

# ISTITUTO NAZIONALE DI FISICA NUCLEARE

Sezione di Milano

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**INFN/TC-88/9**

17 Febbraio 1988

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THEORY**

Servizio Documentazione  
dei Laboratori Nazionali di Frascati

**Slippage And Superradiance In The High-Gain FEL : Linear Theory**

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ABSTRACT

We describe the linear regime of a Free-Electron-Laser Amplifier taking into account propagation effects (slippage). We demonstrate analytically, (for a simple case) the existence of two different solutions of pulse propagation equations, which in suitable limits describe the Steady-State and the Superradiant regimes.

1. - INTRODUCTION

The Free Electron Laser (FEL) uses a relativistic high-current electron beam to amplify a co-propagating electromagnetic wave. The optical wave and the electrons are coupled as they pass through an undulator with a periodic, transverse magnetic field.

A theoretical analysis based on equations of the traveling-wave-tube type, shows the existence of two completely different regimes in the dynamics of a Compton FEL, namely, a stable region in which one finds an interference gain (the well known small gain regime), and an unstable cooperative regime, experimentally demonstrated<sup>(1)</sup>, in which the gain grows exponentially (High-Gain regime). This regime, in which the FEL is operating in a single-pass configuration, has been the object of detailed analysis in previous works<sup>(2)</sup>; it is named Amplified Spontaneous Emission (ASE) regime if the lasing process is developing from noise.

In the ASE regime, one neglects the path difference accumulated between the photons and the electrons during the interaction time (slippage): this is correct if the length of the electron bunch  $L_e$  is sufficiently greater than the slippage  $N_0 \lambda_r$  (where  $N_0$  is the number of periods of the undulator and  $\lambda_r$  is the radiation wavelength).

A different regime, in the same single-pass High-Gain Amplifier configuration, occurs in the opposite situation; namely, in a short-bunch limit, when slippage effects are dominant. It is the Superradiant

Regime, in which the radiation power turns out to be proportional to the squared number of electrons. In Ref. 5 the slippage effect was modelled introducing a loss term in the Maxwell's equation; in this simple dissipative model was defined a Superradiant limit in which the field variables can be adiabatically eliminated, and was stated the condition under which a Superradiant emission should be observed.

In this paper, we linearize the equations describing the one dimensional evolution of the radiation and the electrons, taking into account the slippage effects. After defining the initial and the boundary conditions, we describe analytically the evolution of the field and the electron distribution inside an electron pulse of finite length, passing through the undulator. In a simple but meaningful case, we find that in the electron pulse two regions can be identified where the field and electron variables evolve in two completely different ways.

In one of them, the radiation is spatially uniform, and every particle emits radiation in the same way, giving a steady-state behaviour. This is the ASE regime, in which the electrons can reabsorb the emitted radiation. We re-obtain the ASE solution of the equations without slippage effects<sup>(2)</sup>.

In the other region, near the trailing edge of the electron pulse, the field is not uniform, and the electron do not evolve identically; this occurs because the electron pulse experiences less radiation re-absorption than in the first 'steady-state' region, so that slippage inhibits saturation. In this case we introduce a proper Superradiant limit and we obtain results in good qualitative agreement with the previous dissipative model<sup>(5)</sup>.

## 2.-SOLUTION OF THE PROPAGATION EQUATIONS IN THE LINEAR REGIME

The equations describing the one-dimensional evolution of the electron variables and the complex amplitude of the radiation field are

$$\frac{d\theta_j}{dt} = p_j \quad (1a)$$

$$\frac{dp_j}{dt} = - (Ae^{i\theta_j} + c.c.) \quad (1b)$$

$$\left( \frac{\partial}{\partial \bar{t}} + \frac{\partial}{\partial \bar{z}} \right) A = \langle e^{-i\theta} \rangle \quad (1c)$$

( $j = 1, \dots, N_e$ ) where, using the same notation of Ref. 2, to which we refer for more details,

$$\begin{aligned} \theta_j &= \frac{2\pi}{\lambda_o} z_j - \frac{2\pi}{\lambda_r} (z_j - ct) \\ p_j &= \frac{1}{\rho} \frac{\gamma_j - \gamma_R}{\gamma_R} \\ |A|^2 &= \frac{1}{\rho} \frac{|E_o|^2 / 4\pi}{n_e \gamma_R m c^2} \\ \bar{z} &= \frac{4\pi\rho}{\lambda_o} z \\ \bar{t} &= \frac{4\pi\rho}{\lambda_o} ct \\ \rho &= \frac{1}{\gamma_R} \left( \frac{a_o \omega_P \lambda_o}{4 \cdot 2\pi c} \right)^{2/3} \\ a_o &= \frac{eB_o \lambda_o}{2\pi m c^2} \end{aligned} \quad (2)$$

and  $\langle \dots \rangle$  is the average  $\frac{1}{N_e} \sum_{j=1}^{N_e} (\dots)$ .

