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A COMPREHENSIVE EIKONAL APPROACH TO HEAVY-ION GRAZING COLLISIONS

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Abstract

A formalism is developed to describe in a unified fashion intermediate-energy grazing collisions between two composite systems. The starting point is the reduction of the Schroedinger equation to an approximated Glauber–like differential equation, still retaining the effects of finite excitation energy in different channels. The basic feature characterizing this method is that the dynamics of the interaction of the two systems is described in terms of an evolution equation resembling, in spatial coordinates, the time dependent "interaction picture" equation. This allows one to introduce into the formalism the method of the phase-shift operator; in this connection several approximated expressions of the scattering amplitude are examined. The formalism is developed with reference to the heavy-ion physics and is proven to be well suited to describe elastic and inelastic scattering both in the framework of microscopic and collective descriptions of the colliding nuclei.

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I. INTRODUCTION

In the last years a large amount of experimental data has become available in the field of grazing collisions between heavy ions at intermediate energies. These experimental data include elastic and inelastic scattering, Coulomb excitation, stripping and pick-up reactions, spin and isospin exchange. Several analyses of these processes have been performed through different methods, encompassing coupled-channel calculations [1], semiclassical approximations [2,3], and modified Glauber approximations [4,5]. Since a unified formalism is lacking and several experiments are planned, it could be useful to set up a comprehensive approach capable of describing all these processes in a unified way.

The purpose of the present paper is to develop the description of multichannel-multistep heavy-ion scattering processes in terms of an extended eikonal approximation, taking into account the finite excitation energy in different channels. Some formal results of the type derived systematically in the present paper have been obtained in previous works of more applied character (see, e.g.,refs. [6,7,4,8,5,9,10]).

We illustrate our method in the framework of the heavy-ion physics, but the formalism, in principle, is more general. It is not our aim to develop here explicit applications, already carried out in previous papers in which related methods have been used; this point will be discussed in sec.V.

To make clear the nature of the problems we are confronted with and the results we obtain, it may be useful to consider first some previous works related to the present attempt. Feshbach and Hüfner [6] some years ago discussed the nucleon-nucleus diffusion process at high energy by introducing the eikonal approximation [11] in a coupled-channel representation of the scattering problem. At variance with the Glauber approximation, the Feshbach-Hüfner formalism takes full account of Q-value effects. In its lowest order approximation, the final scattering amplitude so obtained is expressed through a simple integral on the impact parameter; the essential content of the integrand is the exponential of a matrix describing the channel-channel coupling. The same result has been obtained recently

[5] by summing an eikonalized Lippmann-Schwinger series in a multichannel approach. It should be remarked that, in spite of the formal compactness of these achievements, one should remember that the exponential of a matrix has to be realized by means of a series of powers of the matrix, or through the diagonalization of the matrix. As a matter of fact, it has been found difficult to further develop these approaches to exploit possible dynamical symmetries appearing in a specific scattering problem, as the ones appearing in the collision of vibrational nuclei, and to express the related scattering amplitudes in a closed form.

On the converse, Alder and Winther [3] developed a formalism describing multisteo processes in Coulomb excitation, which takes account of the excitation energy without resorting, in principle, to the coupled channel representation. This is done in the framework of the semiclassical time-dependent approach to scattering theory. The most direct application of this method is the study of the multiple Coulomb excitation of a nucleus described as a harmonic oscillator [12]. By this way very general and compact descriptions of collision processes are obtained, which are currently regarded as the natural reference for the comparison and the discussion of results obtained by different methods. For a recent application of the time-dependent approach to the description of multistep excitation processes in the coupled-channel approach see, e.g., ref. [13].

Other models incorporate elements of both the approaches of ref. [11] and [3]. As an example, to study multiple Coulomb excitation in heavy-ion collisions at intermediate energies Baur [14] introduced in a Glauber-like formalism some relations derived in the time-dependent formalism.

