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A COMPLETE STOCHASTIC DESCRIPTION OF INTERNAL TARGETS EFFECTS ON ELECTRON RINGS

M.Taiuti[1], V.Muccifora[2], C.Costa[1], M.Anghinolfi[1], N.Bianchi[2], P.Corvisiero[1], E.DeSanctis[2], G.Gervino[3], P.LeviSandri[2], V.Lucherini[2], V.I.Mokeev[4], E.Polli[2], G.Ricco[1], M.Ripani[1], P.Rossi[5], A.Zucchiatti[1]

[1] Dipartimento di Fisica dell'Università di Genova or I.N.F.N.-Sezione di Genova, I-16146, Genova, Italy.
[2] I.N.F.N.-Laboratori Nazionali di Frascati, P.O.Box 13, I-00044, Frascati , Italy.
[3] Dipartimento di Fisica dell'Università di Torino, Italy.
[4] Institute of Nuclear Physics, Moscow State University.
[5] Laboratorio di Fisica dell'Istituto Superiore di Sanità and I.N.F.N.-Sezione Sanità, I-00185, Roma, Italy.

ABSTRACT

The effects of internal targets on the beam accumulated in electron storage rings are studied with a linear stochastic model which describes the beam emittance and the beam phase space growth and provides simple formulae for the longitudinal and transverse beam emittance. The description of facilities with and without beam damping is carried out in a consistent way. The obtained results are comparable to those obtained with other models and numerical simulations.

1. INTRODUCTION

The interest to storage rings, and to recirculating facilities, has recently grown due to the use of very thin targets ($<10^{-7}$ g/cm^2) placed inside the ring: a technique that, in the case of electron beam, requires a detailed study of the beam-target interaction. In fact, while internal targets installed in anti-proton storage rings[1] and low energy electron storage rings[2] have shown to work with negligible perturbations, in the case of thick targets interacting with electron beams the radiative emissions could result in appreciable perturbations. So far, these interactions (Single Scattering, Møller Scattering and Bremsstrahlung Radiation) have been studied by means of Monte Carlo codes[3-7] or analytical models.[3,8] Both techniques give however an approximate description of the interaction. Numerical simulations are not able of giving exact predictions when the beam damping is present due to computers limitations.[7] Analytical formulations generally make use of the multiple scattering theory with the crude assumption that n crossings through a target of a given thickness ρx are equivalent to one crossing through a target of thickness $n\rho x$.

In this paper we describe by a linear stochastic model the static properties of an electron beam in presence of an internal target (mean lifetime, spreading, divergence), and the transients occurring after the injection. The theory of stochastic processes is well established and its foundations can be found in almost any text book of probability theory.[9]

In our approach we first describe (section 2) a stored beam without internal targets. Then we review (section 3) the Single Scattering, the Møller Scattering and the Bremsstrahlung Radiation as a background for the description of the beam in presence of internal targets. The beam cooling is introduced and the effects of the electron interaction with very thin internal targets are obtained as the solution of the stochastic betatron and synchrotron oscillations (section 4). The full results are summarized in section 5 where the expressions for the transverse and longitudinal beam emittance, together with their time dependence, are given. In particular, in the limit of no beam cooling, the transport matrix and the stochastic equation give the same results. Finally, in section 6 we compare our results with those from other calculations.

2. THE BEAM DESCRIPTION

The motion of a stored electron is described by means of the variables x, y (radial and vertical displacement from the equilibrium orbit s) and ε (deviation from the nominal energy).[10] Both the radial betatron oscillation x_β and the deviation x_ε due to electron energy losses affect the radial deviation x, while the radial betatron oscillation y_β only contributes to y.

In the absence of damping, the change in one revolution of the variable x_β (and, in the same way, y_β), due to the betatron oscillation can be described by means of the transport matrices:[3]

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{n+1} = \begin{pmatrix} \cos(2\pi\nu) & \beta_x \sin(2\pi\nu) \\ -\frac{1}{\beta_x}\sin(2\pi\nu) & \cos(2\pi\nu) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_n = T_1 \begin{pmatrix} x \\ x' \end{pmatrix}_n , \qquad (2.1)$$

where β_x is the betatron function, ν is the fractional part of the betatron number ν_x and $x' = \frac{dx}{ds}$ is the slope of the actual orbit relative to the standard one. The effect of n revolutions is described by the matrix:

$$T_n = T_1^n = \begin{pmatrix} \cos(2n\pi\nu) & \beta_x \sin(2n\pi\nu) \\ -\dfrac{1}{\beta_x}\sin(2n\pi\nu) & \cos(2n\pi\nu) \end{pmatrix} . \tag{2.2}$$

The damping effect can be taken into account by the equations:[10]

$$\frac{d^2x_\beta(t)}{dt^2} + 2\alpha_x \frac{dx_\beta(t)}{dt} + \omega_{\beta_x}^2 \; x_\beta(t) = 0 \ , \tag{2.3}$$

$$\frac{d^2\varepsilon(t)}{dt^2} + 2\alpha_\varepsilon \frac{d\varepsilon(t)}{dt} + \omega_0^2 \; \varepsilon(t) = 0 \ , \tag{2.4}$$

where α_x (α_ε) and ω_{β_x} (ω_0) are the damping coefficients for the betatron (synchrotron) oscillations and the betatron (synchrotron) frequency, respectively. In equation (2.3) the average value $\bar{\beta}_x = c\ \omega_{\beta_x}^{-1}$ of the betatron function is used and results obtained from eqs. (2.2) and (2.3) have to be corrected before comparisons. The energy deviation ε contributes to the spreading in the radial direction x as follows:[10]

$$x_\varepsilon = \eta \frac{\varepsilon}{E_0} \ , \tag{2.5}$$

being η the dispersion of the machine.

3. THE BEAM-TARGET INTERACTION: generality

3.1 The Single Scattering

The electrons in a storage ring cross many times the internal target, and their scattering by the nuclear screened Coulomb field increases the beam divergence. The interaction of the electrons with a target is described for a mean value of scatterings higher than 10 by the multiple scattering theory[11] and for a mean value lower than 1 by a theory that considers the probabilities of no scattering, and single and double scattering.[12] So far there are no theories for the scattering process if the mean value of scatterings is between 1 and 10.

The Single Scattering process is described by the Born screening angle χ_μ, and by the angle χ_c beyond which the screening is negligible.[3,11] For relativistic electrons, χ_μ and χ_c are given by:[3]

$$\chi_\mu = 4.215 * 10^{-3} \frac{\mu Z^{1/3}}{p} \qquad \text{[rad]} \ , \tag{3.1}$$

$$\chi_c = \left[0.157 \frac{Z(Z+1)}{A\ p^2} \rho x\right]^{1/2} \quad [\mathrm{rad}] \ , \tag{3.2}$$

where p is the electron momentum in mass units, μ a constant whose empirical value is 1.8 [11] and ρx the target thickness in gr/cm^2. The value of the mean number of scatterings Λ_S is given by:[3]

$$\Lambda_S = \frac{\chi_c^2}{\chi_\alpha^2} = 8837 \frac{\sqrt[3]{Z}\ (Z+1)}{\mu^2 N_0} N_t \ , \tag{3.3}$$

where χ_α is the modified screening angle $\chi_\alpha = \chi_\mu \left(1+0.04902\chi_\mu\right)$ and N_0 is the Avogadro number. N_t is the target thickness in atoms/cm^2.

