum statistical matrix. After k turns we have:

$$\vec{P}_f = \langle \vec{S}_f \rangle = \langle R(\epsilon)^k \vec{S}_i \rangle = \langle R(\epsilon)^k \rangle \langle \vec{S}_i \rangle = \langle R(\epsilon)^k \rangle \vec{P}_i \quad (2.9)$$

where the independence of the two averages has been used.

Let now (n_α, n_β, n) be a base in which R_0 is diagonal. The perturbed rotation can be written as:

$$R(\epsilon) = T_\epsilon \begin{vmatrix} e^{\delta i} & 0 & 0 \\ 0 & e^{-\delta i} & 0 \\ 0 & 0 & 1 \end{vmatrix} T_\epsilon^T \quad (2.10)$$

Here $e^{\pm \delta i}$ are the complex conjugate eigenvalues of $R(\epsilon)$, T_ϵ is the rotation that overlaps \vec{n}_ϵ , real eigen-vector of $R(\epsilon)$, to \vec{n} . The matrix after k iterations can then be evaluated to be:

$$R(\epsilon)^k = T_\epsilon \begin{vmatrix} e^{\delta k i} & 0 & 0 \\ 0 & e^{-\delta k i} & 0 \\ 0 & 0 & 1 \end{vmatrix} T_\epsilon^T \quad (2.11)$$

Performing the average over ϵ , one has to calculate:

$$\langle R(\epsilon)^k \rangle = \langle T_\epsilon \rangle \begin{vmatrix} \langle e^{\delta k i} \rangle & 0 & 0 \\ 0 & \langle e^{-\delta k i} \rangle & 0 \\ 0 & 0 & 1 \end{vmatrix} \langle T_\epsilon \rangle^T \quad (2.12)$$

The conservation of $\vec{P} \cdot \vec{n}$ is related to T_ϵ , which is a characteristic of a single turn. Following the language of dynamical systems we may say that $\vec{P} \cdot \vec{n}$ is stable in the Ljapunov sense: for any value $\kappa > 0$ we can find an ϵ such that:

$$|\vec{P}_f \cdot \vec{n} - \vec{P}_i \cdot \vec{n}| < \kappa$$

As far as the conservation of the others two components is concerned, it becomes important the average of the eigenvalues of $R(\epsilon)^k$:

$$\langle e^{\delta i} \rangle = \int d\epsilon g(\epsilon) e^{\delta i k} = e^{2\pi\nu_s k} \int d\epsilon g(\epsilon) e^{2\pi\Delta\nu(\epsilon) k i} \quad (2.13)$$

where $g(\epsilon)$ is the distribution function and $\Delta\nu = \delta/2\pi - \nu_s$ is the spin-tune shift.

If we do assume the distribution as uniform in the interval $-\frac{2}{k} \leq \Delta\nu \leq \frac{2}{k}$ the average becomes:

$$k \int_{-2/k}^{2/k} d\Delta\nu e^{2\pi\Delta\nu k i} = \int_{-1}^1 d\xi e^{2\pi\xi i} = 0 \quad (2.14)$$

and the transverse polarization is completely lost.

This is of course an extreme case; if $\Delta g(\epsilon)$ is sharply peaked at zero there is a conservation also of the transverse polarization component for a high number of turns. As a general conclusion we can state that the \vec{n} component of the spin is stable, and the depolarization in this case is related to the rotation of the precession axis and is independent of the number of turns k we wait before measuring the polarization. For the two transverse components instead the depolarization is related to the entity of the spin-tune spread $\Delta\nu(\epsilon)$ multiplied by the number of revolutions k .

If the machine is planar, we have a great simplification: \vec{B} is always vertical, $\vec{n}(s)$ corresponds to \hat{z} and the spin tune is $\nu_s = a\gamma$. In the general case one has to compose the spin rotations due to the single elements of the machine, and then to calculate R_0 and its spectrum.

In particular when in the ring there is a spin rotator, called Siberian Snake, realized by a solenoid or a sequence of vertical and horizontal bending magnets, the calculation of the \vec{n} -axis is the first step of the spin dynamics analysis. It is easy to prove that in the configuration of Fig. 1, with a ring equipped with a device that rotates the spin of 180° around the particle trajectory, \vec{n} is longitudinal in the point opposite to the rotator and the spin tune is half-integer.

3. Requirements for the Spin Splitter

In its basic configuration the Spin Splitter ^{[1],[2],[3]} consists of a strong doublet with a solenoid that rotates of about 180° the spin in between. Due to the quadrupole gradient the (anti)protons experience a small kick in a direction that depends upon the sign of \vec{S} , and because of the spin rotation the kick of the second quadrupole adds up. The solenoid acts as the Siberian Snake mentioned in the previous section, so that \vec{n} lies on the horizontal plane.

If the polarization is kept in the direction \vec{n} the kick repeats at the same azimuth turn after turn as it happens for the imperfections which affect the betatron oscillations; thus the effect would add up only if the betatron tune is either integer or half-integer, i.e. in situations where the machine is unstable.

The way to overcome this problem is to make the polarization \vec{P} rotate of an angle λ turn after turn; more precisely, the component of \vec{P} that is transverse to the motion at the azimuth of the Spin-Splitter should rotate with this rate. If this is the case, the condition for a coherent adding up of the kicks becomes:

$$\lambda = 2\pi\Delta Q_H \tag{3.1a}$$

or

$$\lambda = 2\pi\Delta Q_V \tag{3.1b}$$

where

$$\Delta Q_H = |Q_H - (\text{integer and/or half - integer})| \tag{3.2a}$$

$$\Delta Q_V = |Q_V - (\text{integer and/or half - integer})| \tag{3.2b}$$

being Q_H and Q_V the two betatron tunes.

The first condition gives a coherent growing up of the horizontal oscillations, the second of the vertical ones.

There are basically two means to implement the spin rotation λ :

- (a) by constructing the polarization in the direction orthogonal to \vec{n} and choosing ν_s proportional to λ or to $\Delta Q_{H,V}$ (see Ref. 3 and Sect. 4);
- (b) by using an RF solenoid which adiabatically rotates \vec{n} of an amount λ turn after turn.

The approach (a) is much easier to be implemented, but it requires a previous verification that the spin-tune spread in our operating conditions is small enough not to mix the two transverse components in the time we need. This is the reason why we began an experimental verification at IUCF [4] to test the conservation of the transverse spin components in a condition suited for the Spin Splitter.

It is interesting to observe that in a planar machine (i.e. without Siberian Snake) the condition ν_s proportional to $\Delta Q_{H,V}$ corresponds to one of the so called "intrinsic depolarizing resonance", where all the focussing fields contribute coherently to depolarize the beam. The Siberian Snake succeeds in removing the unwanted effects caused by any kind of depolarizing resonance.

4. Coherent Stern-Gerlach Kicks

The Stern-Gerlach kicks must be coherent with the betatron motion in such a way that the transverse momenta be continuously enhanced. To this end, the transverse impulse and the trajectory slope of the test particle must coincide.

We here like to recall that the spin splitter device consists of two quadrupoles with opposite gradient, separated by a solenoid which rotates the spins by 180° , and that for $a\gamma = \text{half-integer}$ the spins precess, after a revolution, by an amount equal to

$$a\gamma 2\pi = \left(n + \frac{1}{2}\right)2\pi = 2n\pi + \pi \quad (n \text{ integer}) \quad (4.1a)$$

Hence the spins capsize after one turn but restore their previous alignments because of the solenoid crossing. In this situation Q_V should be an integer in order to make the trajectory slopes keep the same direction turn after turn.

Similarly Q_V should be half an integer when the spin-precession is

$$a\gamma 2\pi = 2m\pi \quad (m \text{ integer}) \quad (4.1b)$$

In fact, the trajectory slope of the test particle alternates sign every second turn, whereas the spin, though keeping its direction during a revolution, is reversed by the solenoid and then follows the direction of the betatron oscillation.

In order to have a steady growth of the betatron-oscillation amplitudes, under the effect of the Stern-Gerlach force, but staying simultaneously outside both integer and half-integer resonances, several methods have been proposed:

- 1) a periodic variation of the spin-precession frequency (Sect. 8 of Ref. 3);
- 2) a periodic bump of vertical betatron oscillations [6];

- 3) a jump of the whole spin-splitter device properly synchronized with the betatron tune (Sect. 9 of Ref. 3);
- 4) a spin rotation different from π (Sect. 10 of Ref. 3 and Ref. 7).