The meaning of the dimensionless variables and parameters in eqs. (1a)–(1c) is the following:  $\theta_j$  and  $p_j$  are the phase ('position') and the energy variation or 'momentum' variables, respectively, of the  $j$ th electron;  $A$  is the scaled complex field variable of the radiation field,  $\bar{z}$  and  $\bar{t}$  are the scaled position and the time. In eqs. (2) these quantities are expressed in terms of the amplitude  $E_0$  and the wavelength  $\lambda_r$  of the radiation field, the magnetostatic amplitude  $B_0$  and the period  $\lambda_0$  of the undulator whose length is  $L_0 = N_0 \lambda_0$  ( $N_0$  is the number of period);  $\rho$  is the Pierce parameter,  $\omega_P$  is the plasma frequency and  $\alpha_0$  is the undulator parameter.

We assume resonance, i.e.

$$\lambda_r = \lambda_0 \frac{1 - \beta_{\parallel}}{\beta_{\parallel}} \simeq \frac{\lambda_0}{2\gamma R^2} (1 + \alpha_0^2) \quad (3)$$

where  $\beta_{\parallel} = \frac{v_{\parallel}}{c}$ , and  $v_{\parallel}$  is the mean longitudinal electron velocity.

We now linearize eqs.(1) around the initial condition, which is an equilibrium condition,

$$A(0) = 0 \quad p(0) = 0 \quad \langle e^{-i\theta(0)} \rangle \quad (4)$$

By introducing the phase deviation  $\delta\theta_j(\bar{t}) = \theta_j^{\circ} - \theta_j(\bar{t})$ , and considering  $\delta\theta_j$ ,  $p_j$  and  $A$  in (1) as infinitesimal quantities, the linearized system can be written as

$$\left( \frac{\partial}{\partial \bar{t}} + \beta_{\parallel} \frac{\partial}{\partial \bar{z}} \right) b = -i\mathcal{P} \quad (5a)$$

$$\left( \frac{\partial}{\partial \bar{t}} + \beta_{\parallel} \frac{\partial}{\partial \bar{z}} \right) \mathcal{P} = -A \quad (5b)$$

$$\left( \frac{\partial}{\partial \bar{t}} + \frac{\partial}{\partial \bar{z}} \right) A = b \quad (5c)$$

in terms of the electron collective variables<sup>(2)</sup>  $b$ ,  $\mathcal{P}$  :

$$\begin{aligned} b &\equiv -i \langle e^{-i\theta^{\circ}} \delta\theta \rangle \\ \mathcal{P} &\equiv \langle e^{-i\theta^{\circ}} p \rangle \end{aligned} \quad (6)$$

In (5) we have introduced a fluid description for the electron variables by  $\frac{d}{d\bar{t}} = \frac{\partial}{\partial \bar{t}} + \beta_{\parallel} \frac{\partial}{\partial \bar{z}}$ .

The eqs.(5) depend only on the  $\beta_{\parallel}$  parameter, and are defined for  $\bar{t} > 0$  and only in the region occupied by the electrons. We assume that the electrons are continuously distributed in a finite length  $L_e = N_b \lambda_r$  ( $N_b$  is the number of radiation wavelength in  $L_e$ ). So the eqs.(5) are defined in the reference frame moving with the velocity  $v_{\parallel}$ , in the region of space occupied by the electron bunch, whereas outside the bunch the field propagates in the vacuum according to eq.(5,c) without the source term on the l.h.s.

We introduce

$$\begin{cases} t' = \bar{t} \\ z' = \bar{z} - \beta_{\parallel} \bar{t} \end{cases} \quad (7)$$

so that the eqs.(5) become :

$$\frac{\partial b}{\partial t'} = -i\mathcal{P} \quad (8a)$$

$$\frac{\partial \mathcal{P}}{\partial t'} = -A \quad (8b)$$

$$\frac{\partial A}{\partial t'} + (1 - \beta_{\parallel}) \frac{\partial A}{\partial z'} = b \quad (8c)$$

For sake of simplicity, we neglect the transient interval of time during which the bunch of electron goes into the undulator (suppose  $L_e \ll L_0$ ) and we set  $t' = 0$  when the trailing edge of the bunch is at  $\bar{z} = 0$  in the undulator. Hence eqs.(8) are defined for

$$t' \geq 0 \quad 0 \leq z' \leq \bar{z}_0 = \frac{4\pi\rho}{\lambda_0} L_e \quad (9)$$

with the initial condition at  $t' = 0$

$$\begin{aligned} A(z', 0) &= A_0(z') \\ \mathcal{P}(z', 0) &= 0 \\ b(z', 0) &= b_0(z') \end{aligned} \quad (10)$$

and with  $A(z' = 0, t') = A_0(0)$  in the trailing edge of the bunch.  $A_0(z')$  and  $b_0(z')$  are the known initial field and electron distribution inside the bunch.