The nature of scattering depends on the number of atoms and on the atomic number Z of the target. In the limit of an interaction probability $\Lambda \ll 1$ this process, and the others studied in this paper, can be considered Poisson-like and included in (2.2) (see Appendix A) and (2.3) (see section 4). In Fig.1 the target thickness, at which the contribution of two or more scatterings is lower than 10%, is reported as a function of Z. Below this limit, the effect of the scattering can be, however, better described by means of the projection of the scattering angle θ on the planes $\hat{zx}$ and $\hat{zy}$, the former containing the standard orbit and the latter being perpendicular to it. The projected scattering angle ψ is described by the distribution:[3]

$$f_p(\psi_\alpha, \Lambda_S)\ d\psi_\alpha = \frac{1}{1+\Lambda_S}\left[\delta(\psi_\alpha) + \frac{\Lambda_S}{2\left(1+\psi_\alpha^2\right)^{3/2}}\right] d\psi_\alpha \ , \tag{3.4}$$

having defined $\psi_\alpha = \psi/\chi_\alpha$. In this paper the mean value $\langle\psi\rangle$ and the rms $\langle\psi^2\rangle_k$ of the Single Scattering angle ψ were used. The first was obtained from the Single Scattering distribution function:

$$\langle\psi\rangle = \chi_\alpha \int_{-\infty}^{+\infty} \frac{\psi_\alpha}{2\left(1+\psi_\alpha^2\right)^{3/2}} d\psi_\alpha = 0 \ . \tag{3.5}$$

To obtain the second we used a Monte Carlo simulation to evaluate the total effects of k scatterings that add linearly and incoherent
ly. The Monte Carlo results show a continuous widening as a function of k (Fig. 2) with a standard deviation logarithmically divergent. The best fit is provided by the following expression

$$\langle\psi^2\rangle_k = k\left[0.754 + 0.551 * \ln(k)\right] \chi_\alpha^2 \ . \tag{3.6}$$

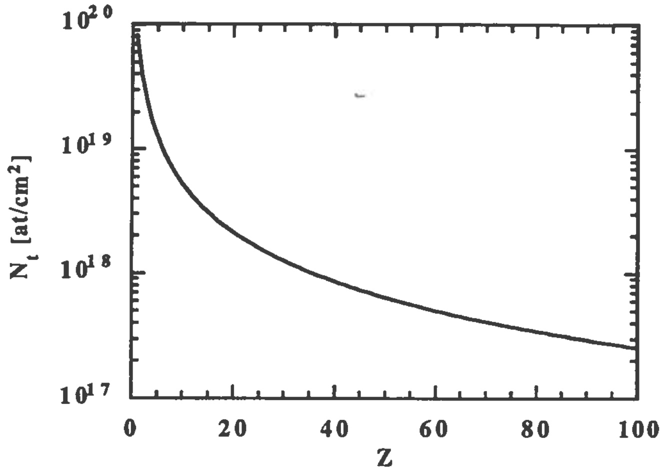

FIG. 1 - The target thickness N_t (in at/cm^2) corresponding to the upper limit for the Single Scattering treatment plotted as a function of the target Z.

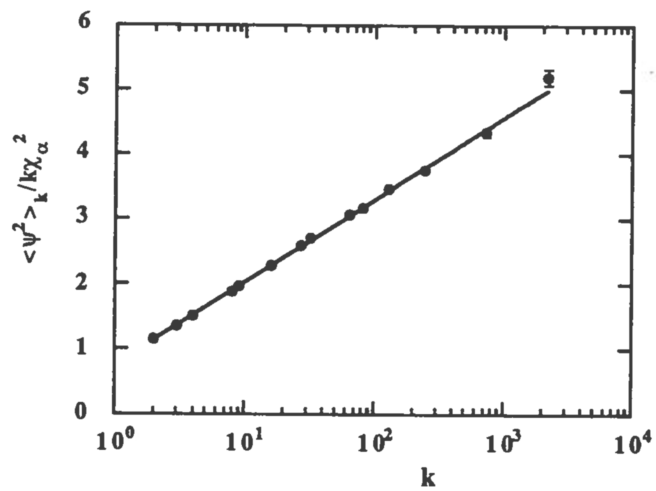

FIG. 2 - $<\psi^2>_k/k\chi_\alpha^2$ obtained by fitting with a gaussian distribution the results of a Monte Carlo simulation of the effect of k consecutive single scatterings. The line represents the best fit (equation (3.6)).

3.2 The Bremsstrahlung Radiation

The electron crossing the internal target loses energy mainly by bremsstrahlung radiation. The effect of the bremsstrahlung radiation is twofold: electrons that radiate photons with energy E_γ higher than the energy acceptance K of the ring are definitely lost and produce a reduction of the beam mean lifetime,[7] while electrons that radiate photons with energy lower than K oscillate in the momentum phase space. The average number $\Lambda_{B\xi}$ of emitted photons with energy $\kappa \leq E_\gamma \leq K$ in a single revolution is a function of the target thickness ρx: [13]

$$\Lambda_{B\xi} = \frac{\rho x}{X_0} \int_\kappa^K \left(\frac{4}{3} - \frac{4E_\gamma}{3E_0} + \frac{E_\gamma^2}{E_0^2} \right) \frac{dE_\gamma}{E_\gamma} \equiv \frac{\rho x}{X_0} i_{B\xi} \ , \qquad (3.7)$$

where the radiation length X_0 is related to the target parameters by:[14]

$$\frac{1}{X_0} = \frac{4\alpha r_0^2 N_0 Z^2}{A} f_{X_0}(Z) \ , \qquad (3.8)$$

where α is the fine structure constant and r_0 the classical electron radius.

This probability density, however, shows an infrared divergence: this is the reason why it is necessary to cut the distribution down to a fixed cut-off $\kappa > 0$ and define the average emitted photon number as a function of κ. However, we will show in section 4.2 that the results weakly depend on the cut-off value. The mean and rms values of the energy distribution are

$$<E_\gamma>_\xi = \frac{K}{i_{B\xi}}\left[\frac{4}{3}(1-\xi) - \frac{2}{3}\frac{K}{E_0}(1-\xi^2) + \frac{1}{3}\left(\frac{K}{E_0}\right)^2(1-\xi^3)\right] , \tag{3.9}$$

$$<E_\gamma^2>_\xi = \frac{K^2}{i_{B\xi}}\left[\frac{2}{3}(1-\xi^2) - \frac{4}{9}\frac{K}{E_0}(1-\xi^3) + \frac{1}{4}\left(\frac{K}{E_0}\right)^2(1-\xi^4)\right] , \tag{3.10}$$

having defined $\xi = \kappa/K$.