For sake of compactness, these four methods have been respectively defined as spin-precession kicks, vertical orbit bumps, missed spin-splitter and spin-precession lag (or advance).

In this Section we shall first prove how the induced spin-precession mentioned above, succeeds in giving rise to the desired adding up of the Stern-Gerlach kicks. In order to find a description of this *controlled* precession, based upon the rotation R introduced in Section 2, we shall use the Pauli-matrices formalism ^[8], finding that all the other methods will yield rotations quite similar ^[9] to the one just obtained. Only the missed spin-splitter proposal will be left aside, since it would require quite a big amount of hardware to be realized.

4.a) Spin-precession Kicks

A pulsed ($\vec{E} \perp \vec{B}$) device, synchronized with the RF, is implemented; its aim is to vary the precession angle of the spins by a small amount λ , or precession kick, which coincides with the one of eqs. (3.1a,b), as it will be demonstrated in Appendix A.

We like now to find how the action of the Stern-Gerlach force influences the vertical position of the particle. To this end we define, as done in Sect. 8 of Ref. 3, the variable

$$\bar{z} = z - i\beta_V z l \quad (4.2)$$

while the details of the spin precession in the successive revolutions, together with the 180° rotation inside the solenoid, are discussed in Appendix A.

After N crossings of the spin-splitter device, eq. (4.2) transforms into

$$\bar{z}_N = e^{iN\mu_V} \left\{ \bar{z}_0 - i\beta_V \eta \left[\sum_{h=1}^N e^{-ih(\mu_V + \lambda)} + \sum_{h=1}^N e^{-ih(\mu_V - \lambda)} \right] \right\} \quad (4.3a)$$

$$\bar{z}_N = e^{iN\mu_V} \left\{ \bar{z}_0 - i\beta_V \eta \left[\sum_{h=1}^N (-1)^h e^{-ih(\mu_V + \lambda)} + \sum_{h=1}^N (-1)^h e^{-ih(\mu_V - \lambda)} \right] \right\} \quad (4.3b)$$

with η given by eq. (1.4) and where:

$\mu_V = 2\pi Q_V$ is the phase-advance of the vertical betatron oscillations;

$\bar{z}_0 = z_0 - i\beta_V z l_0$, being $z_0, z l_0$ the coordinates of a generic particle at the initial time.

Bearing in mind what discussed before, we may write:

$$a\gamma = \text{half - integer}, \quad Q_V = k - \Delta Q_V \quad (4.4a)$$

$$a\gamma = \text{integer}, \quad Q_V = k + \frac{1}{2} - \Delta Q_V \quad (4.4b)$$

with $k = \text{integer}$ and ΔQ_V given by eq. (3.2b).

If we want a growing term to appear in both eqs. (4.3a,b), we have to set

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$$\mu_V + \lambda = 2\pi k \quad (4.5a)$$

$$\mu_V + \lambda = 2\pi k + \pi \quad (4.5b)$$

which, bearing in mind eqs. (4.5a,b) and the definition $\mu_V = 2\pi Q_V$, yield:

$$\lambda = 2\pi\Delta Q_V \quad (4.6)$$

Of course eqs. (4.5a,b) and (4.6) imply that

$$\mu_V - \lambda = 2\pi k - 4\pi\Delta Q_V \quad (4.7a)$$

$$\mu_V - \lambda = 2\pi k + \pi - 4\pi\Delta Q_V \quad (4.7b)$$

After straight-forward manipulations of eqs. (4.3a,b) and into account eq. (3.2b) we obtain for both $a\gamma$ =half-integer and $a\gamma$ =integer:

$$\tilde{z}_N \simeq e^{iN\mu_V} \left\{ \tilde{z}_0 - \frac{1}{2}i\beta_V\eta N + \text{fading off terms} \right\} \quad (4.8)$$

or

$$z_N = Z_0 \cos N\mu_V + \beta_V z'_{l_0} \sin N\mu_V \quad (4.9a)$$

$$z'_N = -\frac{z'_{l_0}}{\beta_V} \sin N\mu_V + z'_{l_0} \cos N\mu_V + \frac{1}{2}N\eta \quad (4.9b)$$

Formulae (4.9a,b) clearly show that the $\frac{1}{2}N\eta$ term represents the global kick after N revolutions and we obtain for the separation between the centres of mass of these two groups of opposite polarized particles:

$$\Delta z_{Max} = \frac{1}{2}N\beta_V\eta \quad (4.10)$$

Thus we have proven that, no matter how distant from either an integer or a half-integer resonance of the machine is, a building up of this separation takes place, provided that the conditions (4.4a,b) are fulfilled.

The two polarized bunches we obtain in this manner have their spins pointing ^[3] parallel or antiparallel to a vector which rotates in the horizontal plane (x, y) around the z -axis continuously by steps of λ .

Let us use now the formulae introduced in Appendix B; the polarization vector \vec{P} , after $N = 2n$ revolutions (we remind that one kick gives λ and the next gives $-\lambda$) is

$$\begin{vmatrix} P_x \\ P_y \\ P_z \end{vmatrix}_N = R^n \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} \cos 2n\lambda \\ \sin 2n\lambda \\ 0 \end{vmatrix} \quad (4.11)$$

where

$$R = \begin{vmatrix} \cos 2\lambda & -\sin 2\lambda & 0 \\ \sin 2\lambda & \cos 2\lambda & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad (4.12)$$

is obtained by inserting eq. (B.9b) into the rotation (B.5).

Then we may state that eq. (4.11) describes the wanted continuous precession of the spin, by steps of λ per revolution. Besides the precession axis, in correspondence to the entrance of the solenoid, coincides with the stable solution \vec{n} , i.e. the polarization \vec{P} rotates in the horizontal (x,y) -plane: this can be easily proven by plugging into the definition (B.6a) the coefficients given by eqs. (B.9b,c) obtaining a vector of components $(n_x = 0, n_y = 0, n_z = \mp 1)$. Moreover eq. (B.7) yields as spin-precession tune

$$\nu_s = (2s + 1) \pm \Delta Q_V \quad (4.13)$$

for both cases $a\gamma$ =integer and/or half-integer.

4.b) Vertical Orbit Bump

The same results can be achieved by supplying alternate kicks to the vertical betatron oscillations. In Appendix B there is the demonstration that the spin tune is the one given by eq. (4.13) and that, at the solenoid location, \vec{P} rotates, lying in a plane, around the x-axis, if $a\gamma$ =integer, and about the z-axis if $a\gamma$ =half-integer.

4.c) Spin-Precession Lag or Advance

In this case the spins have to undergo a precession, inside the solenoid, which is different than the usual 180° by an amount equal to $\pm\delta\phi$. Always in Appendix B we may find that we have a continuous precession of the spin by steps of

$$\delta\phi = 2\lambda = 4\pi\Delta Q_V \quad (4.14)$$

and with spin tune

$$\nu_s = \frac{1 - (-1)^m}{2} - \frac{1}{2} + 2\Delta Q_V \quad (4.15)$$

only for $a\gamma$ =integer= m , while for $a\gamma$ =half-integer we have constantly ν_s =half-integer thus the precession \vec{P} restores the same alignment each second revolution; this means that the betatron oscillations make particles receive the Stern-Gerlach kicks alternately with respect to the orbit slopes.

This would be solution of our problems if we could work over a *perfect* half-integer betatron-tune: but this is either impossible to achieve with the due accuracy or dangerous to be implemented since destructive resonances of the betatron oscillations would take place.