The solution of eqs.(8) with the conditions (9), (10) proceeds<sup>(3)</sup> by introducing the Laplace transform in  $t'$  :

$$\bar{X}(z', \lambda) = \int_0^\infty dt' e^{-\lambda t'} X(z', t') \quad (11)$$

This yields the ordinary differential equation

$$\frac{d\bar{A}(z', \lambda)}{dz'} + \frac{\lambda^3 - i}{\lambda^2(1 - \beta_{\parallel})} \bar{A}(z', \lambda) = \frac{1}{1 - \beta_{\parallel}} \left\{ A_0(z') + \frac{b_0(z')}{\lambda} \right\} \quad (12)$$

where  $\bar{A}(0, \lambda) = \frac{A_0(0)}{\lambda}$ . Eq.(12) has the solution

$$\begin{aligned} A(z', \lambda) &= \frac{A_0(0)}{\lambda} e^{-\frac{\lambda^3 - i}{\lambda^2(1 - \beta_{\parallel})} z'} + \\ &+ \frac{1}{1 - \beta_{\parallel}} \int_0^{z'} dz'' e^{-\frac{\lambda^3 - i}{\lambda^2(1 - \beta_{\parallel})} (z' - z'')} \left\{ A_0(z'') + \frac{b_0(z'')}{\lambda} \right\} \end{aligned} \quad (13)$$

Inverting the Laplace transform, we get

$$\begin{aligned} A(z', t') &= \frac{A_0(0)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{d\lambda}{\lambda} e^{\lambda t' - \frac{\lambda^3 - i}{\lambda^2(1 - \beta_{\parallel})} z'} + \\ &+ \frac{1}{1 - \beta_{\parallel}} \int_0^{z'} dz'' A_0(z'') \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\lambda e^{\lambda t' - \frac{\lambda^3 - i}{\lambda^2(1 - \beta_{\parallel})} (z' - z'')} + \\ &+ \frac{1}{1 - \beta_{\parallel}} \int_0^{z'} dz'' b_0(z'') \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{d\lambda}{\lambda} e^{\lambda t' - \frac{\lambda^3 - i}{\lambda^2(1 - \beta_{\parallel})} (z' - z'')} \end{aligned} \quad (14)$$

### 3. - STEADY STATE AND SUPERRADIANT REGIMES

We study eq.(14) for a simple but relevant initial situation, that is, no field excitation ( $A_0 = 0$ ) and constant initial bunching  $b_0$ .

Physically, it corresponds to no laser pump into the undulator and some noise due the non uniform distribution of the electrons.

In this case eq.(4) becomes:

$$A(z', t') = \frac{b_0}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\lambda \frac{\lambda}{\lambda^3 - i} e^{\lambda t'} \left\{ 1 - e^{-\frac{\lambda^3 - i}{\lambda^2(1 - \beta_{\parallel})} z'} \right\} \quad (15)$$

The field amplitude  $A(z', t')$  is the sum of two terms. The first, only time dependent, is given by the residua of the simple three poles  $\lambda_1 = -i$ ,  $\lambda_{2,3} = \frac{i}{2} \pm \frac{\sqrt{3}}{2}$  and is the well known solution of the linear 'steady-state' problem (2):

$$A_1(\bar{t}) = \sum_{i=0}^3 \frac{\lambda_i e^{\lambda_i \bar{t}}}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)} \quad (i \neq j \neq k) \quad (16)$$

The second term of (15) gives a contribution  $-A_1(\bar{t})$  from the simple poles plus another contribution  $A_S$  from the essential singularity  $\lambda = 0$ .