3.3 The Møller Scattering

The electron-electron scattering contributes both to the betatron and synchrotron oscillations. The cross section of the process,[15] the average number Λ_M of scatterings in one revolution with an energy transfer E lower than the machine energy acceptance K, the mean and rms values for the angular distribution for ultra-relativistic electrons at forward angles are given by the following equations (see appendix B for the distribution function and detailed evaluations):

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{Møller}} = 4r_0^2\left(\frac{m_0c^2}{E_0}\right)^2\frac{1}{\theta^4} , \tag{3.11}$$

$$\Lambda_M = 4r_0^2\left(\frac{m_0c^2}{E_0}\right)^2 ZN_t \int_{-\theta_2}^{\theta_2} f_p(\psi)d\psi \equiv 4r_0^2\left(\frac{m_0c^2}{E_0}\right)^2 ZN_t\, i_{M\psi} , \tag{3.12}$$

$$<\psi> = 0 , \tag{3.13}$$

$$<\psi^2> = \frac{1}{i_{M\psi}}\int_{-\theta_2}^{\theta_2}\psi^2 f_p(\psi)d\psi = \frac{\pi}{2i_{M\psi}}\ln\left(\frac{K}{I}\right) . \tag{3.14}$$

From (3.11) the expression for the energy dependence of the Møller cross section can also be derived:

$$\left.\frac{d\sigma}{d\omega}\right|_{\text{Møller}} = 2\pi r_0^2\, m_0c^2\frac{1}{E^2} ; \tag{3.15}$$

this provides a different expression for Λ_M, the average emitted energy $<E>$ and the rms $<E^2>$:

$$\Lambda_M = 2\pi r_0^2\, m_0c^2\, ZN_t \int_I^K \frac{\mathrm{d}E}{E^2} \equiv 2\pi r_0^2\, m_0c^2\, ZN_t\, i_{ME} \ , \tag{3.16}$$

$$<E> = \frac{1}{i_{ME}} \int_I^K \frac{\mathrm{d}E}{E} = \frac{1}{i_{ME}} \ln\left(\frac{K}{I}\right) \ , \tag{3.17}$$

$$<E^2> = \frac{1}{i_{ME}} \int_I^K \mathrm{d}E \approx \frac{K}{i_{ME}} \ . \tag{3.18}$$

4. THE BEAM-TARGET INTERACTION: Stochastic Treatment

4.1 The Transverse Beam Emittance

The Single Scattering, the Møller Scattering and the Bremsstrahlung Radiation act as external forces that induce on the electron beam forced harmonic oscillations whose strength and frequencies are functions of the relevant process probabilities Λ_S, Λ_M and $\Lambda_{B\xi}$. The betatron equation (2.3) can be modified in the following way:

$$\frac{\mathrm{d}^2 x_\beta(t)}{\mathrm{d}t^2} + 2\alpha_x \frac{\mathrm{d}x_\beta(t)}{\mathrm{d}t} + \omega_{\beta_x}^2\, x_\beta(t) = \mathrm{F}_\beta(t, \Lambda_S, \Lambda_M) \ , \tag{4.1}$$

Since the interaction time is negligible compared to the revolution time, the interaction can be treated as instantaneous and the function F_β becomes an incoherent sum of k different single scatterings and h Møller scatterings:

$$\mathrm{F}_\beta(t,\Lambda_S,\Lambda_M) = \sum_{i=1}^{k} S_i\, \delta(t-t_i) + \sum_{j=1}^{h} M_j\, \delta(t-t'_j) \qquad t_1 < \ldots < t_k \leq t$$

$$t'_1 < \ldots < t'_h \leq t \ , \tag{4.2}$$

where the unitary processes are weighted with the coefficient S_i and M_j that have the dimension $\left[\mathrm{LT}^{-1}\right]$ and are related to the variation of the direction of the electron velocity. In the hypothesis of small angles, S_i and M_j are simply related to the scattering angle through the light speed c:

$$\mathrm{F}_\beta(t, \Lambda_S, \Lambda_M) = c\left[\sum_{i=1}^{k} \psi_i\, \delta(t-t_i) + \sum_{j=1}^{h} \psi_j\, \delta(t-t'_j)\right] \qquad t_1 < \ldots < t_k \leq t$$

$$t'_1 < \ldots < t'_h \leq t \ . \tag{4.3}$$

As shown in Appendix C, the properties of the process $F_\beta(t, \Lambda_S, \Lambda_M)$ can be derived from those of the stochastic staircase process $x(t,\Lambda)$ after the following substitutions in (C.12) and (C.13):

$$\lambda \rightarrow c^2 \left[<\psi^2>_S \lambda_S + <\psi^2>_M \lambda_M\right] , \qquad (4.4)$$

$$\lambda^2 \rightarrow c^2 \left[<\psi>_S^2 \lambda_S^2 + <\psi>_M^2 \lambda_M^2\right] , \qquad (4.5)$$

obtaining the following expressions for the mean, the autocorrelation and the power spectrum:

$$<F_\beta(t)> = c \left[<\psi>_S \lambda_S + <\psi>_M \lambda_M\right] = 0 , \qquad (4.6)$$

$$R_{F_\beta F_\beta}(\tau) = c^2 \left[<\psi^2>_S \lambda_S + <\psi^2>_M \lambda_M\right] \delta(\tau) , \qquad (4.7)$$

$$S_{F_\beta F_\beta}(\omega) = c^2 \left[<\psi^2>_S \lambda_S + <\psi^2>_M \lambda_M\right] . \qquad (4.8)$$

In this way the betatron equation (2.3) has become a differential stochastic equation and the statistical properties of x_β can be obtained from those of $F_\beta(t, \Lambda_S, \Lambda_M)$ by means of the system function:[9]

$$H(i\omega) = \frac{1}{(i\omega)^2 + 2\alpha_x (i\omega) + \omega_{\beta_x}^2} . \qquad (4.9)$$

Usually the damping time $1/\alpha_x$ is orders of magnitude larger than the betatron oscillation period ω_{β_x} [10] and, as a consequence, the solution of the stochastic equation (4.1) is a damped oscillation[9] and the process x_β is stationary with the following overall values for mean, power spectrum and autocorrelation:

$$<x_\beta(t)> = \frac{1}{\omega_{\beta_x}^2} <F_\beta(t)> = 0 , \qquad (4.10)$$

$$S_{x_\beta x_\beta}(\omega) = | H(i\omega)|^2 S_{F_\beta F_\beta}(\omega) = \frac{c^2 \left(<\psi^2>_S \lambda_S + <\psi^2>_M \lambda_M\right)}{\left(\omega^2 - \omega_{\beta_x}^2\right)^2 + (2\alpha_x)^2 \omega^2} , \qquad (4.11)$$

$$R_{x_\beta x_\beta}(\tau) = \frac{c^2 \left(<\psi^2>_S \lambda_S + <\psi^2>_M \lambda_M\right)}{2(2\alpha_x) \omega_{\beta_x}^2} \left[\cos\left(\omega_1 \tau\right) + \frac{\alpha_x}{\omega_1} \sin\left(\omega_1 \tau\right)\right] e^{-\alpha_x \tau} , \qquad (4.12)$$

where

$$\omega_1 = \sqrt[2]{\omega_{\beta_x}^2 - \alpha_x^2} \approx \omega_{\beta_x} . \qquad (4.13)$$

$x_\beta(t)$ is then described by a normal distribution with zero mean value and standard deviation:

$$\sigma_{x\beta}^{2} = R_{x\beta x\beta}(0) = \frac{\bar{\beta}_x^{2}}{2\alpha_x}\left(\frac{m_0c^2}{E_0}\right)^2 f_\beta(Z)\,\frac{r_0^2 N_t}{T_0} \,, \tag{4.14}$$

where we have defined:

$$f_\beta(Z) = Z(Z+1)\left[1.26+0.48\left(\frac{\lambda_S}{\alpha_{x/y}}\right)^{0.7}\right]+\pi Z \ln\left(\frac{K}{I}\right) \qquad \frac{\lambda_S}{\alpha_{x/y}} < 4 \,, \tag{4.15a}$$

$$f_\beta(Z) = Z(Z+1)\left[1.26+0.92 * \ln\left(\frac{\lambda_S}{\alpha_{x/y}}\right)\right]+\pi Z \ln\left(\frac{K}{I}\right) \qquad \frac{\lambda_S}{\alpha_{x/y}} \geq 4 \,. \tag{4.15b}$$

The time dependence of the variance is given by:[9]

$$\sigma_{x\beta}^{2}(t) = \sigma_{x\beta}^{2}\left(1 - e^{-2\alpha_x t}\right) \,. \tag{4.16}$$

Equation (4.16) describes the spread of the electron beam at the time the internal target is inserted. The variance of the distribution x_β reaches exponentially the equilibrium value with a time constant equal to half the damping time of the betatron oscillation.