5. Forced Oscillations vs. Damping

Any force, of strength F and azimuthal distribution $f(\theta)$, acting on the (e.g. vertical) betatron oscillations modifies the corresponding motion equation in the following way:

$$\frac{d^2 z}{d\theta^2} + Q_V^2 z = \frac{R^2}{\beta^2 \gamma m c^2} F f(\theta) \quad (5.1)$$

In our case $F = G\mu$, with μ given by eq. (1.2), and $f(\theta)$ is the Dirac function

$$f(\theta) = \frac{l_Q}{2\pi R} \left(1 + 2 \sum_{k=1}^{\infty} \cos k\theta\right) \quad (5.2)$$

since the quadrupole length l_Q is much smaller than the average machine radius R . Hence the motion equation (5.1) becomes:

$$\frac{d^2 z}{d\theta^2} + Q_V^2 z = \eta \frac{R}{2\pi} \left(1 + 2 \sum_{k=1}^{\infty} \cos k\theta\right) \quad (5.3)$$

with η given by eq. (1.4).

If the appropriate synchronism-conditions described in the previous Section are fulfilled, the θ -dependent terms succeed in creating a resonance for the spin-tune, achieving thus the wanted coherent adding up of the kicks. On the contrary, if nothing of specific is done, the right-side of eq. (5.3) gives rise to a tiny closed orbit deformation of the order of η (i.e. of the order of \hbar).

Representing this synchronism as a linear resonance, we reduce eq. (5.3) to the one of a damped harmonic oscillator, of self-frequency ω , driven by a constant forcing term $f_0 \cos \Omega t$:

$$\frac{d^2 z}{dt^2} + \omega^2 z = \frac{f_0}{m} \cos \Omega t \quad (5.4)$$

whose solution is:

$$z(t) = z_{\text{free}} + z_{\text{resonant}} = z_f + z_r \quad (5.5a)$$

with

$$z_f = z_0 \cos \omega t + \frac{1}{\omega} z_0' \sin \omega t \quad (5.5b)$$

and

$$z_r = \frac{f_0}{m(\omega^2 - \Omega^2)} (\cos \Omega t - \cos \omega t)$$

or

$$z_r = \frac{2f_0}{m(\Omega + \omega)(\Omega - \omega)} \sin\left(\frac{\Omega + \omega}{2} t\right) \sin\left(\frac{\Omega - \omega}{2} t\right) \quad (5.5c)$$

In order to have steady blow-up of the oscillation amplitudes, Ω must be as close as possible to ω fulfilling the condition

$$\frac{|\Omega - \omega|}{2} t \ll 1 \quad (5.6)$$

Under these hypotheses eq. (5.5c) reduces to

$$z_r = \frac{f_0}{2m\omega} t \sin \omega t \quad (5.7)$$

showing a linear increase of the amplitude as a function of time.

Since the spin-splitter experiment can last a long time, e.g. of the order of thousand(s) seconds, the condition (5.6) implies that:

$$\frac{|\Omega - \omega|}{\omega} \ll \frac{2}{\omega t} = 4 \times 10^{-11} \quad (5.8)$$

having put, for instance, $\omega = Q_V 2\pi f_{rev} = 5.3 \times 10^7 \text{sec}^{-1}$.

But the condition (5.8) pertains to the case of an external driving force of frequency Ω , like e.g. a RF-system. In our example, each particle, after receiving the Stern-Gerlach kick, undergoes betatron oscillations always keeping its own frequency ω , which becomes coincident with Ω , at least in principle.

Actually, particles receive these kicks turn after turn, with the same uncertainty which affects the revolution period T, say

$$\frac{dT}{T} \simeq 10^{-4} \quad (5.9)$$

It is very well known, from the statistical treatment of measurements, that the relative error of a sum of quantities, carelessly how many they are, is of the same order of magnitude of the relative error of each term; thus the accuracy required for our experiment coincides with the typical accuracy of any machine.

Even in an ion ring, oscillation amplitudes trend to damp down because of the energy distribution from each oscillator to the other ones (Landau damping), to the residual gas molecules and to the electron of a possibly existing e-cool system.

As first hypothesis, let us consider this damping as viscous, i.e. the drag-force is proportional to the velocity; then the motion equation (5.4) is modified into:

$$\frac{d^2 z}{dt^2} + \frac{2}{\tau} \frac{dz}{dt} + \omega^2 z = f(t) = f_0 \sin \Omega t \quad (5.10)$$

Every revolution we have a constant energy-input δU_{kick} supplied by the forcing term, accompanied by an energy-loss δU_{loss} , related to the Landau damping.

As demonstrated in Appendix C, δU_{loss} increase as the oscillation amplitudes grow, giving rise to a saturation (see Fig. 2) of the stored energy

$$U(t) = U_{\infty} (1 - e^{-\frac{t}{\tau_E}}) \quad (5.11)$$

with U_{∞} depending on the ring characteristics and where

$$\tau_E = \frac{1}{2} \tau \quad (5.12)$$

being τ the same quantity appearing in eq. (5.11).

If the driving term is removed, the oscillation amplitude decreases and after n revolutions reduces its value by a factor $e(=2.7182818\dots)$. If N is the number of revolutions required for reaching the saturation, we demonstrate in Appendix C that

$$N \leq \frac{1}{2}n \quad (5.13)$$

or that any further action of the driving force is useless if applied for periods longer than the damping time characterizing the ring.

Since in the proposed experiment we have a sequence of kicks, rather than a driving term acting continuously, i.e. oscillation amplitudes increase suddenly after each Spin-Splitter crossing, and then decrease smoothly (see Fig. 3) with the same damping-time τ as in eq. (5.12). This behaviour can also be described analitically in the following way:

1st turn:

$$a_1 = a_0 e^{-\frac{T}{\tau}}$$

2nd turn:

$$a_2 = (a_1 + a_0) e^{-\frac{2T-T}{\tau}} = (a_0 + a_1) e^{-\frac{T}{\tau}} = a_0 e^{-\frac{T}{\tau}} (1 + e^{-\frac{T}{\tau}})$$

after N revolutions:

$$a_N = (a_{N-1} + a_0) e^{-\frac{NT-(N-1)T}{\tau}} = (a_0 + a_{N-1}) e^{-\frac{T}{\tau}}$$

or

$$a_N = a_0 e^{-\frac{T}{\tau}} [1 + e^{-\frac{T}{\tau}} + e^{-\frac{2T}{\tau}} + \dots + e^{-(N-1)\frac{T}{\tau}}]$$

i.e.

$$a_N = a_0 e^{-\frac{T}{\tau}} \frac{1 - e^{-N\frac{T}{\tau}}}{1 - e^{-\frac{T}{\tau}}} \quad (5.14)$$

Since T is of the order of some microseconds while τ can vary from a few milliseconds (bad vacuum) till hours (very high vacuum), we have in eq. (5.14):

$$1 - e^{-\frac{T}{\tau}} \simeq \frac{T}{\tau}$$

or

$$a_N \simeq a_0 \frac{\tau}{T} (1 - e^{-N\frac{T}{\tau}}) \simeq a_0 \frac{\tau}{T} \quad (5.15)$$

as soon as the number of revolutions N exceeds the ratio $\frac{\tau}{T}$. Fig. 4 illustrates the time dependence of eq. (5.14) for $\tau \simeq 10^{-3} s$ and $T \simeq 10^{-6} s$.

Therefore, in this example of viscous friction, the amplitude may stop growing after a time which can be shorter than the one required by achieving the spin-states separation. In fact the lost energy is proportional to the squared amplitude: thus a stage will always be reached when the energy-loss equalizes the energy-gain.

Indeed, the fast Landau-damping depends on the various kinds of scattering from either the electrons of the e-cool system or the molecules of the residual gas (bad vacuum).

By considering that the momentum transfer in such collisions is rather small, we might assume that the lost energy is quite independent of the oscillation amplitude, as is our purpose to verify with the an experiment of the RF knock-out type. In this hypothesis the growth is steady as shown in Fig. 5 and demonstrated in the following:

1st turn:

$$a_1 = a_0 - \alpha t$$

2nd turn:

$$a_2 = a_1 + a_0 - \alpha t = a_1 + a_1 = 2a_1$$

after N revolutions:

$$a_N = a_{N-1} + a_0 - \alpha t = a_{N-1} + a_1 = Na_1 \quad (5.16a)$$

provided that the condition

$$\alpha \ll \frac{a_0 - a_1}{T} \quad (5.16b)$$

is fulfilled.

