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A NONLINEAR BOUNDARY VALUE PROBLEM FOR THE STEADY-STATE ONE-DIMENSIONAL HEAT CONDUCTION EQUATION

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SUMMARY

The numerical solution is discussed of a two-point boundary value problem, arising in RF superconductivity, for the steady-state one-dimensional heat conduction equation with thermal conductivity depending on temperature and boundary conditions nonlinearly involving temperature and heat flux. The numerical method employed attains a great computational saving in comparison with standard ones by taking advantage of the particular features of our problem and can be likewise applied to other problems. The general mathematical form a problem must have for being solved by the aforementioned method is then laid down.

1. - INTRODUCTION

In our work in RF superconductivity we have met a problem of heat conduction through a cylindrical plate of superconducting material which can be stated as follows

$$\frac{d}{dr}\left(K(T)\,\frac{dT}{dr}\right) + \frac{K(T)}{r}\,\frac{dT}{dr} = 0\,, \tag{1}$$

$$- K(T_I)\,\frac{dT}{dr}\bigg|_{r=r_I} = \varphi(T_I)\,, \tag{2}$$

$$T_E = \psi(-K(T_E) \left. \frac{dT}{dr} \right|_{r=r_E}) \qquad (3)$$

where T is the temperature, K the thermal conductivity, r_I and r_E are respectively the internal and the external radius of the plate, $T_I = T(r_I)$, $T_E = T(r_E)$

$$\varphi(T) = B^2(\alpha + \frac{\beta}{T} e^{-\gamma/T}), \qquad (4)$$

$$\psi(q) = (\delta + \eta q)^{1/4}, \qquad (5)$$

and $\alpha, \beta, \gamma, \delta, \eta$ and B are constants, this latter meaning magnetic field on the internal surface.

Computational techniques mostly used to solve problems like that above mentioned are shooting and finite-difference methods(1, 2). In shooting methods the boundary value problem is transformed in a sequence of initial value problems; in finite-difference methods a nonlinear algebraic system ensues which is then reduced to a sequence of linear systems.

In this work a method is discussed that computes temperatures at boundary points by solving a single nonlinear algebraic equation and then requires the independent solution of one more nonlinear algebraic equation for each additional interior point at which temperature must be found. This method is much less general than standard ones but, in our opinion, proves profitable whenever it can be applied. In our case it turnes out especially well suited because we were mostly interested in just finding T_I and T_E versus B. In fact to this aim this particular problem must be solved several times for different B values and, constant feature, we have no interest in finding T versus r.

If the aforementioned standard methods were used, we could not take advantage of the above feature to get computational saving, as our method does, because T should be computed anyway for several r values to obtain T_I and T_E. In section 2 the mathematical development of the method is carried out for the problem under consideration. In section 3 we discuss features and performances of our numerical implementation. In section 4 we state the class of mathematical problems that the described method, or a slight adaptation of it, can solve.

2. - REDUCTION OF THE DIFFERENTIAL PROBLEM TO ALGEBRAIC EQUATIONS

If we indicate heat flux by q, i.e.

$$q = -K(T) \frac{dT}{dr} \qquad (6)$$

eq. (1) can be rewritten as

$$\frac{dq}{dr} + \frac{q}{r} = 0 \tag{7}$$

and then solved by separation of variables to give

$$q = \frac{r_I}{r} q_I \tag{8}$$

and, for $r = r_E$,

$$q_E = \frac{r_I}{r_E} q_I \tag{9}$$

where $q_I = q(r_I)$ and $q_E = q(r_E)$.

Now by substituting eq. (6) into eq. (8) and then integrating the latter we have

$$\int_T^{T_E} K(T')\,dT' = -r_I q_I \ln \frac{r_E}{r} \tag{10}$$

which we rewrite as follows

$$(T_E - T)\overline{K}(T, T_E) = -r_I q_I \ln \frac{r_E}{r} \tag{11}$$

where

$$\overline{K}(T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} K(T')\,dT' \tag{12}$$

and finally as

$$T = T_E + \frac{r_I q_I}{\overline{K}(T, T_E)} \ln \frac{r_E}{r} . \tag{13}$$

At this point by successive substitutions of eqs. (12), (3), (9) and (2) into eq. (13) for $r = r_I$ we obtain a nonlinear algebraic equation having the form

$$T_I = f(T_I) \tag{14}$$

which must be numerically solved to get T_I; once this latter found q_I, q_E and T_E are easily computed too if we use in turn eqs. (2), (9) and (3). Then heat flux tempera_ture at any interior point can be independently calculated by respectively direct application of eq. (8) and numerical solution of eq. (13) for a suitable r value.

