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LONGITUDINAL COUPLING IMPEDANCE OF AN ANNULAR RING

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Abstract

In this paper we propose a new method to evaluate the longitudinal coupling impedance of a charged particle passing perpendicularly through the centre of a perfectly conducting annular ring. It is shown that the solution of the problem can be expressed as a *double Neumann series*, whose expansion coefficients can be easily computed. Moreover we show the causality of the wake field for a particle travelling at the speed of light.

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1 INTRODUCTION

In this paper we shall describe a theoretical study of the radiation emitted from a point charge (q) moving at constant velocity $v = \beta c$, where c is the speed of light in vacuum and passing through the centre of a perfectly conducting annular ring, with inner radius r_1 and outer radius r_2; we shall assume that the charge moves in the positive $\hat{z}$ direction, as depicted in Figure 1. A charge moving with uniform velocity in vacuum radiates only because of inhomogeneities existing near its path. The radiation is due to the diffraction of the field of the charge at the circular edges. The diffraction problem is described by the field travelling with the charge itself [1] and the reaction of the ring which has a travelling wave behaviour. Accordingly we can represent all the fields and/or potentials as the superposition of two terms: a term generated by the charge in the free space and a term due to the presence of the metallic region of the iris, which together have to satisfy the boundary conditions. The aim of the paper is the evaluation of the longitudinal coupling impedance due to the iris, a parameter determining the performance of an accelerator, defined by [2]

$$Z_{\|}(k) = -\frac{1}{q}\int_0^{\infty} E_z(r, z=0; k)\, e^{jkz/\beta}\, dz\ , \tag{1.1}$$

where $E_z(r, z=0; k)$ represents the longitudinal component of the electric field in the frequency domain due to the reaction of the ring and k is the wavenumber.

Using as unknown the Hankel transform of the radially induced surface current density on the metallic region

$$J(r) = \int_0^{\infty} w\, F(w)\, J_1(wr)\, dw\ , \tag{1.2}$$

the scattered electromagnetic field can be represented by the following integral transformations [2]

$$\begin{cases} H_\varphi(r,z) = -\dfrac{\mathrm{sgn}(z)}{2}\displaystyle\int_0^{\infty} w\, F(w)\, J_1(wr)\exp\left(-|z|\sqrt{w^2-k^2}\right) dw \\[2ex] E_r(r,z) = \dfrac{j\zeta_0}{k}\dfrac{\partial}{\partial z} H_\varphi(r,z) = \dfrac{j\zeta_0}{2k}\displaystyle\int_0^{\infty} w\sqrt{w^2-k^2}\, F(w)\, J_1(wr)\exp\left(-|z|\sqrt{w^2-k^2}\right) dw \\[2ex] E_z(r,z) = -\dfrac{j\zeta_0}{k\,r}\dfrac{\partial}{\partial r}\left[r\, H_\varphi(r,z)\right] = \dfrac{j\zeta_0}{2k}\,\mathrm{sgn}(z)\displaystyle\int_0^{\infty} w^2\, F(w)\, J_0(wr)\exp\left(-|z|\sqrt{w^2-k^2}\right) dw \end{cases} \tag{1.3}$$

where sgn(z) represents the signum function, $\zeta_0 = 120\pi\ \Omega$ is the characteristic impedance of vacuum and the branch cut is chosen such that $\mathrm{Im}\left(\sqrt{w^2-k^2}\right) \geq 0$. By imposing boundary conditions, the electromagnetic problem can be written as the following triple system of integral equations

$$\begin{cases} \displaystyle\int_0^\infty w\, F(w)\, J_1(wr)\, dw = 0 & 0 \le r < r_1 \,, \\ \displaystyle\int_0^\infty w\, F(w) \sqrt{w^2 - k^2}\, J_1(wr)\, dw = A\, K_1(\kappa r) & r_1 < r < r_2 \,, \\ \displaystyle\int_0^\infty w\, F(w)\, J_1(wr)\, dw = 0 & r > r_2 \,, \end{cases} \tag{1.4}$$

where we called

$$A = \frac{jqk^2}{\pi\beta^2\gamma} \,, \qquad \kappa = \frac{k}{\beta\gamma} \,, \qquad \gamma = \frac{1}{\sqrt{1 - \beta^2}} \,, \tag{1.5}$$

γ being the so-called Lorentz factor.The first and the third equations state that there is no induced current outside the ring, while the second one is the boundary condition for the radial component of the electric field on the metallic surface. Such a set of integral equations, all three containing the same unknown function but holding over complementary regions, are known in literature as triple integral equations. In this paper our interest is in their application to the solution of the wave equations for diffraction problems (where the integrals are normally singular). At this point it will not be inappropriate to recall that a quite exhaustive survey of the historical developments and methods of dual equations solution in potential theory can be found in Sneddon's book [3)], but the generalization of the solution of triple problems has not yet reached satisfactory levels. We already applied [4,5)] successfully some methods to the solution of dual integral problems discussed in Sneddon's book.

The longitudinal coupling impedance in terms of the unknown of the problem is given by

$$Z_{||}(k) = \frac{\zeta_0}{q\beta} \int_0^\infty F(w)\, \frac{w^2}{w^2 + \kappa^2}\, dw \,. \tag{1.6}$$

The strategy of solution of triple integral equations (1.4) is similar to the one which was adopted for solving dual integral equations [4,5)] and it consists in finding a complete set of functions, each satisfying the first and third equations (1.4). In this way the electromagnetic problem can be reduced to the only solution of the non homogeneous equation of the system.