6. Quantum Mechanical Implications

Protons (or antiprotons) in an accumulation ring, where the guide magnetic field points into the z-direction, assess themselves in such a way that the beam polarization is defined as:

$$P_z = \frac{N_+ - N_-}{N_+ + N_-} \quad (6.1)$$

where N_+ and N_- are the number of particles with their spins "up" and "down" respectively, thus simplifying the general definition given in Sect. 2. Of course, the case $N_+ = N_-$ corresponds to an unpolarized beam.

The Stern-Gerlach force acts and adds up in a repetitive way, at our leisure, in the quadrupole of the Spin-Splitter device, once an appropriate synchronism (see Sect. 4) is achieved between orbital and spin motions, thus determining kicks in opposite directions for the two groups N_+ and N_- of particles.

We recall here that each particle of the two sets exhibits the adiabatic invariant $I_z + S_z$ and $I_z - S_z$ respectively: these are rigorously motion constants in absence of the coupling between the precessing \vec{S} and the spinless action I_z .

At this stage we must consider the intensity of the *classical* Stern-Gerlach kick in relation with the true features of the particle motion which undergoes the uncertainty principle and follows the laws ^[5] of quantum mechanics.

We therefore recall that in a storage ring made of FODO cells the motion equations, regarding transverse degrees of freedom, are of different types; for the vertical motion we have:

$$\frac{d^2 z}{dt^2} + \omega_Q^2 z = 0 \quad (\text{focusing quadrupoles}) \quad (6.3a)$$

$$\frac{d^2 z}{dt^2} - K_Q^2 z = 0 \quad (\text{defocusing quadrupoles}) \quad (6.3b)$$

$$\frac{d^2 z}{dt^2} = 0 \quad (\text{straight sections and bending magnets}) \quad (6.3c)$$

while for the horizontal motion we have the same equations plus another one for the bending magnets:

$$\frac{d^2 x}{dt^2} + \omega_M^2 x = 0 \quad (6.4)$$

moreover there would be another equation dealing with solenoids and regarding both coordinates x and z :

$$\frac{d^2}{dt^2} \begin{Bmatrix} x \\ z \end{Bmatrix} + \omega_S^2 \begin{Bmatrix} x \\ z \end{Bmatrix} = 0 \quad (6.5)$$

The frequencies and the time-constant appearing in the previous equations are defined as follows:

$$\omega_Q^2 = -K_Q^2 = \frac{e\beta cG}{\gamma m} \quad (6.6a)$$

$$\omega_M^2 = \frac{e\beta cB_M}{\gamma m\rho} \quad (6.6b)$$

$$\omega_S = \frac{eB_S}{\gamma m} \quad (6.6c)$$

where B_M, B_S are respectively the magnetic field in the bending magnets and in the solenoid(s) and ρ is the bending radius.

Therefore, in all the elements where eqs. (6.3a),(6.4),(6.5) hold on, the (anti)proton stationary eigenfunctions are the ones of the harmonic oscillator, i.e.

$$u_n(z) = u_n^*(z) = N_n H_n(\alpha z) e^{-\frac{1}{2}\alpha^2 z^2} \quad (6.7)$$

being $H_n(\alpha z)$ the n -th Hermite polynomial with

$$N_n^2 = \frac{\alpha}{\sqrt{\pi 2^n n!}} \quad (6.8)$$

$$\alpha^2 = \frac{m\omega}{\hbar} \quad (6.9)$$

where ω is one of the frequencies given by eqs. (6.6a,b,c).

Instead, the "corresponding state" functions are:

$$\psi(z, t) = \frac{\sqrt{\alpha}}{\pi^{\frac{1}{4}}} \exp\left(-\frac{1}{2}(\xi - \xi_0 \cos \omega t)^2 - i\left(\frac{1}{2}\omega t + \xi \xi_0 \sin \omega t - \frac{1}{4}\xi_0 \sin 2\omega t\right)\right) \quad (6.10a)$$

$$\psi^*(z, t) = \frac{\sqrt{\alpha}}{\pi^{\frac{1}{4}}} \exp\left(-\frac{1}{2}(\xi - \xi_0 \cos \omega t)^2 + i\left(\frac{1}{2}\omega t + \xi \xi_0 \sin \omega t - \frac{1}{4}\xi_0 \sin 2\omega t\right)\right) \quad (6.10b)$$

with $\xi_0 = \alpha x_0$, being x_0 the oscillation amplitude.

Bearing in mind the usual quantum indetermination Δp_z of p_z

$$\Delta p_z = \sqrt{\langle p_z^2 \rangle - \langle p_z \rangle^2} \quad (6.11)$$

taking into account the stationary eigenfunction(s) (33) and easily verifying that $\langle p_z \rangle = 0$, we obtain:

$$\Delta p_z^2 = \langle p_z^2 \rangle = -\hbar^2 \int_{-\infty}^{+\infty} u_n \frac{d^2 u_n}{dz^2} dz = \Delta p_0^2 \left(n + \frac{1}{2} \right) \quad (6.12)$$

with

$$\Delta p_0 = \hbar \alpha = \sqrt{\hbar m \omega} \quad (6.13)$$

While reiterating the same operations with the "corresponding state" eigenfunctions, we attain:

$$\langle p_z \rangle = -\Delta p_0 \xi_0 \sin \omega t \quad (6.14a)$$

$$\langle p_z^2 \rangle = \Delta p_0^2 \left(\frac{1}{2} + \xi_0^2 \sin^2 \omega t \right) \quad (6.14b)$$

which, inserted into eq. (6.11), yield:

$$\Delta p_z = \frac{\Delta p_0}{\sqrt{2}} \quad (6.15)$$

that corresponds to the ground state ($n=0$ in eq. (6.12)) of the harmonic oscillator.

We may proceed in a similar way for the conjugate variable z , obtaining:

$$\Delta z = \Delta z_0 \sqrt{n + \frac{1}{2}} \quad (\text{stationary eigenfunctions}) \quad (6.16a)$$

$$\Delta z = \frac{\Delta z_0}{\sqrt{2}} \quad (\text{"corresponding state" eigenfunctions}) \quad (6.16b)$$

with

$$\Delta z_0 = \sqrt{\frac{\hbar}{m \omega}} \quad (6.17)$$

As discussed in Refs. 1,2,3 the (anti)proton-energy must fulfil the requirements $a\gamma = \text{half-integer}$ or $a\gamma = \text{integer}$. Since $a = 1.793\dots$, the smallest kinetic energy attainable is $W = 0.108 \text{ GeV}$ ($p = 0.464 \text{ GeV}/c = 2.481 \times 10^{-19} \text{ Kgms}^{-1}$), as the minimum half-integer with physical meaning is $\frac{3}{2} = 1.5$, i.e. $W = 1.394 \text{ GeV}$.

After an approximate estimation of the IUCF Cooler Ring ^[4] parameters at $W = 0.108 \text{ GeV}$, we may insert into eqs. (6.6a,b,c) $G \simeq 4.6 \text{ Tm}^{-1}$, $B_M \simeq 0.645 \text{ T}$, $\rho \simeq 2.4 \text{ m}$, $B_S \simeq 2.5 \text{ T}$, obtaining:

$$\omega_M \simeq 5.54 \times 10^7 \text{ rad/s} < \omega_S \simeq 2.15 \times 10^8 \text{ rad/s} < \omega_Q \simeq 2.29 \times 10^8 \text{ rad/s} \quad (6.18)$$

Disregarding ω_M , since we treat vertical motion only, and putting ω_Q into eqs. (6.13) and (6.17) we get:

$$\Delta z_0 = 1.66 \times 10^{-8} m \quad (6.19a)$$

$$\Delta p_0 = 6.35 \times 10^{-27} Kg ms^{-1} \quad (6.19b)$$

The latter can be manipulated to yield the angular indetermination

$$\Delta z'_0 = \frac{\Delta p_0}{p} \simeq 2.56 \times 10^{-8} rad \quad (6.19c)$$

The wave-packet centre corresponds to the "classical" particle: in our case a 0.108 GeV proton crosses a quadrupole of length $L_Q \simeq 0.5 m$ and undergoes "classical" Stern-Gerlach impulse and kick

$$\delta p_z = \frac{G\mu L_Q}{\beta c} \simeq 2.44 \times 10^{-34} Kg ms^{-1} \quad (6.20a)$$

$$\delta z' = \frac{\delta p_z}{p} \simeq 9.83 \times 10^{-16} rad \quad (6.20b)$$

A quick comparison between eqs. (6.19b) and (6.20a) shows that the classical impulse is 7 orders of magnitude smaller than the quantum uncertainty. This confirms the Bohr's old proof that no splitting can be achieved through a single crossing of a region endowed with magnetic gradient.

