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THE 2 RF-CAVITIES PARAMETRIC AMPLIFIER

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Abstract

We shall illustrate a parametric amplifier, made of two coupled RF cavities, whose non-linear element is given by the Stern-Gerlach interaction between a polarized beam and a TE RF cavity, tuned in a suitable way. An experimental verification is suggested to be carried out at the South Hall Ring of MIT-Bates.

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1 Introduction

The Stern-Gerlach interaction, between a moving magnetized particle and a RF electromagnetic field, can be detected by making us of two cavities coupled[1] as a parametric amplifier. Here we intend to illustrate the proposed method quite in detail.

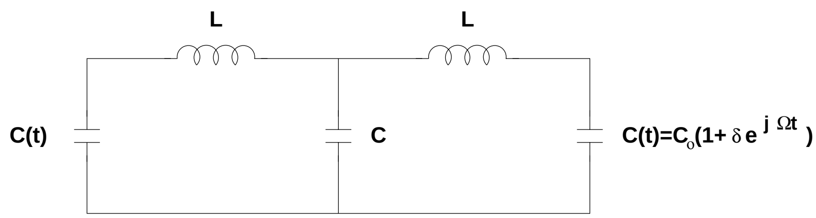

Figure 1: Coupled lumped circuits.

This device is assimilated[2][3] to a system made of two coupled lumped LC circuits (see Fig. 1), where currents flowing in the same wise correspond to cavities in their symmetric mode, while counter-flowing currents are congruous with the antisymmetric mode.

The two RF cavities are coupled through a static element such a hole in the common wall, a wave guide, etc., while the actually time varying items will be just these cavities, via the variation of their tune(s) due to the Stern-Gerlach induced energy exchange.

Therefore, the coupling capacitor C in the analogous lumped system has to be constant, while the two capacitors will vary in time as

$$C(t) = C_0(1 + \delta e^{j\Omega t}) \tag{1}$$

The circuit equations are

$$L\frac{d^2 q_1}{dt^2} + \frac{q_1}{C_0(1 + \delta e^{j\Omega t})} - \frac{q_2 - q_1}{C} = 0 \tag{2}$$

$$L\frac{d^2 q_2}{dt^2} + \frac{q_2}{C_0(1 + \delta e^{j\Omega t})} + \frac{q_2 - q_1}{C} = 0 \tag{3}$$

where q_1 and q_2 are the electric charges flowing in the two circuits nd

$$\Omega = \omega_2 - \omega_1 \tag{4}$$

is the difference between the two frequencies

$$\omega_1 = \sqrt{\frac{1}{LC_0}} = \frac{1}{\sqrt{LC_{11}}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{1}{LC_0} + \frac{2}{LC}} = \frac{1}{\sqrt{LC_{22}}} \tag{5}$$

characterizing each circuit.

Before proceeding in integrating these equations, the energy stored in the two normal modes, i.e. co-flowing and counter-flowing currents, will be evaluated[2] under the very sensible hypothesis of momentarily neglecting the term δ, which is indeed very small. Moreover, we define as mode amplitudes

$$\begin{cases} a_1 = \frac{1}{2}\sqrt{L}(I_1 + j\omega_1 C_{11} V_1) \simeq A_1 e^{j\omega_1 t} \\ a_2 = \frac{1}{2}\sqrt{L}(I_2 + j\omega_2 C_{22} V_2) \simeq A_2 e^{j\omega_2 t} \end{cases} \tag{6}$$

where

$$I_1 = \frac{d}{dt}(q_1 + q_2),\ I_2 = \frac{d}{dt}(q_2 - q_1),\ V_1 = \frac{(q_1 + q_2)}{C},\ V_2 = (q_2 - q_1)\left(\frac{1}{C} + \frac{2}{C_0}\right)$$

In this approach, twice the modulus squared of the amplitudes (6) represents the stored energy; in fact

$$2|a_1|^2 = 2a_1 a_1^* = \frac{1}{2}LI_1^2 + \frac{1}{2}L\omega_1^2 C_{11} V_1^2 = \frac{1}{2}LI_1^2 + \frac{1}{2}C_{11}V_1^2 = U_1 \tag{7}$$

and the same result can be found for U_2.