2 SOLUTION OF THE PROBLEM

A candidate to expand the unknown spectrum F(w) is the set $\Phi(w) = [\varphi_1(w)\ ,\ \varphi_2(w)\ ,\ \cdots]^T$ of functions

$$\varphi_n(w) = \frac{1}{w}\, J_n\!\left(\frac{r_1 + r_2}{2}\, w\right) J_n\!\left(\frac{r_2 - r_1}{2}\, w\right) .$$

In fact these functions behave [6)] as previously required, namely

$$\int_0^\infty w\, \varphi_n(w)\, J_1(wr)\, dw = \begin{cases} \dfrac{2\sqrt{(r^2 - r_1^2)(r_2^2 - r^2)}}{n\pi r\,(r_2^2 - r_1^2)}\, U_{n-1}\!\left(\dfrac{r_1^2 + r_2^2 - 2\,r^2}{r_2^2 - r_1^2}\right) & r_1 \le r \le r_2\ , \\ 0 & \text{elsewhere}\ . \end{cases} \qquad (2.1)$$

In addition, they exhibit, according to Meixner’s condition, the correct edge behaviour of the current; the functions $U_{n-1}(x)$ are the Chebyshev polynomials of the second kind. Moreover the factor representing the edge behaviour is independent from the index n and can be factorized. At present, this set of functions represents the best tool to expand the unknown Hankel transform of the current F(w) according to the series

$$F(w) = \frac{jq}{\pi\beta} \sum_{n=1}^{\infty} F_n\, \varphi_n(w) = \frac{jq}{\pi\beta}\, \mathbf{F}^T\, \Phi(w)\ , \qquad (2.2)$$

which is called double Neumann series and $\mathbf{F} = [F_1\ ,\ F_2\ ,\ \cdots]^T$. Since the series (2.2) intrinsically satisfies homogeneous equations of the system (1.4), all the electromagnetic problem is contained in the second one which becomes

$$\frac{jq}{\pi\beta} \sum_{n=1}^{\infty} F_n \int_0^\infty \sqrt{w^2 - k^2}\; w\, \varphi_n(w)\, J_1(wr)\, dw = A\, K_1(\kappa r)\ . \qquad (2.3)$$

This integral equation may be solved according to the method of Rietz-Galerkin: we project it on a complete set of functions, in the domain $[r_1,\ r_2]$, getting an infinite system of linear algebraic equations. As a test function we use just the function on the right hand side of the equation (2.1). The projection consists into an inverse Hankel transform. Resorting to the integral representation of the modified Bessel function [6)]

$$K_1(\kappa r) = \frac{1}{\kappa} \int_0^\infty J_1(wr)\, \frac{w^2}{w^2 + \kappa^2}\, dw\ , \qquad (2.4)$$

and to the properties of the inverse Hankel transform of the test functions, we get the following system of linear algebraic equations

$$\mathbf{A}\,\mathbf{F} = \mathbf{S}\,, \tag{2.5}$$

where the generic element of the coefficients' matrix **A** and the free term vector **S** have been defined as

$$A_{nm} = A_{mn} = \int_0^\infty u\,\sqrt{u^2 - 1}\;\varphi_n(k\,r_2\,u)\,\varphi_m(k\,r_2\,u)\,du\,,\quad S_m = I_m\left(\frac{1-a}{2}\,\kappa r_2\right) K_m\left(\frac{1+a}{2}\,\kappa r_2\right) \tag{2.6}$$

We note that the dimensionless wavenumber kr_2 and the aspect ratio $a = r_1/r_2 < 1$ have been introduced and that the relevant integral [6]

$$\int_0^\infty w\,\frac{J_n(xw)\,J_n(yw)}{w^2 + \kappa^2} = I_n(\kappa x)\,K_n(\kappa y) \qquad (x < y)\,, \tag{2.7}$$

has been used to evaluate the free term of the system (2.5).

As a preliminary conclusion, we can say that the problem of a particle passing perpendicularly through an annular ring has been transformed into a system of algebraic equations, whose coefficients' matrix is symmetric.

The system (2.5) has been successfully adopted to evaluate the expansion coefficients in a wide range of frequencies and for various values of the particle energy. Examples are reported in figures 2, 3 and 4 for different values of the bunch velocity, from the low value $\beta\gamma = 0.1$ up to the ultra-relativistic one $\beta\gamma = 10$; accordingly, the wavenumbers are varied from $kr_2 = 0.2$ to $kr_2 = 20$. These figures clearly indicate that matrices of small size have to be inverted to evaluate the expansion coefficients and that only few of them give an accurate evaluation of all the relevant electromagnetic quantities, such as the current distribution density. This can be easily obtained by means of the Hankel transform (1.2) and it is represented by the following expansion

$$J(r) = \frac{2jq}{\pi^2\beta r}\,\frac{\sqrt{(r^2 - r_1^2)(r_2^2 - r^2)}}{r_2^2 - r_1^2}\sum_{n=1}^{\infty}\frac{F_n}{n}\,U_{n-1}\left(1 - 2\,\frac{r^2 - r_1^2}{r_2^2 - r_1^2}\right). \tag{2.8}$$

An example of the current density distribution is shown in Figure 5. It is worth noting that the current density distribution (2.8) exhibits two zeros for $r = r_1$ and $r = r_2$, which correspond to the singularities of the radial and longitudinal components of the electric field (1.3); this behaviour can be successfully used to assess the completeness of the proposed expansion and the uniqueness of the solution.