Nevertheless, if we succeed [3] in adding up coherently these kicks, the size of the quantum uncertainty is reached after 26×10^6 revolutions or 17 s, as easily inferred from eqs. (6.19c) and (6.20b).

Then, at the azimuth where $\beta_V = (\beta_V)_{Max} \simeq 46 m$, a vertical displacement $\delta z \simeq 10^{-3}$ mm should be attained; therefore, wanting a detectable shift of the order of 0.1 mm at least, 1.3×10^9 turns, or roughly 14 minutes, are required. Notice that a factor 2 is gained as the Spin-Splitter supplies two kicks.

Outside the focusing quadrupoles the steady-state Schroedinger equation is

$$\frac{1}{u} \frac{d^2 u}{dz^2} = -\frac{2m}{\hbar^2} [E - V(z)] < 0$$

since

$$V(z) = \left\{ \begin{array}{l} 0 \\ \frac{1}{2} e\beta c G z^2 \end{array} \right\} = \left\{ \begin{array}{l} \text{straight sections + bending magnets} \\ \text{defocusing quadrupoles} \end{array} \right\}$$

Therefore no discrete levels can exist in such situations and a continuous spectrum is attained. Consequently, the free-motion wave-packet endures a time dependent dilation

$$\Delta z(t) = \Delta z_0(t) \sqrt{1 + \frac{1}{\Delta x_0^4} \left(\frac{\hbar t}{2m} \right)^2} \quad (6.21)$$

where t is the time taken by the wave-packet centre, or proton, to cover the average distance L between two successive focusing quadrupoles.

Taking into account eqs. (6.6a) and (6.17), the already estimated values of B_M , G , ρ , the definition $p = \beta\gamma mc = eB_M \rho$ and assuming:

$$L = \frac{\text{Ring Circumference}}{\frac{1}{2}[\text{Number of F-quads}]} = \frac{86.83 m}{9} = 9.65 m$$

eq. (6.21) becomes:

$$\Delta z(L) = \sqrt{\frac{\hbar\gamma}{e\sqrt{B_M G \rho}}} \sqrt{1 + \frac{GL^2}{4B_M \rho}} \simeq 4.74 \times 10^{-7} m \quad (6.22)$$

The result of the same order of magnitude, achieved by using the angular quantum indetermination (6.19c),

$$\Delta z_0(L) = L\Delta z'_0 \simeq 2.45 \times 10^{-7} m \quad (6.23)$$

makes sure that the size of the wave packet resets to the ground state when the particle enters the focusing quadrupoles.

7. Conclusions

The Spin-Splitter method should work, at least in principle, since we have demonstrated that there are not any quantum mechanical constraints against this method.

Those same arguments can also be valid for counteracting the criticism arising from problems related to accuracy. In fact, as in the quantum mechanical approach we referred [5] to a "probability cloud", in the example of classical randomness we may assimilate all the imperfections to a noise and then tackle the problem as one of stochasticity threshold. Moreover, if we consider that quantum mechanics can be interpreted as the deterministic motion of the wave-packet centre surrounded by a random-walk, the analogy between quantum and classical randomnesses is complete.

Other source of errors can be either periodic tune-modulations, easy to be dealt with, or slow drifts of the machine tune; the latter could constitute a problem, at least in principle, but we are studying the possibility of adapting some stochastic cooling technique to this process. However, we may state that the required accuracy of 4 part in 10^{11} does not regard our experiment, since eq. (4.11) applies in an oversimplified example which substantially differs from the actual mechanism of the cumulative Stern-Gerlach effect.

The various methods proposed in Sect. 4 all imply a beam polarization aligned to the stable solution \vec{n} . In a recent experiment [10] carried out at IUCF we have demonstrated that quite an amount of polarization is preserved even if the spins are orthogonal to \vec{n} . Then a first step towards the feasibility of the Spin-Splitter method has been moved.

Next experiments should verify the actual possibility of accumulating the Stern-Gerlach kicks, against counteracting effects like multiple scattering on molecules of residual gas, intrabeam scattering, damping due to beam-environment interactions, etc. In order to do so, we need a very high vacuum (e.g. 10^{-12} torr instead of 10^{-9} torr used so far) and, above all, a Spin-Splitter device with quadrupole gradients much stronger than the ones currently used in a low energy storage ring. The latter requirement implies a deep modification of the optics of any existing ring but, if fulfilled, all the drawbacks related to the smallest of the effect could be disregarded.

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APPENDIX A: Spin Components Evolution

Let the spin, lying on the horizontal plane, form an angle α with the y-axis (parallel to the) motion direction at the spin-splitter entrance, after one turn this angle is transformed into:

$$\left[\left(2k + \frac{1}{2} \right) 2\pi - \alpha + \lambda \right]_{a\gamma=\text{half-integer}}$$

$$[2m\pi - \alpha + \lambda]_{a\gamma=\text{integer}}$$

since α is reversed by the solenoid and the crossed-fields device has given a (e.g. positive) kick. After another revolution we obtain:

$$[-4k\pi - \pi + \alpha - \lambda + 4k\pi + \pi - \lambda = \alpha - 2\lambda]_{a\gamma=\text{half-integer}}$$

$$[-4m\pi + \alpha - \lambda + 4m\pi - \lambda = \alpha - 2\lambda]_{a\gamma=\text{integer}}$$

having now considered a negative kick, together with the usual π reversal. By iterating this procedure it is easy to show that the angles between spins and motion direction vary as follows:

$a\gamma=\text{half-integer}$

$$\alpha, -\alpha + \pi + 4k\pi + \lambda, \alpha - 2\lambda, -\alpha + \pi + 4k\pi + 3\lambda$$

$a\gamma=\text{integer}$

$$\alpha, -\alpha + 2m\pi + \lambda, \alpha - 2\lambda, -\alpha + 2m\pi + 3\lambda$$

In this case the Stern-Gerlach kicks affect the component P_x of the polarization vector \vec{P} , that is

$a\gamma=\text{half-integer}$

$$\sin \alpha, \sin(\alpha - \lambda), \sin(\alpha - 2\lambda), \sin(\alpha - 3\lambda) \dots$$

$a\gamma=\text{integer}$

$$\sin \alpha, -\sin(\alpha - \lambda), \sin(\alpha - 2\lambda), -\sin(\alpha - 3\lambda) \dots$$

or for $\alpha = \frac{\pi}{2}$

$a\gamma=\text{half-integer}$

$$1, \cos \lambda, \cos 2\lambda, \cos 3\lambda \dots$$

$a\gamma=\text{integer}$

$$1, -\cos \lambda, \cos 2\lambda, -\cos 3\lambda \dots$$

APPENDIX B: Matrix Treatment

A very useful tool to describe spin manipulations is the SU(2) representation of rotations. The polarization vector \vec{P} can be associated to a two dimensional complex spinor $\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}$ through the relation

$$\vec{P} = \Psi^\dagger \vec{\sigma} \Psi \quad (B.1)$$

being $\Psi^\dagger = [\Psi_1^* \ \Psi_2^*]$ and where $\vec{\sigma}$ is a 3-dimension vector having as components the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The evolution of \vec{P} from $\theta=0$ to a generic azimuth θ of the ring is a rotation that can be represented either by the orthogonal matrix R, defined in Sect. 2, which acts upon \vec{P} :

$$\vec{P}(\theta) = R\vec{P}(0) \quad (B.2a)$$

or by a unitary matrix M acting on ψ :

$$\psi(\theta) = M\psi(0) \quad (B.2b)$$

the relation between the two being:

$$R\vec{\sigma} = M^\dagger \vec{\sigma} M \quad (B.2c)$$

Since the composition of Pauli matrices is rather easy, it is often convenient to use eq. (B.2b), having expressed the matrixes M as

$$M = IC_0 - i\vec{\sigma} \cdot \vec{C} \quad (B.3)$$

where

$$C_0 \text{ and } \vec{C} \equiv (C_x, C_y, C_z) \quad (B.4)$$

and $C_0^2 + C_x^2 + C_y^2 + C_z^2 = 1$ because of the unitarity of M.