From the distribution of the process x\s\do4(β) the distribution for the process x'\s\do4(β) was obtained:[9] it is a normal distribution with mean value zero and variance:

$$\sigma_{x'\beta}^{2} = R_{x'\beta x'\beta}(0) = \frac{1}{\bar{\beta}_x^{2}} R_{x\beta x\beta}(0) \,, \tag{4.17}$$

and the same time dependence as (4.16).

It is to be noted that in equation (4.14) the average value $\bar{\beta}_x$ of the betatron function is present. Then the stochastic model provides results averaged along the ring, and local effects can be computed rescaling the results by the factor $\beta_x/\bar{\beta}_x$.

4.2 The Longitudinal Beam Emittance

The synchrotron equation (2.4) is modified due to the presence of the Møller Scattering and the Bremsstrahlung Radiation processes as follows:

$$\frac{d^2\varepsilon(t)}{dt^2} + 2\alpha_\varepsilon \frac{d\varepsilon(t)}{dt} + \omega_0^2\, \varepsilon(t) = F_\varepsilon(t, \Lambda_{B\xi}, \Lambda_M) \,. \tag{4.18}$$

Since the interaction time is negligible compared to the revolution time, the interaction can be treated as instantaneous and the function F_ε becomes an incoherent sum of k different Bremsstrahlung radiations and h Møller scatterings:

$$F_\varepsilon(t, \Lambda_{B\xi}, \Lambda_M) = \sum_{i=1}^{k} B_i\, \delta(t-t_i) + \sum_{j=1}^{h} M_j\, \delta(t-t'_j) \qquad t_1 < \ldots < t_k \leq t$$

$$t'_1 < \ldots < t'_h \leq t \ , \tag{4.19}$$

where B_i and M_j, similarly to the betatron equation, are the rate at which the electron loses energy:

$$F_\varepsilon(t, \Lambda_{B\xi}, \Lambda_M) = \omega_0 \left[\sum_{i=1}^{k} E_{\gamma_i}\, \delta(t-t_i) + \sum_{j=1}^{h} E_j\, \delta(t-t'_j) \right] \qquad t_1 < \ldots < t_k \leq t$$

$$t'_1 < \ldots < t'_h \leq t \ . \tag{4.20}$$

If κ is chosen large enough to make $\Lambda_{B\xi} < 1$, the contribution of the Bremsstrahlung Radiation to $F_\varepsilon(t, \Lambda_{B\xi}, \Lambda_M)$ becomes a Poisson process with $\lambda_{B\xi} = \Lambda_{B\xi}/T_0$ and the properties of the process $F_\varepsilon(t, \Lambda_{B\xi}, \Lambda_M)$ can be derived from those of the stochastic staircase process $x(t,\Lambda)$ after the following substitutions in (C.12) and (C.13):

$$\lambda \rightarrow \omega_0^2 \left[<E_\gamma^2>_\xi \lambda_{B\xi} + <E^2>\lambda_M \right] \ , \tag{4.21}$$

$$\lambda^2 \rightarrow \omega_0^2 \left[<E_\gamma>_\xi^2 \lambda_{B\xi}^2 + <E>^2\lambda_M^2 \right] \ . \tag{4.22}$$

The expressions for the mean, the autocorrelation and the power spectrum result to be:

$$<F_\varepsilon(t)> = \omega_0 \left[<E_\gamma>_\xi \lambda_{B\xi} + <E>\lambda_M \right] \ , \tag{4.23}$$

$$R_{F_\varepsilon F_\varepsilon}(\tau) = \omega_0^2 \left[<E_\gamma>_\xi^2 \lambda_{B\xi}^2 + <E>^2\lambda_M^2 \right] + \omega_0^2 \left[<E_\gamma^2>_\xi \lambda_{B\xi} + <E^2>\lambda_M \right] \delta(\tau) \ , \tag{4.24}$$

$$S_{F_\varepsilon F_\varepsilon}(\omega) = 2\pi\omega_0^2 \left[<E_\gamma>_\xi^2 \lambda_{B\xi}^2 + <E>^2\lambda_M^2 \right] \delta(\omega) + \omega_0^2 \left[<E_\gamma^2>_\xi \lambda_{B\xi} + <E^2>\lambda_M \right] . \tag{4.25}$$

The products $<E_\gamma>_\xi \lambda_{B\xi}$ and $<E_\gamma^2>_\xi \lambda_{B\xi}$, that appear in these equations, depend weakly on the cut-off value and, in the limit $\xi \ll 1$, are equal to:

$$<E_\gamma>\lambda_B \approx \frac{4}{3} \frac{\rho x}{T_0 X_0} K \ , \tag{4.26}$$

$$<E_\gamma^2>\lambda_B \approx \frac{2}{3}\frac{\rho x}{T_0 X_0} K^2 \ . \tag{4.27}$$

From the stochastic differential equation all the properties of the process ε can be deduced: the mean value

$$<\varepsilon(t)> \ = \frac{1}{\omega_0^2} <F_\varepsilon(t)> = \frac{<E_\gamma>\lambda_B + <E>\lambda_M}{\omega_0} \ , \tag{4.28}$$

and the autocorrelation function

$$R_{\varepsilon\varepsilon}(\tau) \ = \frac{<E_\gamma>^2 \lambda_B^2 + <E>^2 \lambda_M^2}{\omega_0^2} +$$

$$+ \frac{1}{2\alpha_\varepsilon}\left(\frac{<E_\gamma^2>\lambda_B + <E^2>\lambda_M}{2}\right)\left[\cos\left(\omega_2\tau\right) + \frac{\alpha_\varepsilon}{\omega_2}\sin\left(\omega_2\tau\right)\right] e^{-\alpha_\varepsilon\tau} \ , \tag{4.29}$$

where

$$\omega_2 = \sqrt[2]{\omega_0^2 - \alpha_\varepsilon^2} \approx \omega_0 \ . \tag{4.30}$$

According to the upper limit for the target thickness N_t, reported in Fig. 1, the mean value $<\varepsilon(t)>$ is less than $10^{-4}\,E_0$ and then negligible with respect to the ring energy resolution. The $\varepsilon(t)$ distribution is then normal with zero mean value and standard deviation

$$\sigma_\varepsilon^2 \ = R_{\varepsilon\varepsilon}(0) = \frac{1}{2\alpha_\varepsilon} f_\varepsilon(Z)\, K^2 \frac{r_0^2 N_t}{T_0} \ , \tag{4.31}$$

with a time dependence

$$\sigma_\varepsilon^2\,(t) = \sigma_\varepsilon^2\left(1 - e^{-2\alpha_\varepsilon t}\right) \ , \tag{4.32}$$

where we have defined

$$f_\varepsilon(Z) = \frac{4}{3}\alpha Z^2 f_{X_0}(Z) + \pi\, Z \frac{m_0 c^2}{K} \ . \tag{4.33}$$