2 Energy Considerations

At this stage we find the energy transferred from a mode to the other one, also taking the dissipation into account. In order to achieve this, it is convenient to make use of the most general relations[2] governing the coupling between modes:

$$\begin{cases} \frac{da_1}{dt} = j\omega_1 a_1 + c_{12} e^{j\Omega t} a_2 \\ \frac{da_2}{dt} = j\omega_2 a_2 + c_{21} e^{j\Omega t} a_1 \end{cases} \tag{8}$$

where

$$\begin{cases} c_{12} = -j\frac{\omega_1}{4}\frac{\Delta C}{\sqrt{C_{11}C_{22}}} \\ c_{21} = j\frac{\omega_2}{4}\frac{\Delta C}{\sqrt{C_{11}C_{22}}} \end{cases} \tag{9}$$

are the coupling coefficients and

$$\Delta C = C_0 \delta \tag{10}$$

is deduced from eq. (1). Besides, bearing in mind the rightmost side of eq. (6), eq. (8) can be written as

$$\begin{cases} \frac{dA_1}{dt} \simeq c_{12} A_2 \\ \frac{dA_2}{dt} \simeq c_{21} A_1 \end{cases} \tag{11}$$

An energy transfer from the mode 1 to the mode 2, in a realistic dissipative regime, can be described by introducing the dissipative time constant

$$\Gamma = \frac{\omega_1}{2Q_1} = \frac{\omega_2}{2Q_2} \tag{12}$$

where

$$Q_1 = \omega_1 \frac{U_1}{P_1} \quad \text{and} \quad Q_2 = \omega_2 \frac{U_2}{P_2} \tag{13}$$

are the usual quality factors in their most general definition as a quantity proportional to the ratio between the stored energy and the dissipated power. Then, the second equation appearing in (11) must be modified into

$$\frac{dA_2}{dt} = c_{21} A_1 - \Gamma A_2 \tag{14}$$

whose integral is

$$A_2(t) = \frac{c_{21} A_1}{\Gamma} (1 - e^{-\Gamma t}) = (A_2)_{\mathrm{M}} (1 - e^{-\Gamma t}) \tag{15}$$

having assessed $A_1(0) \neq 0$ and $A_2(0) = 0$ as initial conditions. This equation gives us information about the time needed to reach the steady state condition, i.e. the balance between the energy transferred from mode 1 to mode 2 and the energy dissipated in mode 2. Setting as f a certain fraction of $(A_2)_{\mathrm{M}}$, the time required for attaining this value is

$$t^* = \frac{1}{\Gamma} \ln \left(\frac{1}{1-f} \right) = \frac{6.908}{\Gamma} \quad (\text{for } f = 0.999) \tag{16}$$

Bearing in mind eqs. (12) and (16), at the operating frequency of $3\ GHz$ we can gather in Table 1 the loading times for either a room temperature copper cavity or a superconducting cavity.

Table 1: Mode loading-times

Cavity	Copper	SC
Q_2	3×10^4	10^9
$\frac{1}{\Gamma}$	$3.18 \times 10^{-6}\ s$	$0.106\ s$
t^*	$22\ \mu s$	$0.73\ s$

A quick glance at the last rows indicates that the SC solution would imply quite a longer time before the signal in mode 2 reaches the full intensity. On the other hand, making use of eqs. (9), (12) and (15) the maximum energy results to be

$$(U_2)_{\mathrm{M}} \simeq 2|(A_2)_{\mathrm{M}}|^2 = 2\frac{|c_{21}|^2|(A_1)_{\mathrm{M}}|^2}{\Gamma^2} = \frac{|c_{21}|^2(U_1)_{\mathrm{M}}}{\Gamma^2} = Q_2^2\,\frac{(\Delta C)^2}{4C_{11}C_{22}}\,(U_1)_{\mathrm{M}} \tag{17}$$

i.e. quite a huge gain! Therefore a compromise should be found between the need of having a fast answer from the instrumentation, linearly depending on Q_2, and the advantage of a big amplification of a possibly very weak signal, which is proportional to the square of the quality factor.