After trivial but tedious calculations it is possible to evaluate the elements of R as a function of the coefficients (B.4):

$$R = \begin{pmatrix} C_0^2 + C_x^2 - C_y^2 - C_z^2 & 2(C_x C_y - C_0 C_z) & 2(C_x C_z + C_0 C_y) \\ 2(C_x C_y + C_0 C_z) & C_0^2 - C_x^2 + C_y^2 - C_z^2 & 2(C_y C_z - C_0 C_x) \\ 2(C_x C_z - C_0 C_y) & 2(C_y C_z + C_0 C_x) & C_0^2 - C_x^2 - C_y^2 + C_z^2 \end{pmatrix} \quad (B.5)$$

which will be used for the various cases considered.

If R is the spin transfer matrix for one revolution, the coefficients (B.4) provide the components of the periodic or stable solution \vec{n} , according to the following formulae:

$$n_{x,y,z} = \frac{\pm C_{x,y,z}}{\sin \xi} \quad (B.6a)$$

with

$$\sin \xi = \sqrt{1 - C_0^2} \quad (B.6b)$$

Besides, according to what discussed in Sect. 2, we have that

$$\cos \pi \nu_s = \frac{1}{2} \text{Tr}(M) = C_0 \quad (B.7)$$

since the Pauli matrices are traceless.

Referring to the example of the alternate kicks supplied to the spin introduced in Section 4 (**Spin-precession Kicks**), we may find that, over two revolutions and for both $a\gamma$ =half-integer and $a\gamma$ =integer cases, we have:

$$M = \{M(\pi)M(\Phi_2^-)M(\Phi_1)\}\{M(\pi)M(\Phi_2^+)M(\Phi_1)\} \quad (B.8)$$

where

$$\begin{aligned} M(\pi) &= -i\sigma_y \quad [\text{spin rotation inside the solenoid}] \\ M(\Phi_2^-) &= I \cos \frac{1}{2}(\Phi_2 - \lambda) - i\sigma_z \sin \frac{1}{2}(\Phi_2 - \lambda) \quad [\text{orbit-part with negative kick}] \\ M(\Phi_2^+) &= I \cos \frac{1}{2}(\Phi_2 + \lambda) - i\sigma_z \sin \frac{1}{2}(\Phi_2 + \lambda) \quad [\text{orbit-part with positive kick}] \end{aligned}$$

(with $\lambda = 2\pi\lambda Q_V$ as given by eq. (4.7))

$$M(\Phi_1) = I \cos \frac{1}{2}\Phi_1 - i\sigma_z \sin \frac{1}{2}\Phi_1 \quad [\text{completion of the revolution}]$$

being $\Phi_1 = a\gamma\theta$ and $\Phi_2 = a\gamma(2\pi - \theta)$, where θ is the azimuth of the ($\vec{E} \perp \vec{B}$) "undeflecting" kicker; then eq. (B.8) reduces to:

$$M = -I \cos \lambda + i\sigma_z \sin \lambda \quad (B.9a)$$

which yields

$$C_0 = -\cos \lambda, \quad C_x = C_y = 0, \quad C_z = -\sin \lambda \quad (B.9b)$$

and, due to eq. (B.6b),

$$\sin \xi = \sin \lambda \quad (B.9c)$$

Bearing in mind the definition (B.7) of the spin tune, we have:

$$\nu_s = \frac{1}{\pi} \arccos(-\cos \lambda) = (2s + 1) \pm \frac{\lambda}{\pi}$$

or

$$\nu_s = (2s + 1) \pm 2\delta Q_V \quad (B.10)$$

having considered eq. (4.7).

As far as the **Vertical Orbit Bump** is concerned, we have after two revolutions:

$$M = \{M(\pi)M(\Phi)M(-\delta)M(\Phi)\}\{M(\pi)M(\Phi)M(+\delta)M(\Phi)\} \quad (B.11)$$

where

$$\begin{aligned} M(\pi) &= -i\sigma_y && [\text{spin rotation inside the solenoid}] \\ M(\Phi) &= I \cos \frac{\Phi}{2} - i\sigma_z \sin \frac{\Phi}{2} && [\text{half-revolution : } \Phi = a\gamma\pi] \\ M(-\delta) &= I \cos \frac{\delta}{2} + i\sigma_z \sin \frac{\delta}{2} && [\text{downwards vertical kick}] \\ M(+\delta) &= I \cos \frac{\delta}{2} - i\sigma_z \sin \frac{\delta}{2} && [\text{upwards vertical kick}] \end{aligned}$$

then eq. (B.11) splits into the two following ones:

$$M = -I \cos \delta + i\sigma_x \sin \delta \quad (a\gamma = \text{integer}) \quad (B.12a)$$

$$M = -I \cos \delta + i\sigma_x (-1)^k \sin \delta \quad (a\gamma = \frac{2k+1}{2} = \text{half-integer}) \quad (B.12b)$$

which yield respectively:

$$C_0 = -\cos \delta, \quad C_x = -\sin \delta, \quad C_y = C_z = 0 \quad (B.13a)$$

$$C_0 = -\cos \delta, \quad C_x = 0, \quad C_y = (-1)^k \sin \delta, \quad C_z = 0 \quad (B.13b)$$

again eq. (B.6b) yields:

$$\sin \xi = \sin \delta \quad (B.13c)$$

Obviously, eqs. (B.13a,b) prove that the spin tune for both examples is the same as in eq. (B.10), since we have $C_0 = -\cos \delta$ in the three cases.

By inserting the coefficients (B.13a,b) into the rotation (B.5) we shall have two solutions:

$$R = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos 2\delta & -\sin 2\delta \\ 0 & \sin 2\delta & \cos 2\delta \end{vmatrix}_{a\gamma = \text{integer}} \quad (B.14a)$$

and

$$R = \begin{vmatrix} \cos 2\delta & 0 & -(-1)^k \sin 2\delta \\ 0 & 1 & 0 \\ (-1)^k \sin 2\delta & 0 & \cos 2\delta \end{vmatrix}_{a\gamma = \text{half-integer}} \quad (B.14b)$$

Applying the rotation (B.14a) to a spin parallel to the z-axis we find again a continuous precession of the spin, by steps of δ per revolution. At the solenoid entrance this precession takes place around the stable solution $\hat{n} \equiv (\pm 1, 0, 0)$, which coincides with the x-axis; consequently the polarization \vec{P} rotates lying in the (y,z)-plane.

$$\begin{vmatrix} P_x \\ P_y \\ P_z \end{vmatrix}_n = R^n \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ -\sin 2n\delta \\ \cos 2n\delta \end{vmatrix} \quad (a\gamma = \text{integer}) \quad (B.15a)$$

and applying the rotation (B.13a) to a spin parallel to the x-axis we re-obtain a continuous precession of the spin, by steps of δ per revolution, in the (x,z)-plane around a stable solution $\hat{n} \equiv (0, \pm 1, 0)$, i.e. directed as the y-axis at the solenoid entrance; namely:

$$\begin{vmatrix} P_x \\ P_y \\ P_z \end{vmatrix}_n = R^n \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} \cos 2n\delta \\ 0 \\ (-1)^k \sin 2n\delta \end{vmatrix} \quad (a\gamma = \frac{2k+1}{2} = \text{half-integer}) \quad (B.15b)$$

Thus we have two different precession axes, according whether $a\gamma$ is integer or half-integer; instead the spin tune remains the same, as discussed before.