At the equilibrium the energy resolution is given by

$$\sigma_{\varepsilon t}^2 \ = \sigma_{\varepsilon sl}^2 \ + \frac{1}{2\alpha_\varepsilon} f_\varepsilon(Z)\, K^2 \frac{r_0^2 N_t}{T_0} \ , \tag{4.34}$$

being $\sigma_{\varepsilon sl}^2$ the intrinsic spread due to the Synchrotron Light effect.[10]

5. THE BEAM EMITTANCE GROWTH

Trough equation (2.5) the Bremsstrahlung and Møller Scattering make the spread in the radial direction a gaussian distribution with zero mean value and standard deviation:[9]

$$\sigma_{x_\varepsilon}^2(t) = \left(\frac{\bar{\eta}}{E_0}\right)^2 \sigma_{x_\varepsilon}^2 \left(1 - e^{-2\alpha_\varepsilon t}\right) . \qquad (5.1)$$

This effect adds incoherently to the previous Single Scattering and Møller Scattering effects: the electron stored beam, in presence of an internal target, increases its emittance to a new equilibrium values σ_{x_t}, $\sigma_{x'_t}$, σ_{y_t} and $\sigma_{y'_t}$ resulting from the quadratic sums

$$\sigma_{x_t}^2 = \sigma_{x_{sl}}^2 + \left[\frac{\bar{\beta}_x^2}{2\alpha_x}\left(\frac{m_0c^2}{E_0}\right)^2 f_\beta(Z) + \frac{\bar{\eta}^2}{2\alpha_\varepsilon}\left(\frac{K}{E_0}\right)^2 f_\varepsilon(Z)\right]\frac{r_0^2 N_t}{T_0} , \qquad (5.2)$$

$$\sigma_{x'_t}^2 = \sigma_{x'_{sl}}^2 + \frac{1}{2\alpha_x}\left(\frac{m_0c^2}{E_0}\right)^2 f_\beta(Z)\,\frac{r_0^2 N_t}{T_0} , \qquad (5.3)$$

$$\sigma_{y_t}^2 = \sigma_{y_{sl}}^2 + \frac{\bar{\beta}_y^2}{2\alpha_y}\left(\frac{m_0c^2}{E_0}\right)^2 f_\beta(Z)\,\frac{r_0^2 N_t}{T_0} , \qquad (5.4)$$

$$\sigma_{y'_t}^2 = \sigma_{y'_{sl}}^2 + \frac{1}{2\alpha_y}\left(\frac{m_0c^2}{E_0}\right)^2 f_\beta(Z)\,\frac{r_0^2 N_t}{T_0} , \qquad (5.5)$$

where $\sigma_{x_{sl}}^2$, $\sigma_{x'_{sl}}^2$, $\sigma_{y_{sl}}^2$ and $\sigma_{y'_{sl}}^2$ are the intrinsic spreads due to the Synchrotron Light effect.[10]

The radiofrequency plays in this model a central role being the damping constants α_ε and $\alpha_{x/y}$ directly related to the field gradient inside the radiofrequency cavity itself. The beam time evolution, in absence of any damping effects, has been derived from eqs. (4.16) and (4.32) in the limit $\alpha_\varepsilon \to 0$ and $\alpha_{x/y} \to 0$:

$$\sigma_\varepsilon^2(t) = \sigma_{\varepsilon sl}^2 + f_\varepsilon(Z)\, K^2 \frac{r_0^2 N_t}{T_0}\, t , \qquad (5.6)$$

$$\sigma_{x_t}^2(t) = \sigma_{x_{sl}}^2 + \left[\bar{\beta}_x^2 \left(\frac{m_0c^2}{E_0}\right)^2 f_\beta(Z) + \bar{\eta}^2\left(\frac{K}{E_0}\right)^2 f_\varepsilon(Z)\right]\frac{r_0^2 N_t}{T_0}\, t , \qquad (5.7)$$

$$\sigma_{x'_t}^2(t) = \sigma_{x'_{sl}}^2 + \left(\frac{m_0c^2}{E_0}\right)^2 f_\beta(Z)\,\frac{r_0^2 N_t}{T_0}\, t , \qquad (5.8)$$

$$\sigma_{y_t}^2(t) = \sigma_{y_{sl}}^2 + \bar{\beta}_y^2 \left(\frac{m_0c^2}{E_0}\right)^2 f_\beta(Z) \frac{r_0^2 N_t}{T_0} t\,, \tag{5.9}$$

$$\sigma_{y'_t}^2(t) = \sigma_{y'_{sl}}^2 + \left(\frac{m_0c^2}{E_0}\right)^2 f_\beta(Z) \frac{r_0^2 N_t}{T_0} t\,. \tag{5.10}$$

Equations (4.15) and (4.33) offer the possibility of evaluating the relative weight of the three studied interaction channels; the Møller contribution can be considered as a χ fraction adding to the Bremsstrahlung and Single Scattering contributions as follows:

$$\sigma_\varepsilon = \sigma_B \left(1+\chi_\varepsilon\right)\,, \tag{5.11}$$

$$\sigma_\beta = \sigma_S \left(1+\chi_\beta\right)\,. \tag{5.12}$$

In Figs. 3-4 the contribution $\chi_{\varepsilon/\beta}$ of Møller Scattering is reported; in both cases the Møller Scattering becomes sizeable only for low Z targets and dominates the spatial resolution only at small $\dfrac{\lambda_S}{\alpha_{x/y}}$ values.

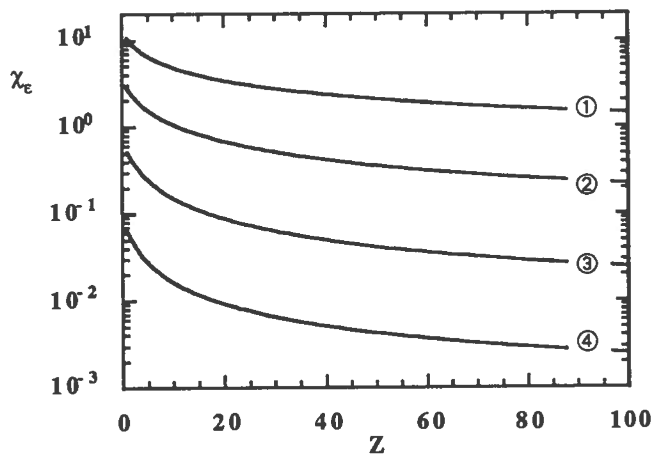

FIG. 3 - Møller contribution χ_ε to energy resolution compared with Bremsstrahlung for different K values:
① $K = 10^{-1}$ MeV, ② $K = 1$ MeV,
③ $K = 10$ MeV, ④ $K = 10^2$ MeV.