Writing (15) in the original coordinates  $\bar{z}$  and  $\bar{t}$  :

$$A(\bar{z}, \bar{t}) = A_1(\bar{t}) - \frac{b_0}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{d\lambda \lambda}{\lambda^3 - i} e^{\lambda \left( \frac{\bar{t} - \bar{z}}{1 - \beta_{\parallel}} \right) + \frac{i}{\lambda^2} \left( \frac{\bar{z} - \beta_{\parallel} \bar{t}}{1 - \beta_{\parallel}} \right)} \quad (17)$$

we see that the second term is identically zero if  $\bar{z} > \bar{t}$ , so we can write

$$A(\bar{z}, \bar{t}) = A_1(\bar{t}) \{1 - \Theta(\bar{t} - \bar{z})\} + \Theta(\bar{t} - \bar{z}) A_S(\bar{z}, \bar{t}) \quad (18)$$

where

$$A_S(\bar{z}, \bar{t}) = b_0 \left( \frac{\bar{z} - \beta_{\parallel} \bar{t}}{1 - \beta_{\parallel}} \right) \sum_{n=0}^{\infty} \frac{i^n}{(2n)!(n+1)!} \left\{ \left( \frac{\bar{z} - \bar{t}}{1 - \beta_{\parallel}} \right)^2 \left( \frac{\bar{z} - \beta_{\parallel} \bar{t}}{1 - \beta_{\parallel}} \right) \right\}^n \quad (19)$$

and  $\Theta(x)$  is the step function ( $\Theta = 1$  if  $x > 0$  and  $\Theta = 0$  if  $x < 0$ ). We have calculated also the bunching parameter  $b(\bar{z}, \bar{t})$  from eqs.(8) following the same steps as for  $A(\bar{z}, \bar{t})$ , obtaining

$$b(\bar{z}, \bar{t}) = b_1(\bar{t}) \{1 - \Theta(\bar{t} - \bar{z})\} + \Theta(\bar{t} - \bar{z}) b_S(\bar{z}, \bar{t}) \quad (20)$$

where

$$b_1(\bar{t}) = \sum_{i=0}^3 \frac{e^{\lambda_i \bar{t}}}{\lambda_i (\lambda_i - \lambda_j)(\lambda_i - \lambda_k)} \quad (i \neq j \neq k) \quad (21)$$

and

$$b_S(\bar{z}, \bar{t}) = b_0 \sum_{n=0}^{\infty} \frac{i^n}{(2n)!n!} \left\{ \left( \frac{\bar{z} - \bar{t}}{1 - \beta_{\parallel}} \right)^2 \left( \frac{\bar{z} - \beta_{\parallel} \bar{t}}{1 - \beta_{\parallel}} \right) \right\}^n \quad (22)$$

From (18) and (20) it turns out that, if  $\bar{t} < \frac{\bar{z}_0}{1 - \beta_{\parallel}}$  (i.e.  $t < \frac{L - e}{c - v_{\parallel}}$ ), we have a region inside the bunch (from  $z = ct$  to the leading edge) where the field and the bunching parameter  $b$  are uniform in  $z$  and growing up exponentially as  $e^{\frac{\sqrt{3}}{2} \bar{t}}$ , where  $\bar{t} = \frac{4\pi\rho}{\lambda_0} t$  is the gain (Steady -State solution).

From the trailing edge to the position  $z = ct$ , the field and the bunching parameter are not uniform and are respectively equal to zero and  $b_0$  on the trailing edge. This is the slippage region (of dimension  $(c - v_{\parallel})t$ ). In this region the electrons cannot reabsorb the radiation emitted at the same rate that in the 'steady-state' region; this occurs because there are no more electrons behind them and so there is less radiation propagating into this region. This regime is investigated in Ref.(4).

Otherwise, if  $t > \frac{L - e}{c - v_{\parallel}}$ , the only contribution to the radiation or bunching is  $A_S$  or  $b_S$ , and we have no more steady-state solution after the time  $\frac{L - e}{c - v_{\parallel}}$ .

Following Ref.(3), we can derive an asymptotic expression for  $A_S$  and  $b_S$ , in a suitable limit on  $\bar{z}$  and  $\bar{t}$ .

We introduce the coordinates of Ref.(4) :

$$\begin{aligned} z_1 &= \frac{\bar{z} - \beta_{\parallel} \bar{t}}{1 - \beta_{\parallel}} = \frac{4\pi\rho}{\lambda_r \beta_{\parallel}} (z - \beta_{\parallel} t) \\ z_2 &= \frac{\bar{z} - \bar{t}}{1 - \beta_{\parallel}} = \frac{4\pi\rho}{\lambda_r} (ct - z) \end{aligned} \quad (23)$$

and we write  $A_S$  and  $b_S$  in the integral form :

$$A_S(z_1, z_2) = b_o z_1 \sum_{n=0}^{\infty} \frac{i^n}{(2n)!(n+1)!} (z_1 z_2^2)^n = -\frac{b_o}{2\pi} \oint_C d\lambda \lambda e^{\lambda z_2 + \frac{i}{\lambda^2} z_1} \quad (24)$$

$$b_S(z_1, z_2) = b_o \sum_{n=0}^{\infty} \frac{i^n}{(2n)!n!} (z_1 z_2^2)^n = \frac{b_o}{2\pi i} \oint_C \frac{d\lambda}{\lambda} e^{\lambda z_2 + \frac{i}{\lambda^2} z_1} \quad (25)$$

where  $C$  is a path surrounding the origin.