3 THE LONGITUDINAL COUPLING IMPEDANCE

The longitudinal coupling impedance of an annular ring can be easily evaluated by substituting the expansion (2.2) into the definition (1.6). Using again the relevant integral (2.7), it is not difficult to conclude that the impedance is given by the following expansion

$$Z_{||}(k) = R_{||}(k)+jX_{||}(k) = \frac{j\zeta_0}{\pi\beta^2} \sum_{n=1}^{\infty} F_n\, I_n\left(\frac{1-a}{2}\,\kappa r_2\right) K_n\left(\frac{1+a}{2}\,\kappa r_2\right) = \frac{j\zeta_0}{\pi\beta^2}\, \mathbf{S}^T \mathbf{A}^{-1}\, \mathbf{S}\,. \qquad (3.1)$$

For a given value of γ, there always exists a frequency, from which **S** decays exponentially with frequency, indipendently from n.

Figures 6 and 7 represent the longitudinal coupling impedance for a slow and for a relativistic travelling particle: in the first case the impedance is mostly reactive, whereas, in the second one, it shows a resistive behaviour.

4 THE PARTICULAR CASE $\beta = 1$

Some remarks are necessary in the case $\beta = 1$, namely when the particle travels at the speed of light , because it is the particular case in most of the accelerator projects and for the slow decay of the impedance (Figure 8). In this case, taking the limit [6)]

$$\lim_{x \to 0} I_n\left(\frac{1-a}{2}\,x\right) K_n\left(\frac{1+a}{2}\,x\right) = \frac{1}{2n}\left(\frac{1-a}{1+a}\right)^n , \qquad (4.1)$$

the expansion (3.1) simplifies as

$$Z_{||}(k) = \frac{j\zeta_0}{2\pi} \sum_{n=1}^{\infty} \frac{F_n}{n}\left(\frac{1-a}{1+a}\right)^n . \qquad (4.2)$$

In such a case, the asymptotic behaviour of **S** with the frequency is not tha same of the previous one, so that it is convenient to subtract the asymptotic behaviour ($k \to \infty$) of the impedance.

First we can reduce the system (1.4) to a unique integral equation. In fact, because of the following limit [6)]

$$\lim_{\kappa \to 0} A\, K_1(\kappa r) = \frac{jqk}{\pi r} , \qquad (4.3)$$

the system can be rewritten as

$$\int_0^\infty w\, F(w)\, J_1(wr)\, dw = \begin{cases} 0 & 0 \le r < r_1 , \\ \displaystyle\int_0^\infty w\, F(w) \frac{jk - \sqrt{w^2 - k^2}}{jk} J_1(wr)\, dw + \frac{q}{\pi r} & r_1 < r < r_2 , \\ 0 & r > r_2 . \end{cases} \tag{4.4}$$

Taking the inverse Hankel transform of the left side of (4.4), we can easily state that the unknown F(w) satisfies the integral equation

$$F(w) = \frac{q}{\pi w}\left[J_0(wr_1) - J_0(wr_2)\right] + \int_0^\infty u\, F(u) \frac{jk - \sqrt{u^2 - k^2}}{jk} L(w,u)\, du , \tag{4.5}$$

where the function L(w,u) is defined as

$$L(w,u) = \int_{r_1}^{r_2} r\, J_1(wr)\, J_1(ur)\, dr =$$
$$= \frac{u\left[r_2 J_1(wr_2) J_0(ur_2) - r_1 J_1(wr_1) J_0(ur_1)\right] - w\left[r_2 J_0(wr_2) J_1(ur_2) - r_1 J_0(wr_1) J_1(ur_1)\right]}{w^2 - u^2} \tag{4.6}$$

An high frequency approximation of the equation (4.5) can be obtained in the limit of k going to infinity

$$F(w) \to \frac{q}{\pi w}\left[J_0(wr_1) - J_0(wr_2)\right] , \quad \text{when } k \to \infty . \tag{4.7}$$

As a consequence, the longitudinal impedance (1.6) becomes [6)]

$$Z_{||}(k) \to \frac{\zeta_0}{q} \int_0^\infty F(w)\, dw = \frac{\zeta_0}{\pi} \ln\left(\frac{r_2}{r_1}\right) = -\frac{\zeta_0}{\pi} \ln a , \quad \text{when } k \to \infty . \tag{4.8}$$

At this point we can state that we are now able to evaluate the impedance in a wide range of frequencies; this fact allows us to get an accurate estimate of the wake field, defined as the inverse Fourier transform of the impedance

$$\varepsilon_0 W_{||}(\tau) = \frac{1}{2\pi\zeta_0}\int_{-\infty}^{+\infty} Z_{||}(k)\exp(jkc\tau)dk = \frac{1}{\pi\zeta_0}\int_{0}^{\infty}\left[R_{||}(k)\cos(kc\tau) + X_{||}(k)\sin(kc\tau)\right]dk \quad (4.9)$$

In a previous paper [7)] we evaluated the wake of a particle travelling through the centre of a round aperture in a perfectly conducting plane; in such a case, the wake was not a causal function when $\beta = 1$, but the reason of this paradox was not understood. Now we can state that the cause was the infiniteness of the plane .
The Figure 10 clearly shows that the function (4.9) is a causal function, vanishing for $t<0$, in the case of a particle travelling at the speed of the light (the arrow indicates the presence of a pulse function); in other words, the wake field is different from zero only when the electromagnetic field interacts with the iris, interaction starting in $t=0$.