3. - NUMERICAL SOLUTION OF THE ALGEBRAIC EQUATIONS

Several different methods could be used to solve eq. (14)[3, 4]. We have used the iteration

$$T_I^{(n+1)} = f(T_I^{(n)}) \tag{15}$$

which converges only to that solutions for which the condition

$$\left| f'(T_I) \right| < 1 \tag{16}$$

is satisfied[3, 4].

The method above was chosen because by graphical solution of extreme cases of our problem we verified that in our problem condition (16) is always satisfied except just for all that solutions which cannot occur in practice being physically unstable. However our choice is perhaps the simplest but by no means neither the sole possible nor the best.

For sake of preciseness we report below the iteration (15) written in full.

$$\begin{aligned} & q_I^{(n)} = \varphi(T_I^{(n)}) , \qquad q_E^{(n)} = \frac{r_I}{r_E} q_I^{(n)} , \qquad T_E^{(n)} = \psi(q_E^{(n)}) , \\ & T_I^{(n+1)} = T_E^{(n)} + \frac{r_I q_I^{(n)}}{\bar{K}(T_I^{(n)}, T_E^{(n)})} \ln \frac{r_E}{r_I} \ . \end{aligned} \tag{17}$$

Also to compute temperature at interior points by eq. (13) the same method has been used; the corresponding iteration reads as

$$T^{(n+1)} = T_E + \frac{r_I q_I}{\bar{K}(T^{(n)}, T_E)} \ln \frac{r_E}{r} \ . \tag{18}$$

The computation of $\bar{K}$ has been fastened by calculating and storing values of

$$\int_0^T K(T')\,dT' \tag{19}$$

at the beginning of the program; then during iteration $\bar{K}$ is obtained as difference between two interpolated values of (19).

To obtain T_I versus B we solve the problem several times by increasing each time B by a fixed amount up to a maximum value and by using as initial approximation for iteration (17) the solution obtained for the previous lesser B value. The solution for $B = 0$, i. e. $T_I = \delta^{1/4}$, which is easily found by analytical tools provides a suitable starting point for the whole process. Computations for the same values of B are then carried out in the reverse order (this time by using as iniyial approximation the solution for the just greater B value) to obtain the other branch of the T versus B curve if hysteresis occurs.

In our experience convergence of iteration (17) within a relative variation of 10^{-5} between subsequent approximations requires few cycles only except near to some critical B values where a few tens of cycles are needed. However the CPU time consumed in a whole computation with 100 B values never exceeded 2 or 3 seconds on a CDC 170/835 computer. Although T versus r for fixed B was of no interest in our problem we nevertheless have checked also iteration (18). With the initial approximation obtained by linear interpolation between T_I and T_E the iteration converged within relative variations up to 10^{-10} in a very few cycles: 2 typical, 8 the worst case.

4. - SOME POSSIBLE GENERALIZATIONS

The method previously described for our particular case can be directly applied to solve a problem provided this has the following mathematical form

$$F(u, u', x) = 0 \tag{20}$$

$$u = f(v) \frac{dv}{dx} \tag{21}$$

$$u_1 = g(v_1) \tag{22}$$

$$u_2 = h(u_2) \tag{23}$$

and eq. (20) must be such that it has a closed form solution integrable by analytical tools in the range of interest.

In solution of (20) is closed form but not analitically integrable the method can be used too but a numerical integration must be carried out in the iterations solving equations corresponding to (13) and (14). Again at the cost of an additonal computational effort we can still apply our method when boundary conditions cannot be fitted into forms (22) and (23) but have the more general forms

$$G(u_1, v_1) = 0 , \tag{24}$$

$$H(u_2, v_2) = 0 . \tag{25}$$

To do so numerical iterations for solving (24) at a fixed v_1 and (25) at a fixed u_2 have to be nested in the outer iteration which solves equation corresponding to (14).

Finally let us point out again that other numerical methods than that of section 3 can be used as well to solve the algebraic equations ; moreover the numerical method should be carefully chosen in each particular case.

5. - CONCLUSIONS

The numerical method for two-point boundary value problems which we used in this work although much less general than usual ones is simpler to implement and requires less calculations too whenever it can be applied. Moreover once a problem has been solved for the boundary points the solution can be computed at any intermediate point independently each from other. As a consequence the solution can be obtained at grid points of unevenly spaced and coarse grid without increasing either computational complexity or error and without computing it at not even a single unnecessary point.

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