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The Slowly Varying Envelope Approximation Revised

R. Bonifacio and R. M. Caloi
INFN and Università di Milano
Via Celoria, 16, 20133 Milano, Italy.

Abstract

We derive the limit of validity of the Slowly Varying Envelope Approximation (SVEA) as a function of the "bulk" velocity v of the radiating system, which reads $\ell_p \gg \lambda(1 - v/c)$, being ℓ_p the radiation pulse length. This condition reduces to the usual SVEA in the limit $v/c \ll 1$, whereas it is sensibly relaxed in the relativistic limit. The example of a Free Electron Laser is discussed.

1 Introduction

In the study of the interaction of radiation with matter, the Maxwell wave equation is of fundamental importance. This is a second order partial differential equation in space and time coordinates which, under some conditions that we shall investigate, reduces to a first order partial differential equation.

Let us consider the one-dimensional case for one component of the transverse electric field $\overline{E}(z,t)$ and for the transverse current density $\overline{J}(z,t)$

$$\frac{\partial^2 \overline{E}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \overline{E}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial \overline{J}}{\partial t} \tag{1}$$

Usually, when a primary radiation wavelength λ arises in the study of a physical problem and the main propagation is in the $+z$ direction, as the spontaneous emission in a FEL, it is useful to introduce the complex amplitudes $E(z,t)$ and $J(z,t)$ defined such as

$$\overline{E}(z,t) = E(z,t)e^{ik(z-ct)} \tag{2}$$
$$\overline{J}(z,t) = J(z,t)e^{ik(z-ct)} \tag{3}$$

where $k = \frac{2\pi}{\lambda}$ and λ is the radiation wavelength.

E and J have an immediate physical meaning in the case in which they do not vary sensibly over a wavelength: they represent the envelope of the electric field and of the current respectively.

Replacing (2) and (3) into (1) we have

$$\left(\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}\right) + 2ik\left(\frac{\partial E}{\partial z} + \frac{1}{c}\frac{\partial E}{\partial t}\right) = \frac{4\pi}{c^2}\left(\frac{\partial J}{\partial t} - ickJ\right) \tag{4}$$

Equation (4) can be simplified to, as a first approximation, a first order differential equation keeping only the largest terms on each side.

2 The usual SVEA

Let us briefly reconsider the hypotheses underlying the usual SVEA [1], which consist in assuming

$$\left|2ik\left(\frac{\partial E}{\partial z} + \frac{1}{c}\frac{\partial E}{\partial t}\right)\right| \gg \left|\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}\right| \tag{5}$$

and

$$|ckJ| \gg \left|\frac{\partial J}{\partial t}\right|$$

so that (4) becomes

$$\frac{\partial E}{\partial z} + \frac{1}{c}\frac{\partial E}{\partial t} = -\frac{2\pi J}{c} \tag{6}$$

Equation (6) is the so called SVEA counterpart of eq.(1). A sufficient condition for the validity of (5) is

$$\left|\frac{\partial^2 E}{\partial z^2}\right| \ll 2k\left|\frac{\partial E}{\partial z}\right| \; , \; \left|\frac{\partial^2 E}{\partial t^2}\right| \ll 2ck\left|\frac{\partial E}{\partial t}\right| \; , \; \left|\frac{\partial J}{\partial t}\right| \ll ck|J| \tag{7}$$

neglecting the cases in which the terms on the l.h.s. cancel each other and the terms on the r.h.s. do not. Assuming the scaling argument

$$\frac{\partial E}{\partial z} \simeq \frac{1}{c}\frac{\partial E}{\partial t} \simeq \frac{1}{\ell_p}E \,, \quad \frac{\partial J}{\partial z} \simeq \frac{1}{c}\frac{\partial J}{\partial t} \simeq \frac{1}{\ell_J}J \tag{8}$$

where ℓ_p and ℓ_J define the scale of variation of the pulse and the current, we can write (7) as

$$\ell_p \gg \lambda \,; \; \ell_J \gg \lambda \tag{9}$$

i.e. the radiation and the current pulse show a slow variation over a wavelength scale.