5 CONCLUSIONS AND PERSPECTIVES

We presented a simple and refined way to evaluate the longitudinal coupling impedance of a perfectly conducting annular ring. The relevant electromagnetic quantities have been expanded into a double Neumann series, and therefore a triple boundary value problem has been transformed into a system of linear equations, able to give an accurate solution with matrices of small dimensions. We have also shown that the wake field, namely the inverse transform of the impedance, is a causal function if the particle travels at the speed of light.

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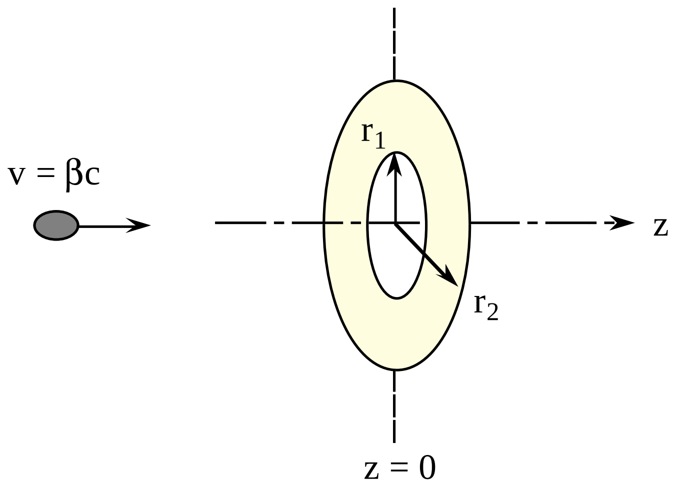

FIG. 1: The geometry of the problem.

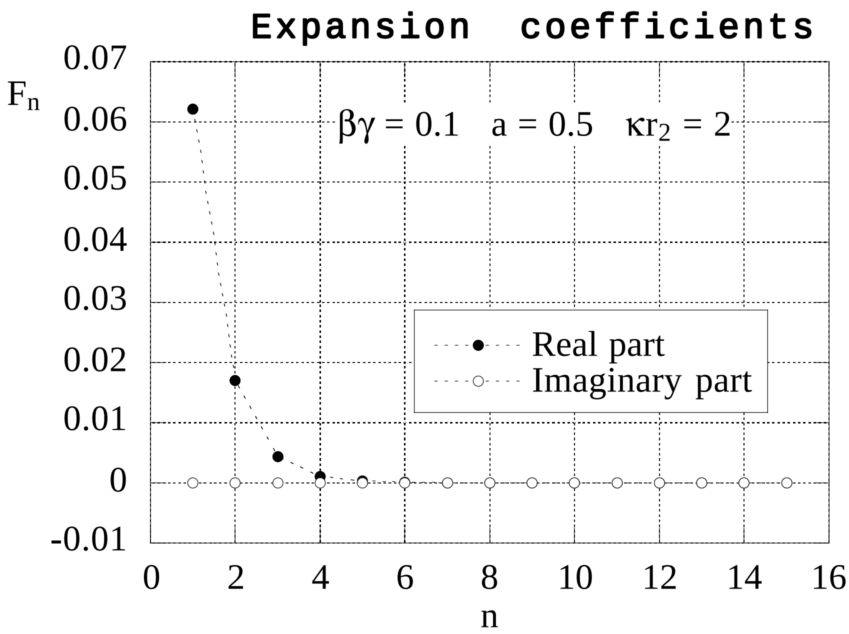

FIG 2: Expansion coefficients as a function of the matrix dimensions for a small value of the particle speed.

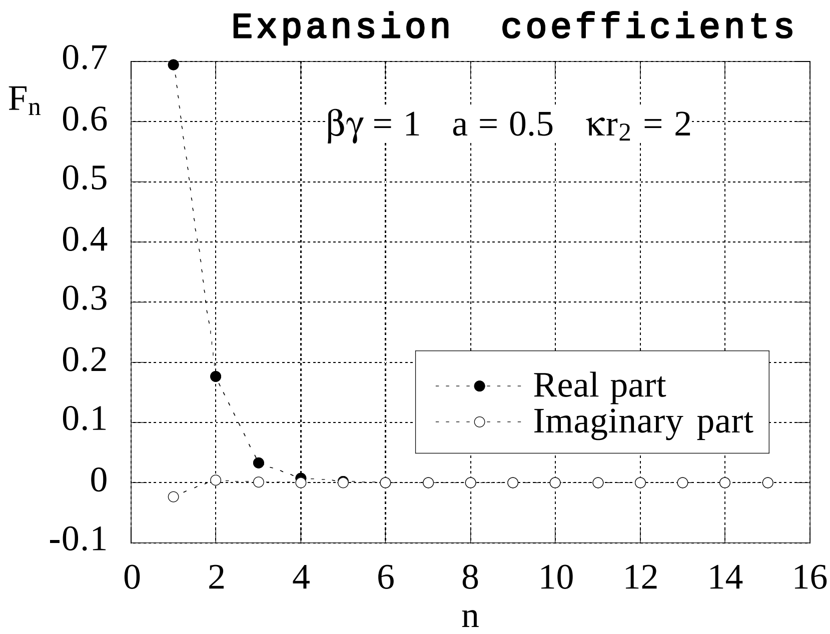

FIG. 3: Expansion coefficients as a function of the matrix dimensions for an intermediate value of the particle speed.