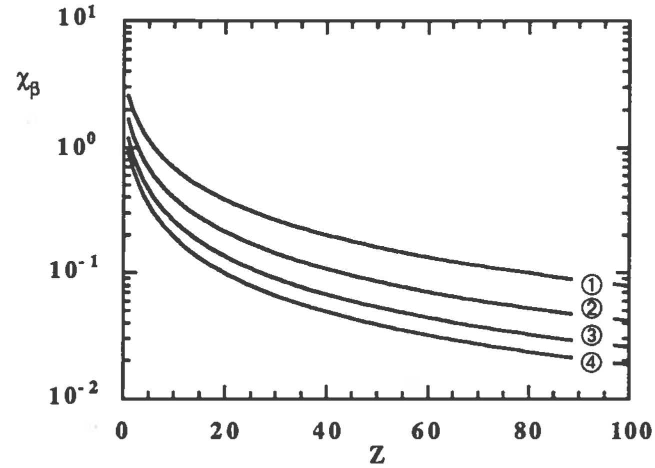

FIG. 4 - Møller contribution χ_β to betatron oscillation compared with Single Scattering for different values of the average number of Single Scatterings in one damping time:
① $\dfrac{\lambda_S}{\alpha_{x/y}} = 1$, ② $\dfrac{\lambda_S}{\alpha_{x/y}} = 10$,
③ $\dfrac{\lambda_S}{\alpha_{x/y}} = 10^2$, ④ $\dfrac{\lambda_S}{\alpha_{x/y}} = 10^3$.

6. COMPARISON WITH OTHER SIMULATIONS

The emittance calculations performed so far, take into account only the Single Scattering effects. In particular we have focused onto those of B.Craft and C.F.Williamson[3] and Muccifora et al.[7] which appear to be the most complete in several aspects.

B.Craft and C.F.Williamson[3] report Monte Carlo and analytical evaluations for the phase space growth P defined as the area of the ellipse in the x-θ_x plane containing 90% of the stored beam. From our model P is computed as the area of the integration path for the phase space distribution

$$f_{x_\beta x'_\beta}(t) = \frac{1}{2\pi\sigma_{x_\beta}\sigma_{x'_\beta}}\, e^{-x_\beta^2/2\sigma_{x_\beta}^2}\, e^{-x'^2_\beta/2\sigma_{x'_\beta}^2}\, dx_\beta\, dx'_\beta \;, \tag{6.1}$$

that corresponds to the integral value 0.9 and is given by:

$$P(t) = 4.6\pi \frac{\sigma_{x_\beta}^2(t)}{\beta_x} \;. \tag{6.2}$$

In Fig. 5 the time dependence of the phase space growth computed for a 2 GeV electron beam, an electron machine with β=7 m and no damping and assuming a 43.5 $\mu g/cm^2$ Hydrogen internal target[3] is in good agreement with our $P(t)$ value derived from (6.2).

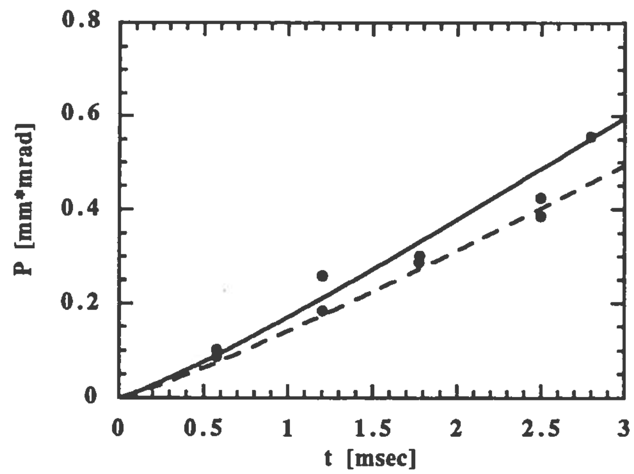

FIG. 5 - Comparison of the phase space growth for a 2 GeV stored electron beam with a 43.5 $\mu g/cm^2$ Hydrogen internal target. Dotted points and dashed line, obtained in ref. 3, respectively, with a Monte Carlo simulation and an analytical evaluation that takes into account the multiple scattering effect, represent the phase space area containing 90% of the stored beam. Continuous line corresponds to $P(t)$ obtained with (6.2).

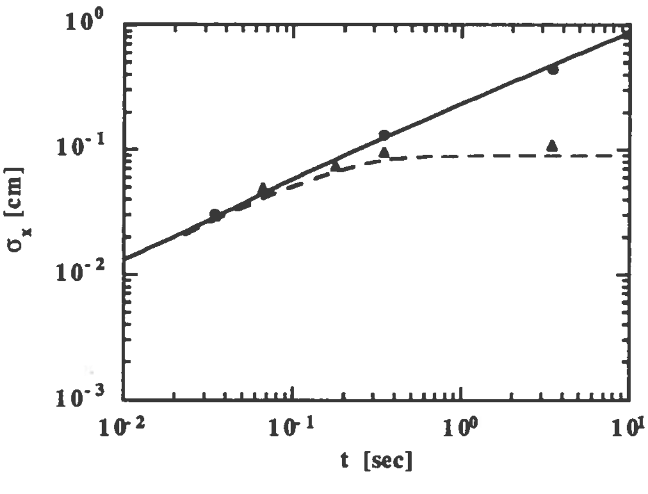

FIG. 6 - Comparison of the radial beam growth with (dashed line) and without (continuous line) damping with results obtained with the code of ref. 7 in the hypothesis of a 0 emittance beam.

We have also compared the present analytical results# with those obtained with our simulation code[7] including or not the damping mechanism. In Fig. 6 the results are reported for a 500 MeV electron beam interacting in the ADONE Storage Ring with a 10 ng/cm^2 Argon internal target with the hypothesis of initial phase space equal to zero. Triangles and dotted points correspond to the Monte Carlo simulation with and without damping. The dashed line is the result of equation (4.16) and the continuous line is the results of the same equation evaluated in the limit $\alpha_x \to 0$. As it is seen there is a complete agreement between the two calculation techniques.

7. CONCLUSIONS

We have proposed a linear stochastic model that, through the modification of the betatron and synchrotron equation, is able to describe the emittance and resolution growth due to an internal target in electron storage rings with or without energy cooling. We have evaluated in a consistent way the contribution to the beam resolution and emittance of intrinsically diverging processes, like the Single Scattering.

In particular, this model shows that the Single Scattering, the Møller Scattering and the Bremsstrahlung Energy Loss result in the continuous growth of the electron energy resolution and beam emittance up to values which are related to the target thickness and machine parameters by simple equations. These values, being related to the betatron and synchrotron damping times, depend on E_0^{-5} while the growth rate depends on E_0^{-2}. The beam energy resolution increases linearly with the target thickness while the beam emittance increases more than linearly due to the presence of diverging tails in the Single Scattering. A similar behavior is expected when the damping is not taken into account. The energy resolution is a function of $t^{1/2}$ while the emittance increases more than linearly with the elapsed time.

The model agrees with other theories and Monte Carlo calculations but provides relatively simple analytical formulae that allow a fast and valuable description of many experimental situations.