In the case of the **Spin-Precession Lag or Advance** the spin rotation induced by the solenoid is no longer π but

$$\Phi_{\parallel} = \pi \pm \delta\phi \quad (B.16)$$

then the matrix $M(\pi)$, met before, must be replaced by the matrix $M(\Phi_{\parallel})$. Let us choose the spin-advance case; for $a\gamma = m = \text{integer}$, we have over a revolution:

$$M = [I \cos \frac{1}{2}(\pi + \delta\phi) - i\sigma_y \sin \frac{1}{2}(\pi + \delta\phi)](I \cos a\gamma\pi - i\sigma_z \sin a\gamma\pi)$$

or

$$M = (-I \sin \frac{\delta\phi}{2} - i\sigma_y \cos \frac{\delta\phi}{2}) \cos m\pi \quad (B.17)$$

which yields:

$$C_0 = (-1)^m \sin \frac{\delta\phi}{2}, \quad C_x = 0, \quad C_y = (-1)^m \cos \frac{\delta\phi}{2}, \quad C_z = 0 \quad (B.18a)$$

and

$$\sin \xi = \sin \frac{\delta\phi}{2} \quad (B.18b)$$

The insertion of eq. (B.17a) into the rotation (B.5) gives:

$$R = \begin{vmatrix} -\cos \delta\phi & 0 & \mp \sin \delta\phi \\ 0 & 1 & 0 \\ \pm \sin \delta\phi & 0 & -\cos \delta\phi \end{vmatrix} \quad (B.19)$$

where the upper sign, when appearing, refers to the spin-advance.

By applying the rotation (B.19) to a spin parallel to the x-axis we find a continuous precession of the spin, by steps of δ , in the (x,z)-plane around a stable solution $\hat{n} \equiv (0, \pm 1, 0)$, i.e. parallel to the y-axis, with a spin tune

$$\nu_s = \frac{1 - (-1)^m}{2} - \frac{1}{2} + 2\delta Q_V \quad (B.20)$$

in fact:

$$\begin{vmatrix} P_x \\ P_y \\ P_z \end{vmatrix}_n = R^n \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} \cos n\delta\phi \\ 0 \\ \sin n\delta\phi \end{vmatrix} \quad (B.21)$$

Notice how eqs. (B.19) and (B.21) almost coincide with eqs. (B.14b) and (B.15b); then by analogy we may state that

$$\delta\phi = \delta = 4\pi\delta Q_V \quad (B.22)$$

For $a\gamma = \frac{2k+1}{2}$ = half-integer, we have over a revolution:

$$M = -\sigma_x \sin \frac{\delta\phi}{2} - i\sigma_z \cos \frac{\delta\phi}{2} \sin(2k+1)\frac{\pi}{2} \quad (B.23)$$

which yields:

$$C_0 = 0, \quad C_x = (-1)^k \cos \frac{\delta\phi}{2}, \quad C_y = 0, \quad C_z = (-1)^k \sin \frac{\delta\phi}{2} \quad (B.24a)$$

and

$$\sin \xi = 1 \quad (\text{always!}) \quad (B.24b)$$

Therefore the rotation (B.5) becomes:

$$R = \begin{vmatrix} \cos \delta\phi & 0 & \mp \sin \delta\phi \\ 0 & -1 & 0 \\ \mp \sin \delta\phi & 0 & -\cos \delta\phi \end{vmatrix} \quad (B.25)$$

where the upper sign, where appearing, refers to the spin-advance.

By applying the rotation (B.25) to the most general polarization vector $\vec{P} \equiv (P_{0x}, P_{0y}, P_{0z})$, we obtain:

$$\begin{vmatrix} P_x \\ P_y \\ P_z \end{vmatrix}_1 = R \begin{vmatrix} P_{0x} \\ P_{0y} \\ P_{0z} \end{vmatrix} = \begin{vmatrix} P_{0x} \cos \delta\phi \mp P_{0z} \sin \delta\phi \\ -P_{0y} \\ \mp P_{0x} \sin \delta\phi - P_{0z} \cos \delta\phi \end{vmatrix} \quad (B.26)$$

and

$$\begin{vmatrix} P_x \\ P_y \\ P_z \end{vmatrix}_2 = R \begin{vmatrix} P_x \\ P_y \\ P_z \end{vmatrix}_1 = \begin{vmatrix} P_{0x} \\ P_{0y} \\ P_{0z} \end{vmatrix} \quad (B.27)$$

i.e. the polarization repeats itself every second revolution. Notice how the stable solution at the solenoid has components $n_x = (-1)^k \cos \frac{\delta\phi}{2}$, $n_y = 0$, $n_z = (-1)^k \sin \frac{\delta\phi}{2}$.

APPENDIX C: Damping(s)

By definition of Q-value we have:

$$Q = \omega \frac{U}{P} = -\frac{\omega U}{\frac{dU}{dt}} \quad (C.1)$$

with U =Stored Energy and $P = -\frac{dU}{dt}$ = Dissipated Power; then eq. (C.2) becomes

$$\frac{dU}{U} = -\frac{\omega}{Q} dt \quad (C.3)$$

whose integral is

$$U(t) = U_{\infty}(1 - e^{-\frac{t}{\tau_E}}) \quad (C.4)$$

being

$$\tau_E = \frac{2}{\tau} = \frac{Q}{\omega} = \frac{Q}{2\pi} T \quad (C.5)$$

Eq. (C.4) means that the stored energy, and then the oscillation amplitudes, cannot grow indefinitely but a saturation energy U_{∞} is reached.

Over one period T , we may set $dU = -\delta U_{Loss}$ and $dt \simeq T$, then eq. (C.3) becomes

$$\frac{\delta U_{Loss}}{U} \simeq \frac{\omega T}{Q} = \frac{2\pi}{Q} \quad (C.6)$$

For $t \ll \tau$ eq. (C.4) reduces to

$$U(t) \simeq U_{\infty} \frac{t}{\tau_E} \quad (\text{steady growth}) \quad (C.7)$$

Besides it is reasonable to assess that, over a time interval t , the stored energy $U(t)$ cannot exceed the energy-loss times the number of revolutions, i.e.

$$U(t) \geq |\delta U_{Loss}| \frac{t}{T} \quad (C.8)$$

then by combining eqs. (C.7) and (C.8) we obtain:

$$U_{\infty} \frac{t}{\tau_E} \simeq |\delta U_{Loss}| \frac{t}{T}$$

or

$$U_{\infty} \simeq \frac{\tau}{T} |\delta U_{Loss}| = \frac{Q}{2\pi} |\delta U_{Loss}| \quad (C.9)$$

on the other hand

$$U_{\infty} = N\delta U_{kick} \quad (C.10)$$

where N is the number of revolutions required for reaching the saturation value U_{∞} via a constant energy input equal to δU_{kick} .

Bearing in mind another definition of Q :

$$Q = \pi n \quad (C.11)$$

where n is the number of revolutions required for reducing the oscillation amplitude by a factor e ($=2.7182818\dots$), when the resonance is removed.

An easy combination of eqs. (C.9) and (C.10) gives us the following relation:

$$N\delta U_{kick} \geq \frac{n}{2}\delta U_{Loss}$$

or

$$N \leq \frac{n}{2} \quad (C.12)$$

since it is trivial that δU_{kick} must be bigger than or equal to δU_{Loss} .

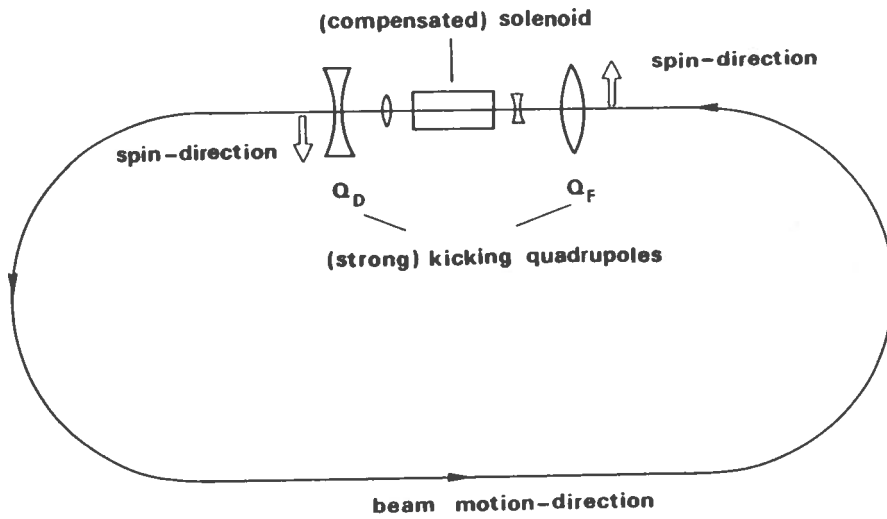


Fig. 1: Spin-Splitter configuration.