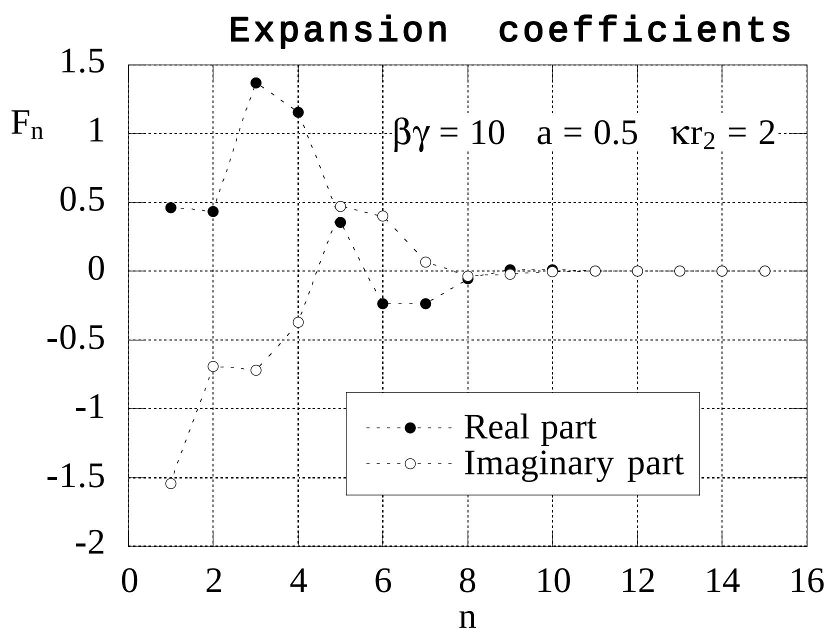

FIG. 4: Expansion coefficients as a function of the matrix dimensions for an ultra-relativistic particle.

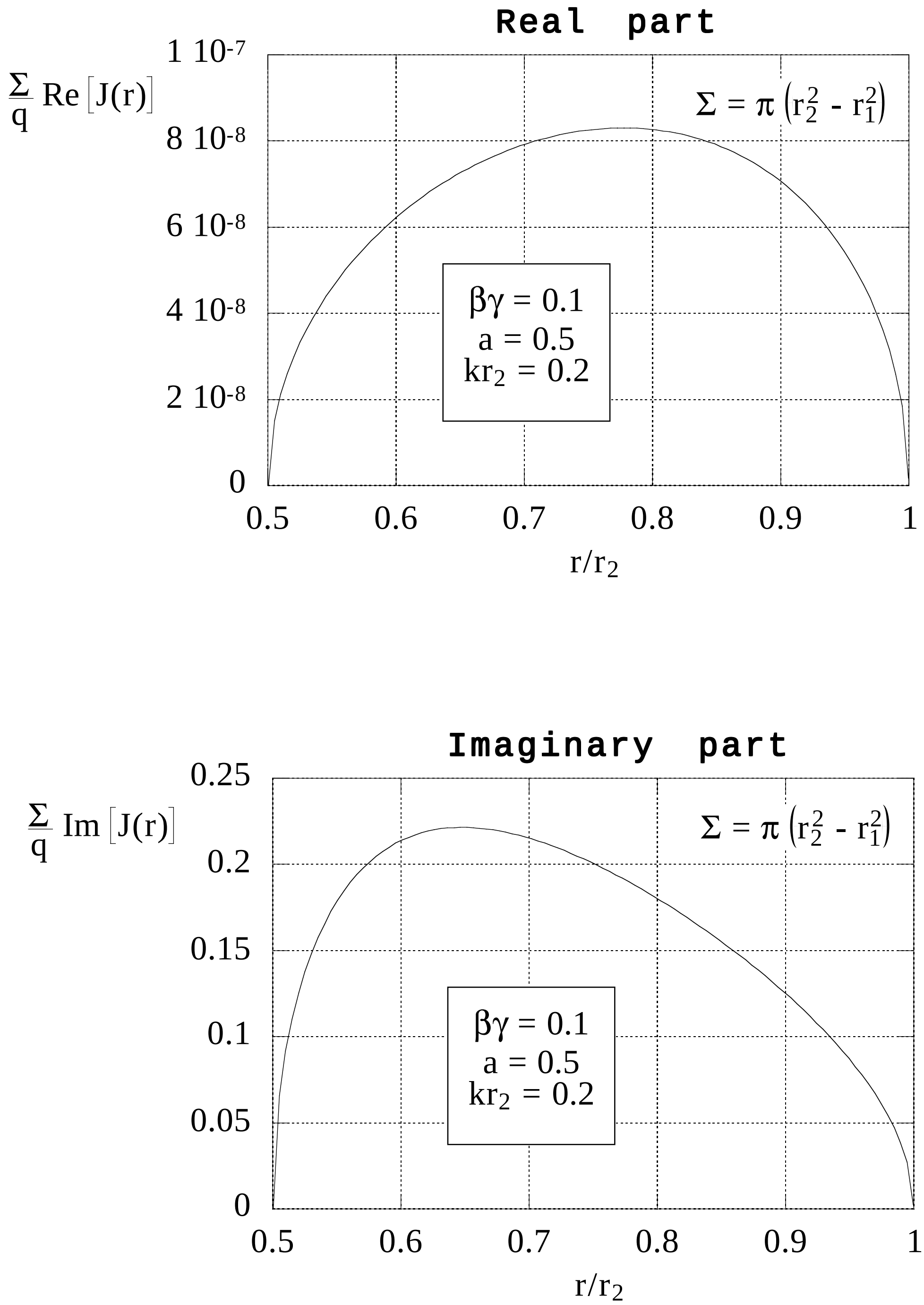

FIG. 5: An example of the current distribution density.