APPENDIX A - The transport matrices and the Single Scattering

At the interaction point the Single Scattering acts only on the direction x' of the electron velocity, leaving the deviation ε from the nominal energy and deviation x from the standard orbit unaffected. Using the matrix description, its effect is described by means of the vector $\begin{pmatrix} 0 \\ \psi \end{pmatrix}$ where ψ is the projected scattering angle. After n turns the Single Scattering effect becomes

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In a previous paper [16] we gave a wrong result because the Bremsstrahlung contribution to energy resolution and phase space growth was overestimated due to an erroneous evaluation of equations (3.10) and (3.13).

$$\begin{pmatrix} x \\ x' \end{pmatrix}_n = T_n \begin{pmatrix} 0 \\ \psi \end{pmatrix} . \qquad \text{(A.1)}$$

Since the scattering angle ψ is very small, two or more scatterings that occurs before the n-th turn add themselves linearly and the overall effect can be described as

$$\begin{pmatrix} x \\ x' \end{pmatrix}_n = \sum_{\substack{\text{number of} \\ \text{scatterings}}} T_{n-i} \begin{pmatrix} 0 \\ \psi_i \end{pmatrix} \qquad i < n , \qquad \text{(A.2)}$$

or

$$x_n = \beta_x \sum_{\substack{\text{number of} \\ \text{scatterings}}} \psi_i \sin\big(2(n-i)\pi \nu\big) \qquad i < n , \qquad \text{(A.3)}$$

$$x'_n = \sum_{\substack{\text{number of} \\ \text{scatterings}}} \psi_i \cos\big(2(n-i)\pi \nu\big) \qquad i < n , \qquad \text{(A.4)}$$

where i indicates the turn at which the electron scatters on the target. The effect of the betatron oscillation, when the damping is switched off, is a phase contribution that reduces the variance of the x and x' distribution. The mean value and the standard deviation, after k scatterings and n turns, are given by:

$$\langle x_n \rangle_k = \beta_x \sum_{i=1}^{k} \langle \psi_i \rangle \, \langle \sin\big(2(n-i)\pi \nu\big) \rangle = 0 , \qquad \text{(A.5)}$$

$$\langle x'_n \rangle_k = \sum_{i=1}^{k} \langle \psi_i \rangle \, \langle \cos\big(2(n-i)\pi \nu\big) \rangle = 0 , \qquad \text{(A.6)}$$

$$\sigma_x^2\,(n,k) = \langle x_n^2 \rangle_k = \beta_x^2 \sum_{i=1}^{k} \langle \psi_i^2 \rangle \, \langle \sin^2\big(2(n-i)\pi \nu\big) \rangle = \frac{1}{2} \beta_x^2 \langle \psi^2 \rangle_k , \qquad \text{(A.7)}$$

$$\sigma_{x'}^2\,(n,k) = \langle x'^2_n \rangle_k = \sum_{i=1}^{k} \langle \psi_i^2 \rangle \, \langle \cos^2\big(2(n-i)\pi \nu\big) \rangle = \frac{1}{2} \langle \psi^2 \rangle_k . \qquad \text{(A.8)}$$

Generally, considering N_e electrons crossing n times the internal target, the fraction f_k of electrons that interacts k times is given by the Poisson formula:

$$f_k = \frac{\big(\Lambda_S n\big)^k e^{-\Lambda_S n}}{k!} , \qquad \text{(A.9)}$$

and, describing the interaction as a function of the revolution time T_0, the elapsed time $t = nT_0$ and the mean rate of scattering $\lambda_S = \Lambda_S/T_0$, the total $\sigma_x^2(t)$ is given by the quadratic sum of the $\sigma_x^2(t,k)$ weighted by f_k:

$$\sigma_x^2(t) = \sum_{k=0}^{\infty} \sigma_x^2(t,k) f_k = \frac{1}{2}\beta_x^2 \langle\psi^2\rangle_S \lambda_S t \quad , \tag{A.10}$$

$$\sigma_{x'}^2(t) = \sum_{k=0}^{\infty} \sigma_x^2(t,k) f_k = \frac{1}{2}\langle\psi^2\rangle_S \lambda_S t \quad . \tag{A.11}$$

The value of $\langle\psi^2\rangle_S\lambda_S t$ was derived from (3.7) considering that $\lambda_S t$ is the average value k of scattering occurring in the time t. The value of $\langle\psi^2\rangle_S$ has been numerically evaluated and fitted with the following expressions (see Fig. 7):

$$\langle\psi^2\rangle_S = \left[0.754 + 0.289\left(\lambda_S t\right)^{0.7}\right]\chi_\alpha^2 \qquad \lambda_S t < 4 \ , \tag{A.12a}$$

$$\langle\psi^2\rangle_S = \left[0.754 + 0.551 \cdot \ln\left(\lambda_S t\right)\right]\chi_\alpha^2 \qquad \lambda_S t \geq 4 \ , \tag{A.12b}$$

logarithmically increasing with the elapsed time.

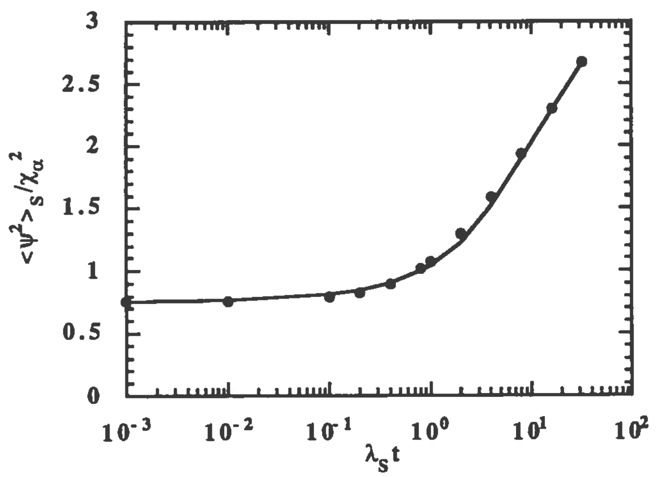

FIG. 7 - $\langle\psi^2\rangle_S/\chi_\alpha^2$ obtained from a numerical evaluation of the gaussian distribution. The line represents the best fit reported as equation (A.12).

APPENDIX B - Møller Scattering: Distribution function of the projected scattering angle.

The angular dependence of the Møller cross section for relativistic electrons and forward angles is:

$$f(\theta,\phi)\theta d\theta d\phi \propto \frac{d\theta d\phi}{\theta^3} \quad , \tag{B.1}$$

and can be written in term of the projected angle ψ, related to the θ angle by $\psi = \theta\cos(\phi)$, as:

$$f(\psi,\phi)d\psi d\phi = \frac{\cos^2(\phi)}{\psi^3} d\psi d\phi \ . \qquad \text{(B.2)}$$

The projected angle ψ is described by the distribution function $f_p(\psi)d\psi$ that, in the interval $\theta_1 \leq \theta \leq \theta_2$ corresponding to an energy transfer less then the machine energy acceptance K, can be extracted from (B.2) with an integration over ϕ as follows:

$$f_p(\psi)d\psi = \frac{2d\psi}{\psi^3}\left(\int_0^{\mathrm{acos}(\psi/\theta_2)} \cos^2(\phi)d\phi - \int_0^{\mathrm{acos}(\psi/\theta_1)} \cos^2(\phi)d\phi\right)$$

$$= \frac{d\psi}{\psi^3}\left[\mathrm{acos}\left(\frac{\psi}{\theta_2}\right) + \left(1 - \frac{\psi^2}{\theta_2^{\,2}}\right)^{1/2}\frac{\psi}{\theta_2} - \mathrm{acos}\left(\frac{\psi}{\theta_1}\right) + \left(1 - \frac{\psi^2}{\theta_1^{\,2}}\right)^{1/2}\frac{\psi}{\theta_1}\right]$$