3 Further Solutions

For a better understanding, it is convenient to integrate eqs. (2) and (3) with the best approximation possible; then, in order to do that, we add and subtract these equations, obtaining:

$$\frac{d^2}{dt^2}(q_1+q_2) + \frac{\omega_1^2}{C_0(1+\delta e^{j\Omega t})}(q_1+q_2) = 0$$

$$\frac{d^2}{dt^2}(q_2-q_1) + \frac{\omega_1^2}{C_0(1+\delta e^{j\Omega t})}(q_2-q_1) + 2\omega_1^2(q_2-q_1) = 0$$

or, setting

$$G = q_1 + q_2 \quad \text{and} \quad F = q_1 - q_2, \tag{18}$$

$$\frac{d^2G}{dt^2} + \frac{\omega_1^2}{C_0(1+\delta e^{j\Omega t})}G = 0 \tag{19}$$

$$\frac{d^2F}{dt^2} + \frac{\omega_1^2}{C_0(1+\delta e^{j\Omega t})}F + 2\omega_1^2 F = 0 \tag{20}$$

where ω_1 has already been defined in eq. (5) and

$$\omega_C = \frac{1}{\sqrt{LC}} \tag{21}$$

As the coupling is weak, we may write

$$G \simeq Ae^{j\omega(1+\alpha e^{j\Omega t})t} = Ae^{f(t)} \tag{22}$$

$$F \simeq Be^{j\omega(1+\alpha e^{j\Omega t})t} = Be^{f(t)} \tag{23}$$

Inserting e.g. eq. (22) into eq. (19) and making up our mind on the following derivatives

$$\frac{d^2G}{dt^2} = A\left[\frac{d^2f}{dt^2} + \left(\frac{df}{dt}\right)^2\right]e^{f(t)}$$

$$\frac{df}{dt} = j\omega(1+\alpha e^{j\Omega t}) - \omega\Omega\alpha(e^{2j\Omega t})t$$

$$\left(\frac{df}{dt}\right)^2 = -\omega^2(1+\alpha e^{j\Omega t})^2 + \omega^2\Omega^2\alpha^2(e^{2j\Omega t})t^2 - 2j\omega^2\Omega\alpha(1+\alpha e^{j\Omega t})e^{j\Omega t}$$

$$\frac{d^2f}{dt^2} = -\omega\Omega\alpha(1+\alpha e^{j\Omega t})e^{j\Omega t}$$

we obtain:

$$(-2\omega\Omega\alpha - j\omega\Omega^2\alpha)e^{j\Omega t} - \omega^2 - 2\omega^2\alpha e^{j\Omega t} - \omega^2\alpha^2e^{2j\Omega t} + \omega^2\Omega^2\alpha^2t^2e^{2j\Omega t} - 2\omega^2\Omega\alpha te^{j\Omega t} - 2j\omega^2\Omega\alpha^2te^{2j\Omega t} + \omega_1^2 - \omega_1^2\delta e^{j\Omega t} = 0$$

Neglecting all terms like $\Omega\alpha$, $\Omega^2\alpha$, α^2, $\Omega^2\alpha^2$, $\Omega\alpha^2$, we find

$$-\omega^2 - 2\omega^2\alpha e^{j\Omega t} + \omega_1^2 - \omega_1^2\delta e^{j\Omega t} = 0$$

or

$$\omega \simeq \omega_1[1 - (\alpha + \frac{1}{2}\delta)e^{j\Omega t}] \tag{24}$$

Therefore, bearing in mind eq. (24), eq. (22) becomes

$$G \simeq Ae^{j\omega_1(1-(\frac{1}{2}\delta)e^{j\Omega t})t} \tag{25}$$

having neglected the term $\alpha\frac{1}{2}\delta$ too. We can now state that the solution consists of a wave of frequency ω_1 modulated by a signal characterized by a frequency Ω and an amplitude depending on the value of the coupling term δ, as can be easily understood by slightly modifying eq. (25) into

$$G \simeq Ae^{j\omega_1 + j\beta\omega_1 t\cos\Omega t} \tag{26}$$

with $\beta = -\frac{1}{2}\delta$.

