ISTITUTO NAZIONALE DI FISICA NUCLEARE
Sezione di Trieste

INFN/TC - 89/2
13 Giugno 1989

Extended verification of two conjectures on prime numbers

A. De Angelis[1], F. Gonella[2], and M. Penzo[3]

Abstract

Two conjectures on prime numbers, due to N.L. Gilbreath and C. Goldbach, have been verified up to 2^{31} and $2 \cdot 10^9$ respectively.

[1]Istituto di Fisica dell'Universita' di Udine and INFN Trieste, Italy
[2]Dipartimento di Fisica dell'Universita' di Padova, Italy
[3]Apollo Computer S.P.A., Padova, Italy

1 Introduction

The verification of some conjectures on primes up to very large numbers requires both high computing speed and effectiveness in addressing huge vectors.

Apollo workstation DN10000, based on new RISC architecture PRISM (Parallel Reduced Instruction Set Multiprocessor), fulfills these requirements, coupling a speed of 25 MIPS/CPU to the capability of addressing directly up to 4 GB in virtual memory. During performance tests of this workstation, we extended the verification of two conjectures on prime numbers: the Gilbreath's conjecture and the Goldbach's conjecture.

- Gilbreath's conjecture

  If p_k is the sequence of prime numbers, define d_i by $d_i = p_{i+1} - p_i$, where p_i is the i-th prime number.

  Define then d_n^k by

$$d_n^1 = d_n$$
$$d_n^{k+1} = |d_{n+1}^k - d_n^k|$$

  (that is, the successive absolute differences of the sequence of primes, see Fig.1).

  N.L. Gilbreath conjectured that $d_1^k = 1$ for all k. The conjecture had been verified by Killgrove and Ralston for all primes $< N_G = 792{,}722$ [1].

- Goldbach's conjecture

  Goldbach's conjecture, perhaps the most famous one about primes, states that every even number greater than 4 can be expressed as the sum of two odd primes.

  The conjecture had been verified for even numbers $< 10^8$ by Stein and Stein and by Light, Forrest, Hammond and Roe [2].

2 The CPU

The system can support up to four independent 64 bit PRISM CPUs each one with its own RISC Floating Point Processor, RISC integer unit and separate data and instruction caches (64 KB and 128 KB respectively). Each CPU can view up to 128 MB of 16-way interleaved main memory. Communication from CPU to main memory and other processors is over a high speed 150 MB/s 64-bit bus.

Effective data flow techniques used in the compilers allow to take full advantage of local parallelism in the hardware.

3 Algorithms and results

The sequence p_n, needed for both verifications, has been produced using a sieve algorithm up to 2^{31}.

- Gilbreath's conjecture

  The sequence d_n has been calculated from p_n and mapped in virtual memory. Then the successive absolute differences between the vector d_n and the same vector shifted by one position on the right have been calculated; the process was iterated k times, until the maximum of the sequence d_n^k was equal to 2. This fact clearly constrains the elements d_n^k (for every $k \neq 1$) to be either 0 or 2, and thus verifies the conjecture.

  The j-th step verifies the conjecture up to the $(i + j - 1)$-th prime number, where i is the location of the first element d_i^j greater than 2. The sequence of the values assumed by the quantity $(i + j - 1)$ is thus an indicator of the rate of convergence of the algorithm, and trace of this quantity has been kept during the calculation.

  The sequence d_n^{276} has been found to have 2 as maximum value, thus verifying the conjecture up to 2^{31} (i.e. for the first 105,097,565 prime numbers). The rate of approach to the result can be seen in Fig.2, where the quantity $(i+j-1)$ defined in the previous section is plotted versus the iteration number.

- Goldbach's conjecture

  In a first step, the 98,222,286 odd prime numbers $< 2 \quad 10^9$ were summed to the first 100 odd prime numbers $p_2 \dots p_{101}$. The resulting sums, if $< 2 \quad 10^9$, were kept out of the successive search. At the end of this step, a set of $N_1 = 180,648$ even numbers> 4 and $< 2 \quad 10^9$ were left to be tested.

  For each even number E_i in this last subset, in a second step the successive differences $\delta_{ij} = E_i - p_j$, $j > 101$ were calculated, until δ_{ij} was prime.

  This method allowed to verify Goldbach's conjecture for all even numbers $< 2 \quad 10^9$, with j never exceeding the value of 1,000.

4 Performance

The performance of the workstation has been measured during the test of Gilbreath's conjecture.