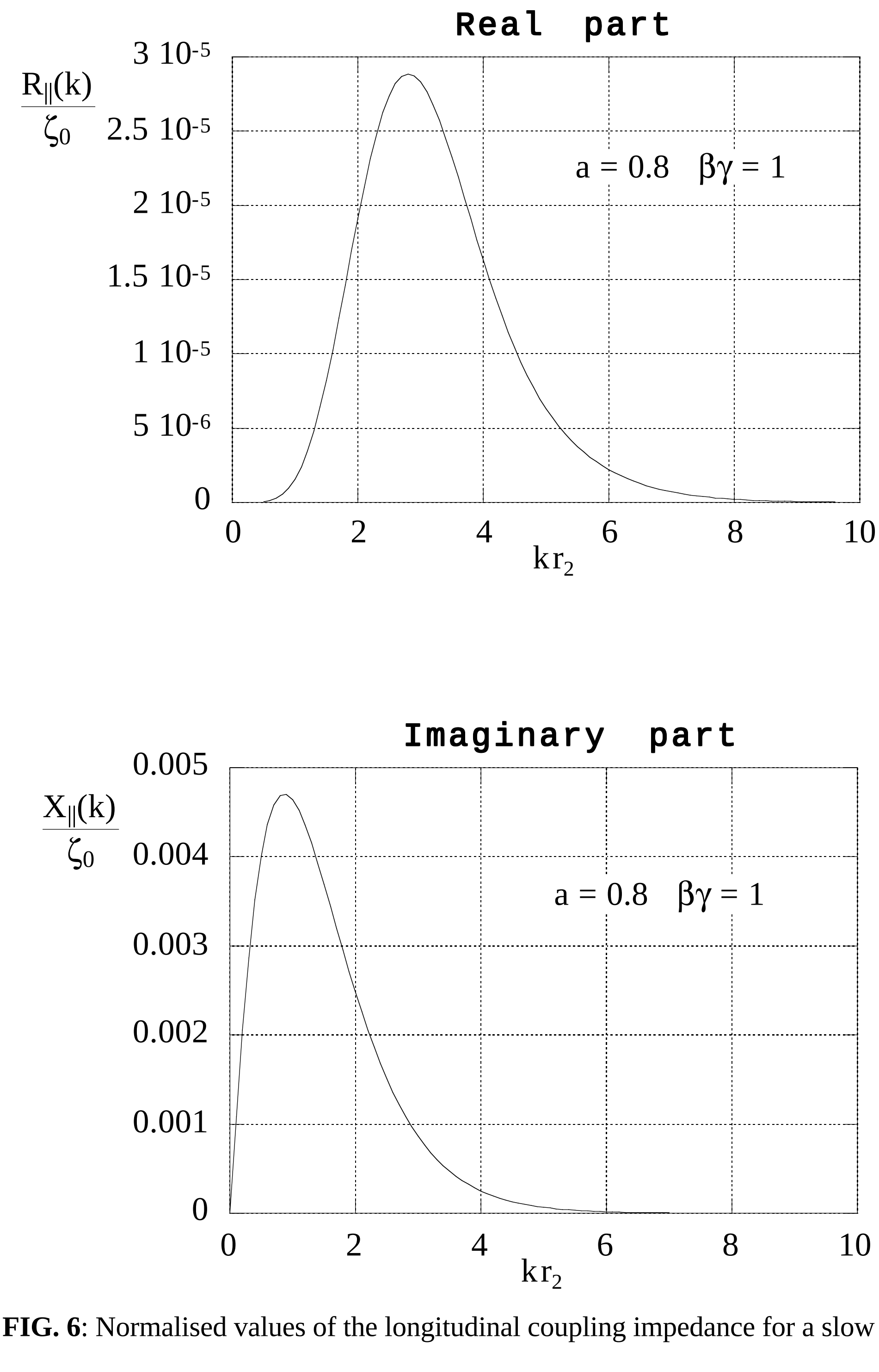

FIG. 6: Normalised values of the longitudinal coupling impedance for a slow particle.