$$= \frac{d\psi}{\psi^3}\left[g\left(\frac{\psi}{\theta_2}\right) - g\left(\frac{\psi}{\theta_1}\right)\right] \qquad 0 \leq \psi \leq \theta_1 \ , \qquad \text{(B.3)}$$

$$f_p(\psi)d\psi = \frac{2d\psi}{\psi^3}\int_0^{\mathrm{acos}(\psi/\theta_2)} \cos^2(\phi)d\phi = \frac{d\psi}{\psi^3} g\left(\frac{\psi}{\theta_2}\right) \qquad \theta_1 \leq \psi \leq \theta_2 \ , \qquad \text{(B.4)}$$

and

$$f_p(\psi) = f_p(-\psi) \ . \qquad \text{(B.5)}$$

From (B.3-5) we have obtained:

$$\int_0^{\theta_2} f_p(\psi)d\psi = \frac{\pi}{2}\left(\frac{1}{\theta_1^{\,2}} - \frac{1}{\theta_2^{\,2}}\right) \ , \qquad \text{(B.6)}$$

$$\int_0^{\theta_2} \psi^2 f_p(\psi)d\psi = \frac{\pi}{2}\ln\left(\frac{\theta_2}{\theta_1}\right) \ . \qquad \text{(B.7)}$$

θ_1 is related the minimum transfer energy that must be higher than the electron binding energy $I \approx 16\, Z^{0.9}$ eV in the target and θ_2 to the maximum transfer energy that correspond to the energy acceptance K:

$$\theta_1 = \left(\frac{2m_0c^2 I}{E_0^{\,2}}\right)^{1/2} \ , \qquad \text{(B.8)}$$

$$\theta_2 = \left(\frac{2m_0c^2K}{E_0{}^2}\right)^{1/2} \quad . \tag{B.9}$$

These values can be obtained using for the transfer energy E the relationship $E = E_0\sin^2(\theta^*/2)$ that, for ultra-relativistic electrons, can be written in terms of the Laboratory θ angle as:

$$E = \frac{E_0{}^2}{2m_0c^2}\,\theta^2 \quad . \tag{B.10}$$

APPENDIX C - The non-unitary staircase stochastic process

The stochastic process $s(t,\Lambda_S)$ that appears in this paper in different expressions, has the form

$$s(t,\Lambda_S) = \sum_{i=1}^{k} S_i\,\delta(t-t_i) \qquad t_1 < \ldots < t_k \leq t \ , \tag{C.1}$$

that corresponds to the stochastic derivative of the Poisson process

$$p(t,\Lambda_S) = \sum_{i=1}^{k} S_i\,U(t-t_i) \qquad t_1 < \ldots < t_k \leq t \ , \tag{C.2}$$

being $U(t-t_i)$ the unit step function, with characteristic parameter $\lambda_S = \Lambda_S/T_0$ and with the following probability density:

$$f(S_1 \ldots S_k, k, t) = \frac{(\lambda_S t)^k\, e^{-\lambda_S t}}{k!} \prod_{i=1}^{k} f(S_i) \quad . \tag{C.3}$$

In text books only the stochastic process

$$x(t,\Lambda) = \sum_{i=1}^{k} U(t-t_i) \qquad t_1 < \ldots < t_k \leq t \ , \tag{C.4}$$

with mean value

$$<x(t)> \ = \lambda\, t \ , \tag{C.5}$$

second order momentum

$$<x(t)^2> \ = \lambda\, t + \lambda^2\, t^2 \ , \tag{C.6}$$

and autocorrelation

$$R_{xx}(t_1,t_2) = \langle x(t_1)x(t_2)\rangle = \lambda\, t_2 + \lambda^2\, t_1 t_2 \qquad t_1 > t_2$$

$$= \lambda\, t_1 + \lambda^2\, t_1 t_2 \qquad t_2 > t_1\ . \qquad \text{(C.7)}$$

and whose shape is a staircase with steps equal to 1, is studied.[9] Then it was necessary to extend the description to the more general process $p(t,\Lambda_S)$. From the probability density it was straightforward to obtain all the statistical properties:

the mean value

$$\langle p(t)\rangle = \sum_{k=1}^{\infty} \int_{-\infty}^{+\infty} dS_1 \ldots \int_{-\infty}^{+\infty} dS_k\ p(t,\Lambda_S,S_1\ldots S_k)\ f(S_1\ldots S_k,k,t)$$

$$= \sum_{k=1}^{\infty} \frac{(\lambda_S t)^k\, e^{-\lambda_S t}}{k!}\, k\, \langle S\rangle = \langle S\rangle\, \lambda_S t\ , \qquad \text{(C.8)}$$

the second order momentum

$$\langle p^2(t)\rangle = \sum_{k=1}^{\infty} \int_{-\infty}^{+\infty} dS_1 \ldots \int_{-\infty}^{+\infty} dS_k\ p^2(t,\Lambda_S,S_1\ldots S_k)\ f(S_1\ldots S_k,k,t)$$

$$= \sum_{k=1}^{\infty} \int_{-\infty}^{+\infty} dS_1 \ldots \int_{-\infty}^{+\infty} dS_k \left(\sum_{i=1}^{k} S_i\right)^2 f(S_1\ldots S_k,k,t)$$

$$= \sum_{k=1}^{\infty} \int_{-\infty}^{+\infty} dS_1 \ldots \int_{-\infty}^{+\infty} dS_k \left(\sum_{i=1}^{k} S_i^2 + \sum_{i\neq j} S_i S_j\right) f(S_1\ldots S_k,k,t)$$

$$= \langle S^2\rangle\, \lambda_S\, t + \langle S\rangle^2\, \lambda_S^2\, t^2\ , \qquad \text{(C.9)}$$

and the autocorrelation

$$R_{pp}(t_1,t_2) = \langle p(t_1)p(t_2)\rangle = \langle p^2(t_2)\rangle + \langle p(t_1)\rangle\langle p(t_1-t_2)\rangle \qquad t_1 > t_2$$

$$= \langle p^2(t_1)\rangle + \langle p(t_1)\rangle\langle p(t_2-t_1)\rangle \qquad t_2 > t_1\ , \qquad \text{(C.10)}$$

that is

$$R_{pp}(t_1,t_2) = \langle p(t_1)p(t_2)\rangle = \langle S^2\rangle\, \lambda_S\, t_2 + \langle S\rangle^2 \lambda_S^2\, t_1 t_2 \qquad t_1 > t_2$$

$$= \langle S^2\rangle\, \lambda_S\, t_1 + \langle S\rangle^2 \lambda_S^2\, t_1 t_2 \qquad t_2 > t_1\ . \qquad \text{(C.11)}$$

The process $p(t,\Lambda_S)$ is equivalent to $x(t,\Lambda)$ with the substitutions

$$\lambda \rightarrow <S^2> \lambda_S \ , \tag{C.12}$$

$$\lambda^2 \rightarrow <S>^2 \lambda_S{}^2 \ , \tag{C.13}$$

and the s process has the following properties

$$<s(t)> = <S> \lambda_S \ , \tag{C.14}$$

$$R_{ss}(t_1,t_2) = <S>^2 \lambda_S{}^2 + <S^2> \lambda_S \, \delta(\tau) = R_{ss}(\tau) \ , \tag{C.15}$$

$$S_{ss}(\omega) = 2\pi <S>^2 \lambda_S{}^2 \, \delta(\omega) + <S^2> \lambda_S \ , \tag{C.16}$$

being $\tau = t_1 - t_2$.

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