As shown in the theory[4] of exponential modulation, the spectral density of a wave modulated with a nonlinear signal can be analysed through an expansion of Bessel functions $J_n(\beta)$; namely

$$G(t) = A\sum_{n=-\infty}^{n=\infty} J_n(\beta)\cos(\omega_1 + n\Omega) \tag{27}$$

or, since $|\beta| \ll 1$,

$$G(t) \simeq A\left[J_{-1}(\beta)\cos(\omega_1 - \Omega)t + J_0(\beta)\cos\omega_1 t + J_1(\beta)\cos(\omega_1 + \Omega)t\right] \tag{28}$$

implying the existence of two side-bands of frequencies $\omega_1 - \Omega$ and $\omega_1 + \Omega$ about the main frequency ω_1.

4 The MIT-Bates Case

Nonlinear couplings, such as the one so far considered, can be profitably treated by the Manley-Rowe[5] relations. In our case, where the accessible frequencies are three only, i.e. ω_1, ω_2 and Ω, these relations reduce to

$$\begin{cases} \frac{P_1}{\omega_1} - \frac{P_\Omega}{\Omega} = 0 \\ \frac{P_2}{\omega_2} + \frac{P_\Omega}{\Omega} = 0 \end{cases} \tag{29}$$

where P_1, P_2 and P_Ω are the powers corresponding to the frequencies ω_1, ω_2 and Ω respectively.

Recalling[1] that the expression of the Stern-Gerlach induced energy exchange is

$$\delta U \simeq 2NP\mu^* B_0 \gamma^2 \tag{30}$$

where N is the number of particle s crossing the cavity 1, for instance, P is the beam polarization, μ^* is the particle magnetic moment, B_0 is the RF magnetic field and γ is the particle Lorentz factor. Let us make the following set of hypotheses:

1. all the N particles are concentrated in a single bunch revolving with an angular frequency $\omega_{\rm rev} = \frac{2\pi}{\tau_{\rm rev}}$;
2. the two RF cavities are fed with two very close each other: i.e. $\omega_1 \simeq \omega_2$;
3. the fundamental choice $\Omega = \omega_{\rm rev}$ is made.

Therefore, from the 1st hypothesis we can deduce that

$$P_\Omega = \frac{\delta U}{\tau_{\rm rev}} \tag{31}$$

while eq. (29) yields

$$P_2 = -\frac{\omega_2}{\Omega} P_\Omega = -\frac{\omega_{\rm RF}}{\omega_{\rm rev}} P_\Omega = -\frac{\tau_{\rm rev}}{\tau_{\rm RF}} P_\Omega = -\frac{\delta U}{\tau_{\rm RF}} \tag{32}$$

or, bearing in mind eq. (30),

$$P_2 = |P_2| \simeq 2NP\frac{\mu^* B_0}{\tau_{\rm RF}} \gamma^2 \tag{33}$$

having disregarded the sign $-$ since we can choose at leisure the direction of the beam polarization. Considering the experimental test to be carried out at the MIT-Bates[6] electron ring, we keep in mind few but useful data only, i.e.

$$\begin{cases} \mu^* = 9.27 \times 10^{-27}\, TJ^{-1} \\ \gamma \simeq 10^3 \\ \tau_{\mathrm{RF}} = 0.35\, ns \\ \tau_{\mathrm{rev}} = 0.634\, ns \end{cases} \tag{34}$$

Then, referring to Table 1, we obtain:

$$N_{\mathrm{turns}} = \frac{t^*}{\tau_{\mathrm{rev}}} = \begin{cases} 35 & \text{(copper cavity)} \\ 10^6 & \text{(SC cavity)} \end{cases} \tag{35}$$