The verification required 39,642 s of single-CPU time, for doing

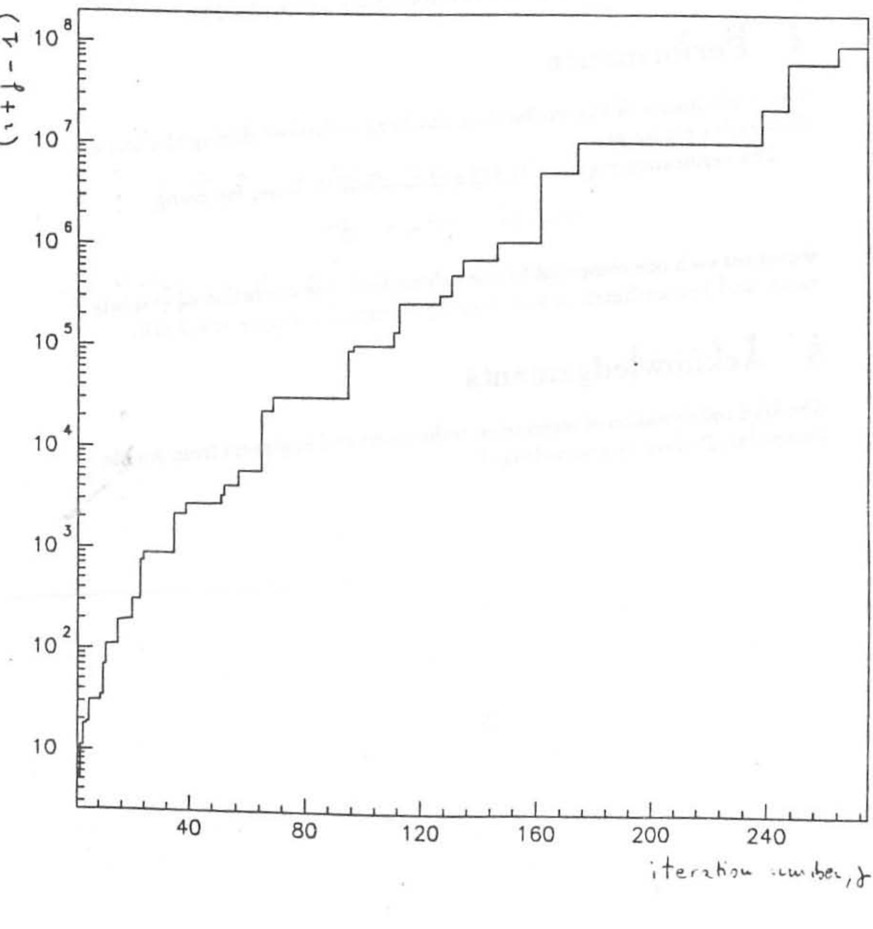
$$N \simeq \pi(2^{31}) \cdot 276 \simeq 3 \quad 10^{10}$$

sequences each one composed by one subtraction, one operation of absolute value, and one arithmetical test, keeping in virtual memory ≈ 0.2 GB.

5 Acknowledgements

The kind collaboration of secretaries, technicians and engineers from Apollo Computer, Padova, is acknowledged.

Fig. 2



(i + j - 1)
10^8
10^7
10^6
10^5
10^4
10^3
10^2
10
40
80
120
160
200
240
iteration number, j

References

1. R.K. Guy, *Unsolved Problems in Number Theory*, Springer-Verlag, NY 1981.

2. P. Ribenboim, *The book of prime number records*, Springer-Verlag, NY 1988.

Figure captions

1. Successive absolute differences of the sequence of primes.

2. Rate of convergence of the verification of Gilbreath's conjecture (see text).

Fig. 1

```
1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
2  3  5  7 11 13 17 19 23 29 31 37 41 43 47 53 59 61 67 71 73 79 83 89
 1  2  2  4  2  4  2  4  6  2  6  4  2  4  6  6  2  6  4  2  6  4  6
  1  0  2  2  2  2  2  2  4  4  2  2  2  2  0  4  4  2  2  4  2  2
   1  2  0  0  0  0  0  2  0  2  0  0  0  2  4  0  2  0  2  2  0
    1  2  0  0  0  0  2  2  2  2  0  0  2  2  4  2  2  2  0  2
     1  2  0  0  0  2  0  0  0  2  0  2  0  2  2  0  0  2  2
      1  2  0  0  2  2  0  0  2  2  2  2  2  0  2  0  2  0
       1  2  0  2  0  2  0  2  0  0  0  0  2  2  2  2  2
        1  2  2  2  2  2  2  2  0  0  0  2  0  0  0  0
         1  0  0  0  0  0  0  2  0  0  2  2  0  0  0
```