By defining  $\omega \equiv \lambda \left( \frac{z_1}{z_2} \right)^{1/3}$  and  $x = (z_1 z_2^2)^{1/3}$ , we have :

$$\begin{aligned} A_S &= -\frac{b_o}{2\pi} \left( \frac{z_1}{z_2} \right)^{2/3} \oint_C d\omega \omega e^{x\Phi(\omega)} \\ b_S &= \frac{b_o}{2\pi i} \oint_C \frac{d\omega}{\omega} e^{x\Phi(\omega)} \end{aligned} \quad (26)$$

where  $\Phi(\omega) = \omega + \frac{i}{\omega^2}$ . So we can do a stationary-phase evaluation of the contour integral at large  $x$ , i.e. for

$$(\sqrt{z_1 z_2})^{2/3} = [\sqrt{z_1}(\bar{t} - z_1)]^{2/3} \gg 1 \quad (27)$$

There are three points of stationary phase, namely the roots of the eq.  $\Phi'(\omega) = 0$ , corresponding to an amplifying, an oscillatory and a decaying exponential. By keeping only the amplifying exponential, we get :

$$A_S(z_1, z_2) = \frac{b_o}{\sqrt{3}\pi} \frac{\sqrt{z_1}}{z_2} e^{\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) \left(\frac{\sqrt{z_1 z_2}}{2}\right)^{2/3} + i\frac{\pi}{4}} \quad (28)$$

$$b_S(z_1, z_2) = \frac{b_o}{\sqrt{3}\pi} \frac{1}{(4\sqrt{z_1 z_2})^{1/3}} e^{\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) \left(\frac{\sqrt{z_1 z_2}}{2}\right)^{2/3} + i\frac{7}{12}\pi} \quad (29)$$

These expressions are more transparent if we observe the field, for times greater than  $\frac{L_e}{c - v_{\parallel}}$ , in the leading edge, i.e. for  $\bar{z} = \bar{z}_o + \beta_{\parallel} \bar{t}$ . if we introduce as in Ref.(14) the Superradiant parameter

$$K = \frac{\bar{z}_o}{1 - \beta_{\parallel}} = \frac{4\pi\rho L_e}{\lambda_o(1 - \beta_{\parallel})} \simeq \frac{4\pi\rho}{\lambda_r} L_e$$

(where  $\beta_{\parallel} \simeq 1$ ), we have on the leading edge

$$|A_S(\bar{t})| = \frac{b_o}{\sqrt{3}\pi} \frac{\sqrt{K}}{K\bar{t} - 1} e^{\frac{\sqrt{3}}{2} \left(\frac{\bar{t} - 1/K}{2\sqrt{K}}\right)^{2/3}} \quad (30)$$

$$|b_S(\bar{t})| = \frac{b_o}{\sqrt{3}\pi} \frac{1}{2} \left(\frac{2\sqrt{K}}{\bar{t} - 1/K}\right)^{1/3} e^{\frac{\sqrt{3}}{2} \left(\frac{\bar{t} - 1/K}{2\sqrt{K}}\right)^{2/3}} \quad (31)$$

Hence for  $\bar{t} \gg \frac{1}{K}$  both the bunching parameter and the field amplitude grow with a rate  $\sqrt{K}$ , as derived in ref.(5) from the dissipative model; however, the exponent is proportional to  $\bar{t}^{2/3}$  (3). Also we note that, for a  $\bar{t} = \frac{1}{K} + 2\sqrt{K}$ , that is at a time  $2\sqrt{K}$  from the beginning of the Superradiant process,  $b_S$  is independent of  $K$  and on the order of unity,  $A_S$  is proportional to  $\frac{1}{K}$  and  $A_S = \frac{b_S}{K}$ . Hence, in the limit  $K \gg 1$  we re-obtain the same adiabatic expression of Ref.(5), where the field is proportional to the bunching parameter after a short initial transient  $\frac{1}{K}$ . We have thus demonstrated the existence of the Superradiant regime, for times  $\bar{t} > 1/K$  and in the limit  $K \gg 1$ .

7. - REFERENCES

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