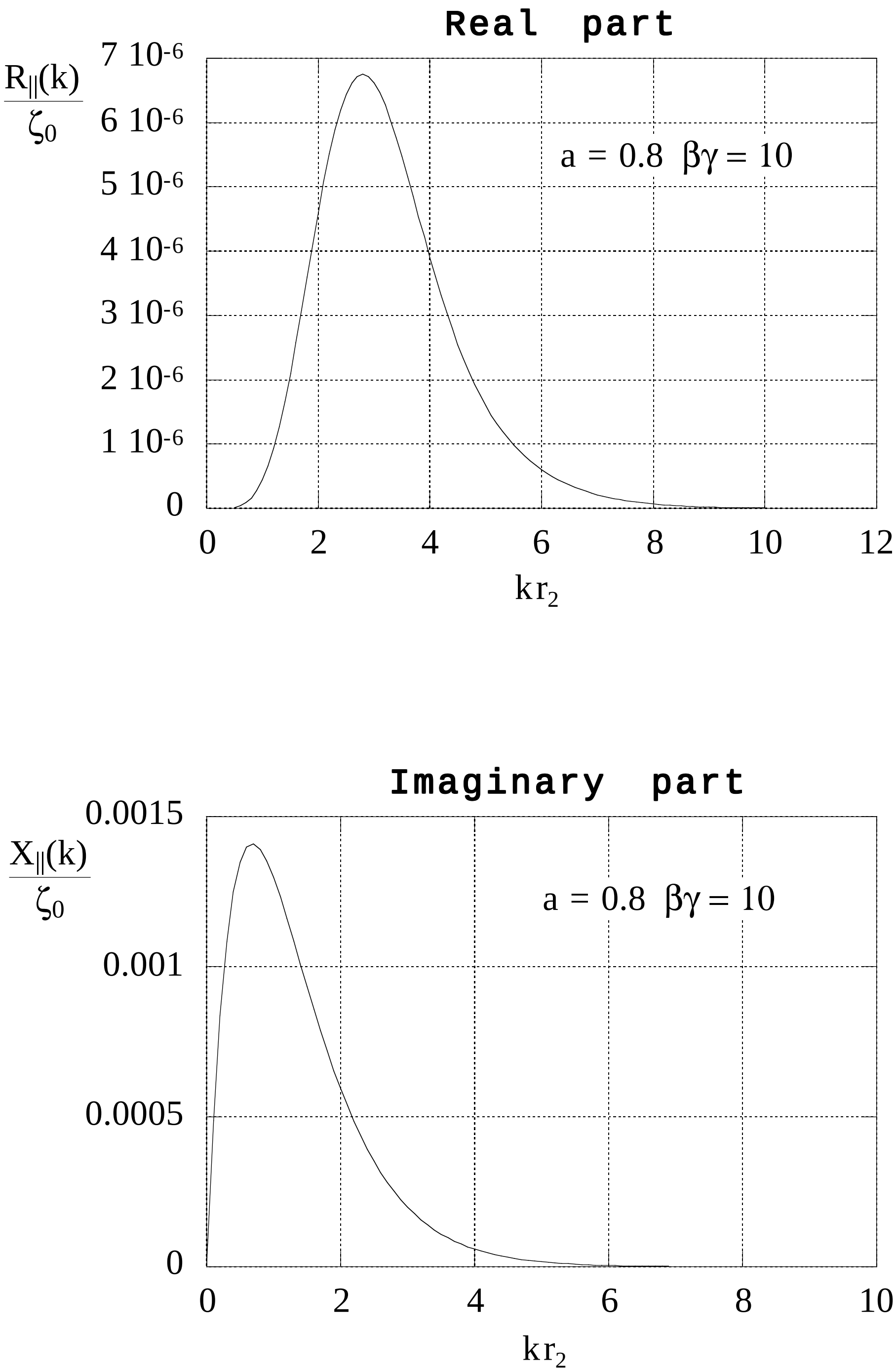

FIG. 7: Normalized values of the longitudinal coupling impedance for an ultra-relativistic particle.

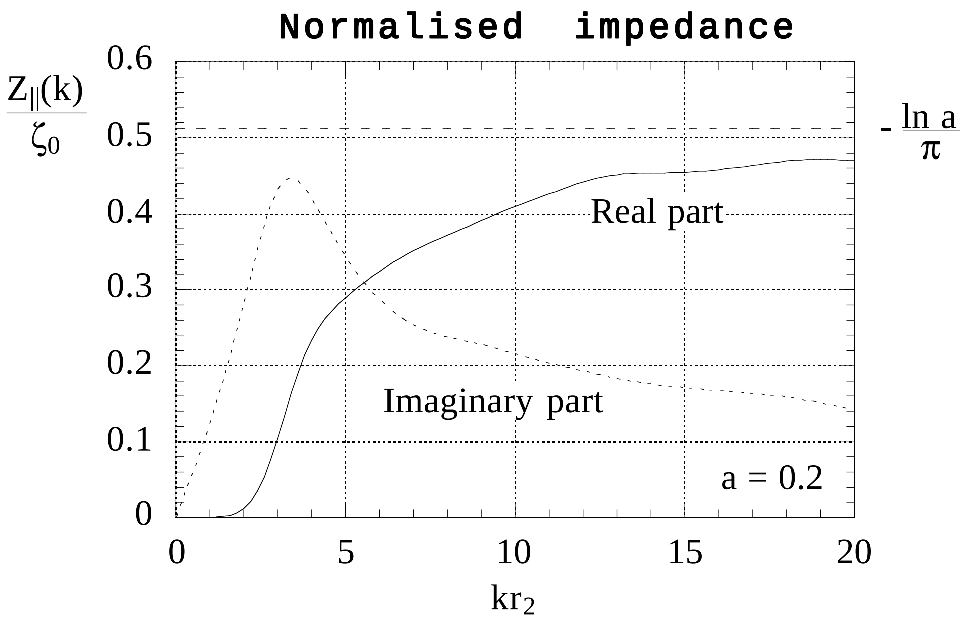

FIG. 8: Normalized values of the longitudinal coupling impedance for a particle travelling at the speed of light.