As far as N and B_0 are concerned, we shall consider first

$$N = 8.1 \times 10^{10} \quad \text{and} \quad B_0 = 0.1\, T \tag{36}$$

which are quite over optimistic, since it would be pretty difficult to squeeze the whole electron beam in a single bunch and such a value of the RF magnetic field can be, perhaps, too big. Hence we make a rather pessimistic choice, lowering B_0 by a factor of 100 and devising a single bucket filled by a reduced number of electrons

$$n = N/(\text{harmonic number}) = \frac{8.1 \times 10^{10}}{1812} = 4.47 \times 10^7 \quad \text{and} \quad B_0 = 10^{-3}\, T \tag{37}$$

Therefore we obtain either

$$\frac{P_2}{P} = 429\, watt \quad \text{or} \quad \frac{P_2}{P} = 2.37\, mwatt \tag{38}$$

meaning that, even in the worst example, feasible measures can be performed.

5 Intuitive Resume of the Measure.

We have already seen some maths regarding our system, now we are resuming the experiment in a simple way from the parametric point of view.

The coupled cavities can be reduced to two oscillators (modes) of angular frequencies $\omega_1 = \sqrt{\frac{1}{LC_0}}$ and $\omega_2 = \sqrt{\frac{1}{LC_0} + \frac{2}{LC}}$ which correspond to cavities ringing with the same angular phase (ω_1) and with a phase separation equal to π (ω_2). In the first mode the energy U_1 is stored and is kept constant by a feedback system which ensures that the fields can work in the correct way while, in the beginning, the second mode is empty.

Now we can consider the bunch crossing the cavity as a non-linear element, in particular a non-linear capacitor driven by the beam. If we consider the frequency domain, we should aspect that particles excite the capacitor with the main frequency equal to the revolution frequency. Then, for the Manley-Rowe theorem the non-linear element will supply[3] power at frequencies $\omega_r = \omega_1 + r\omega_{\mathrm{rev}}$ with r integer.

It is useful to stress that the bunch interaction with the cavity is due both to the Stern-Gerlach force and to the electric field, but our choice of fields in the cavity makes the magnetic contribution much bigger than the electric one.

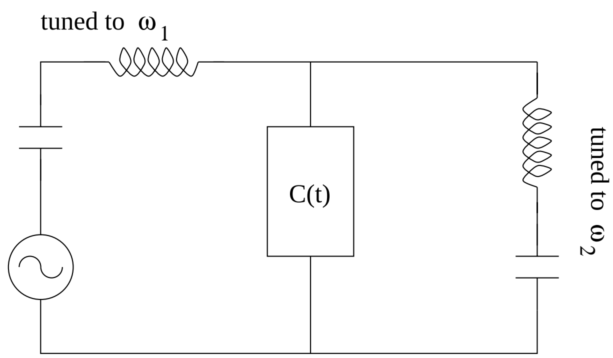

Figure 2: Circuit describing the parametric conversion.

What stated can become clearer looking at Fig. 2, where the beam is “seen” from the cavity as a perturbation $C(t)$ and ω_2 is the idler frequency from which we collect the signal via parametric conversion. The frequency ω_2 has to be equal to $\omega_r = \omega_1 + \omega_{\rm rev}$ and from this relation we can deduce the best value for the coupling between the cavities: in fact

$$\omega_{\rm rev} = \omega_2 - \omega_1 = \sqrt{\frac{1}{LC_0} + \frac{2}{LC}} - \sqrt{\frac{1}{LC_0}} \tag{39}$$

In practice, we may think of tuning the cavity crossed by the beam, conceived as a single bunch, with a frequency $\nu_1 = \omega_1/2\pi = 2.8560\, GHz$ and the other cavity with a frequency $\nu_2 = \omega_2/2\pi = 2.8576\, GHz$, having thus $\nu_2 - \nu_1 = \nu_{\rm rev} = 1.576\, MHz$, which is just the revolution frequency of the South Hall Ring at MIT-Bates.

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