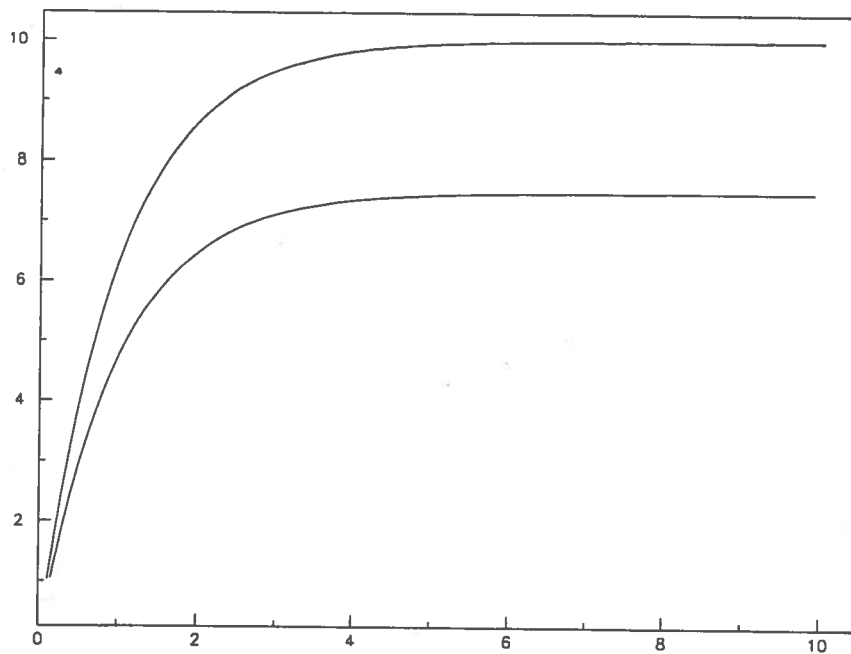


Fig. 2: saturation of the stored energy of a harmonic oscillator in resonant conditions and with viscous damping.

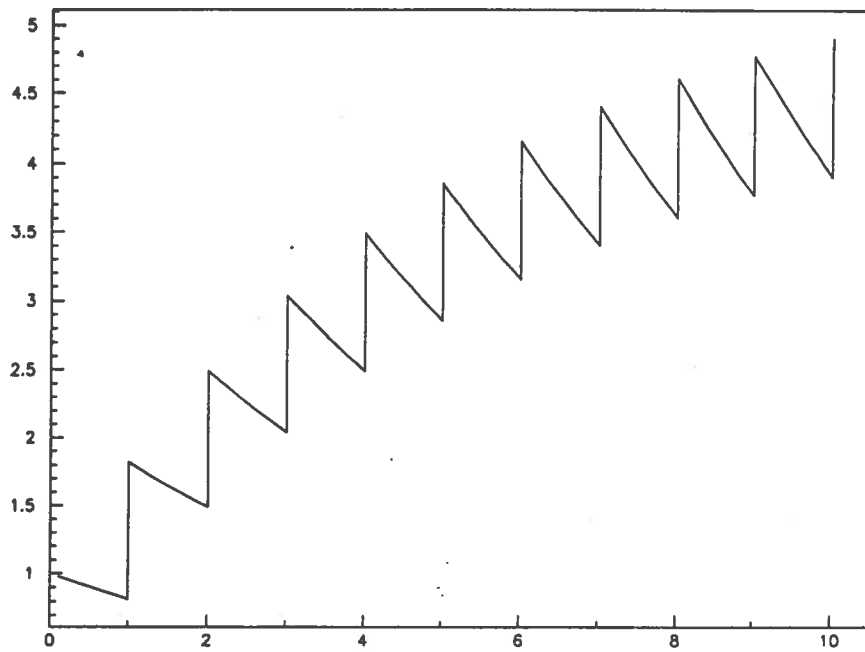


Fig. 3: graphical demonstration of how constant Stern-Gerlach kicks, followed by exponential decays proportional to the amplitude (viscous friction), may not succeed in enhancing the oscillation amplitudes.

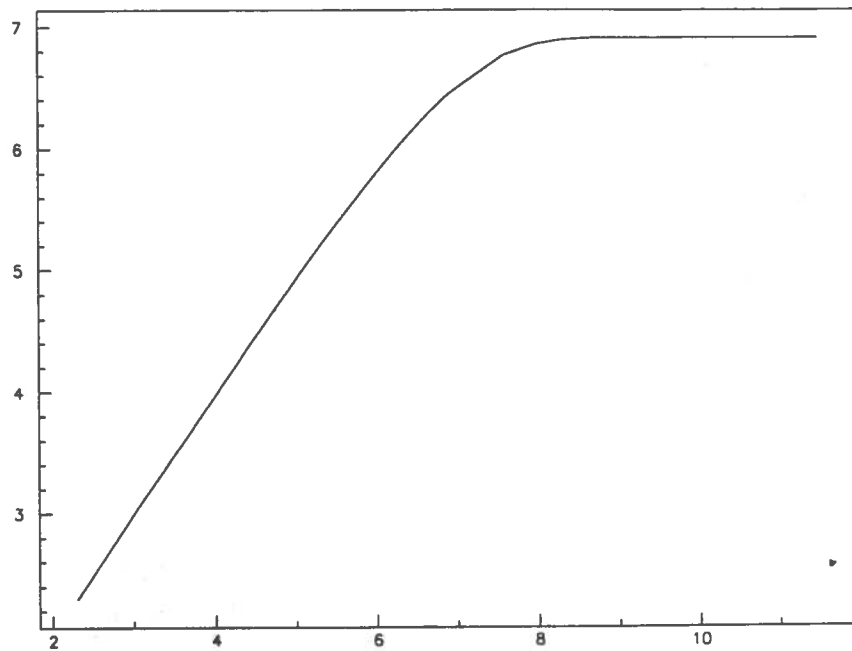


Fig. 4: representation of eq. (5.14) $a_N = a_0 e^{\frac{-T}{\tau}} \frac{1 - e^{-N\frac{T}{\tau}}}{1 - e^{-\frac{T}{\tau}}}$.

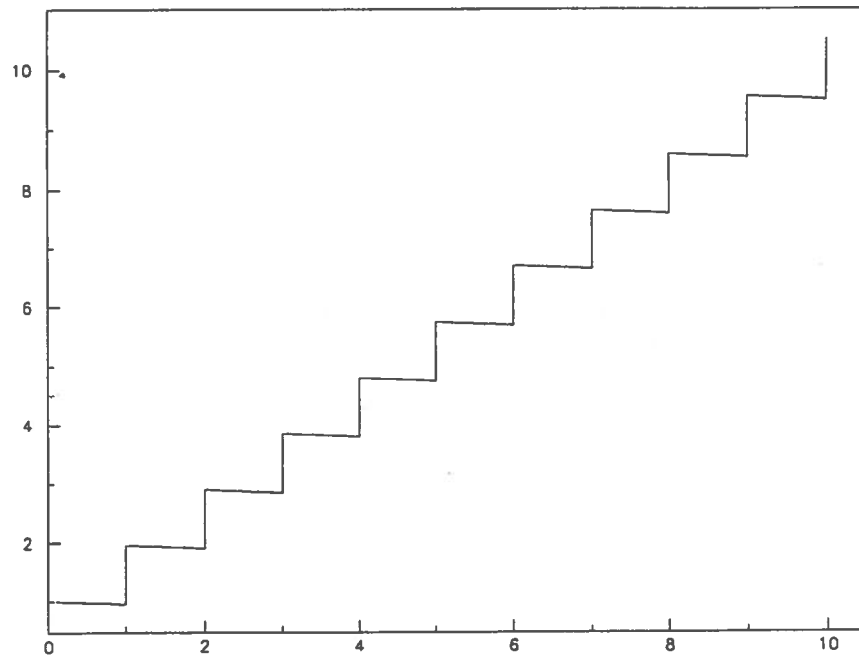


Fig. 5: constant Stern-Gerlach kicks, followed by decays with constant energy-loss (different kinds of scattering), accomplish the required amplitude growth.