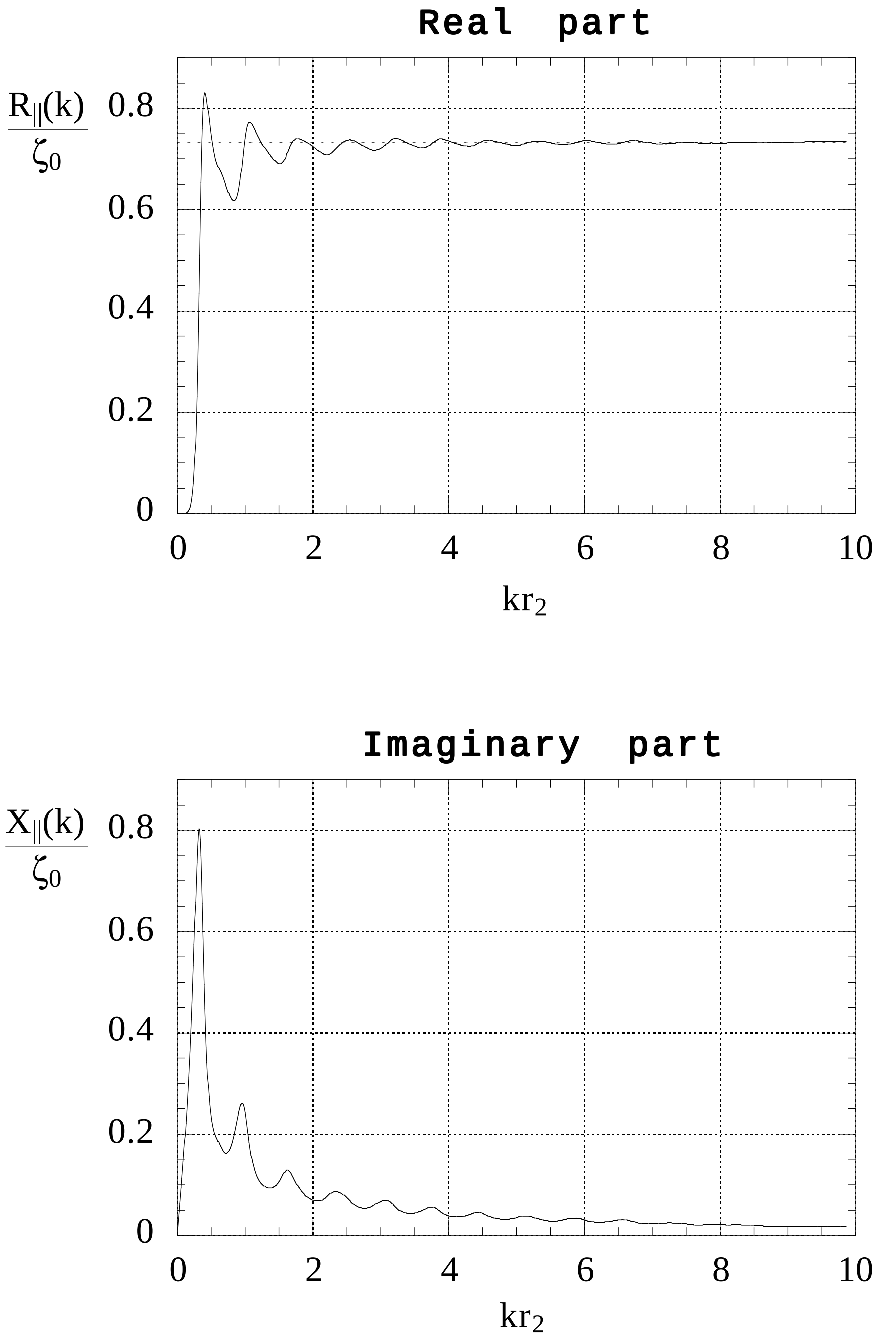

FIG. 9: Normalized values of the longitudinal coupling impedance in the case a = 0.1 and β = 1.

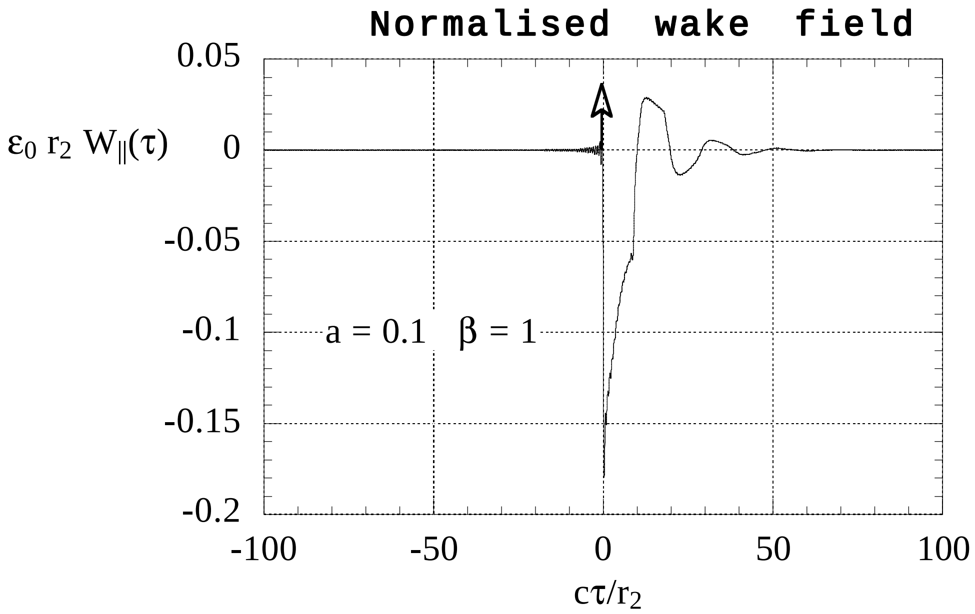

FIG. 10: Normalized values of the wake field.