3 The generalized SVEA

Let us now write (4) in terms of z' and z_1 defined as

$$\begin{cases} z' &= z \\ z_1 &= z - vt \end{cases}$$

where v is the bulk velocity of the radiating system.
One obtains easily

$$\left(\frac{\partial}{\partial z'} + (1-\beta)\frac{\partial}{\partial z_1}\right)\left(\frac{\partial E}{\partial z'} + (1-\beta)\frac{\partial E}{\partial z_1} + 2\beta\frac{\partial E}{\partial z_1} + 2ikE\right) = -\frac{4\pi}{c}\left(\beta\frac{\partial J}{\partial z_1} + ikJ\right) \tag{10}$$

Let us now suppose that

$$\left|\frac{\partial E}{\partial z'} + (1-\beta)\frac{\partial E}{\partial z_1}\right| \ll \left|2\beta\frac{\partial E}{\partial z_1} + 2ikE\right| \tag{11}$$

With this condition, equation (10) can be written as

$$\left(\frac{\partial}{\partial z'} + (1-\beta)\frac{\partial}{\partial z_1}\right)\left(2\beta\frac{\partial E}{\partial z_1} + 2ikE\right) = -\frac{4\pi}{c}\left(\beta\frac{\partial J}{\partial z_1} + ikJ\right) \tag{12}$$

This is equivalent to the SVEA equation:

$$\frac{\partial E}{\partial z'} + (1-\beta)\frac{\partial E}{\partial z_1} = -\frac{2\pi J}{c} \tag{13}$$

In fact, taking the derivative of (13) to respect to z_1, one obtains

$$2\beta\frac{\partial}{\partial z_1}\left[\frac{\partial E}{\partial z'} + (1-\beta)\frac{\partial E}{\partial z_1}\right] = -\frac{4\pi\beta}{c}\frac{\partial J}{\partial z_1}$$

This equation plus eq.(13) multiplied by $2ik$ gives back equation (12). Going back to the initial variables z and t, equation (13) becomes the SVEA equation (6).
Hence a sufficient condition for the validity of (6) or (13) is the inequality (11).
For $\beta = 1$, our analisys becomes equivalent to that of Haselhoff [2].
For $\beta = 0$, since $z_1 = z = z'$, inequality (11) reduces to

$$\left|\frac{\partial E}{\partial z}\right| \ll k|E| \tag{14}$$

which is the usual SVEA.

A sufficient condition for the validity of (11) is that

$$\left|\frac{\partial E}{\partial z'}\right| \ll 2k|E| \ ; \ (1-\beta)\left|\frac{\partial E}{\partial z_1}\right| \ll 2k|E| \tag{15}$$

We are neglecting the case in which $2\beta\frac{\partial E}{\partial z_1}$ and $2ikE$ cancel each other. Defining a gain length ℓ_g and a pulse length ℓ_p such as

$$\frac{E}{\ell_g} \approx \frac{\partial E}{\partial z'} \ ; \ \frac{E}{\ell_p} \approx \frac{\partial E}{\partial z_1} \tag{16}$$

we have

$$\ell_g \gg \lambda \ ; \ \ell_p \gg \lambda(1-\beta) \tag{17}$$

The physical meaning of these conditions is obvious. The first one implies that the field can not be sensibly amplified in a wavelength. The second condition can be derived imposing that the electron-photon interaction time is much larger than the optical period. Note that the last condition reduces to the usual SVEA condition for $\beta \ll 1$ and it is much less restrictive if $\beta \simeq 1$, as in a FEL, since it can be written as $\ell_p \gg \lambda/\gamma^2$.

Futhermore, since in a FEL

$$\ell_p \approx \ell_c = (1-\beta)\ell_g \tag{18}$$

the two conditions (17) reduce to the single one

$$\lambda_w \gg \lambda\rho \tag{19}$$

This can be seen easily using the normalisation of ref.[3]

$$\bar{z}_1 = \frac{z_1}{\ell_c} \ ; \ \bar{z} = \frac{z'}{\ell_g} \tag{20}$$

where

$$\ell_c = \frac{\lambda}{4\pi\rho} \ ; \ \ell_g = \frac{\lambda_w}{4\pi\rho} \ ; \ \lambda = \lambda_w(1-\beta) \tag{21}$$

In this way, inequality (11) becomes

$$\left|\frac{\partial E}{\partial \bar{z}} + \frac{\partial E}{\partial \bar{z}_1}\right| \ll \left|\frac{2\beta}{1-\beta}\frac{\partial E}{\partial \bar{z}_1} + 2i\frac{\lambda_w}{\lambda\rho}E\right| \tag{22}$$

which gives immediately condition (19).

Finally let us note that in the steady–state regime, where $\frac{\partial E}{\partial z_1} = 0$, condition (19) becomes necessary and sufficient for the validity of the SVEA.

4 Conclusions

We have shown that the Slowly Varying Envelope Approximation is valid under conditions (17) which depend on the bulk velocity of the acting medium.

This condition gives a strongly relaxed limit of validity for the SVEA approximation in the case of the FEL, where the electrons move at relativistic velocity.

References

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