The starting point of the present approach is to introduce a modified Glauber-like factorization of the wave function, intended to separate kinematical rapid oscillations from wider range effects originating from the interactions. The standard eikonal factorization $\psi(\mathbf{r}) = \exp(kz)\varphi(\mathbf{r})$ is replaced here by the relation $\psi(\mathbf{r}) = \exp(\mathcal{K}z)\varphi(\mathbf{r})$ where $\mathcal{K} = (1/\hbar)\sqrt{2m(E-H_0)}$. Here z is the component of $\mathbf{r}$ along the the relative momentum $\mathbf{k}$ and H_0 is the sum of the target and projectile Hamiltonians; the impact parameter $\mathbf{b}$ is defined by the relation $\mathbf{r} = [\mathbf{b}, z]$. In this way we obtain for $\varphi(\mathbf{r})$ an approximated first-order differential equation in the variable z quite similar to the interaction picture Schroedinger equation in the time-dependent description. Starting from this equation a general representation of the scattering amplitude is obtained by using the phase-shift operator method [3] [15]. A natural representation of the scattering process is obtained in the framework of a coupled-channel approach. In the concluding section other more abstract descriptions of the scattering process are briefly recalled.

The basic concepts are introduced in sec.II, where the implemented eikonal equation is derived, and the general form of the scattering amplitude is discussed. In sec.III, several different approximations to the scattering amplitude are derived and compared in the coupled-channel representation. In sec.IV the microscopic approach to the description of the target Hamiltonian and of the nucleus-nucleus coupling is introduced into the above formalism; the results so obtained are compared to the standard Glauber approximation. In sec.V we discuss some previous work that can be framed in a systematic way in the formalism here developed, and the attention is drawn to possible developments. Some details of the derivations of sec.II are given in the Appendix.

II. THE IMPLEMENTED EIKONAL APPROXIMATION

A. The wave-function equation

For comparison purposes, let us briefly recall the original Glauber derivation [11]. In the sudden approximation, to describe the scattering of two nuclei with reduced mass m and internal coordinates ζ, coupled by an interaction $V(\mathbf{r},\zeta)$, one first solves the potential–scattering Schroedinger equation

$$[\Delta \,+\, \frac{2m}{\hbar^2}E]\,\psi(\mathbf{r},\zeta) \,=\, \frac{2m}{\hbar^2}V(\mathbf{r},\zeta)\,\psi(\mathbf{r},\zeta) \tag{2.1}$$

assuming that during the collision process the nuclei are frozen in an arbitrary configuration ζ of the internal coordinates. The actual nucleus-nucleus scattering amplitude is then obtained in the final stage of the developments by averaging out the "frozen amplitude" between the relevant initial and final internal states of the two colliding nuclei.

At high energy it is reasonable to assume that the physical solution is essentially a plane wave weakly modulated by the interaction, so that it is convenient to factorize the wave function in the form

$$\psi(\mathbf{r},\zeta) \;=\; e^{ikz}\varphi(\mathbf{r},\zeta) \;=\; e^{ikz}\varphi(\mathbf{b},z;\zeta) \quad ; \tag{2.2}$$

the relative motion coordinate $\mathbf{r}$ has been separated according to $\mathbf{r}=[\mathbf{b},z]$; the z axis is chosen along the direction of the relative momentum and $\mathbf{b}$ is the impact parameter.

The differential equation for φ one obtains by substituting Eq. (2.2) into Eq. (2.1) contains also a term $\Delta\varphi$; this quantity is disregarded in the assumption that φ is a slowly varying function of the relative motion coordinates. By this way one obtains the first-order linear equation [11]

$$\frac{d}{dz}\varphi(\mathbf{b},z;\zeta) \;=\; -\frac{i}{\hbar v}V(\mathbf{b},z;\zeta)\,\varphi(\mathbf{b},z;\zeta) \quad . \tag{2.3}$$

To introduce the eikonal approximation avoiding the sudden approximation one considers the actual Schroedinger equation including H_0, the sum of the internal Hamiltonians of the two colliding nuclei,

$$[\Delta \;+\; \frac{2m}{\hbar^2}(E - H_0(\zeta))]\;\psi(\mathbf{r},\zeta) \;=\; \frac{2m}{\hbar^2}V(\mathbf{r},\zeta)\;\psi(\mathbf{r},\zeta). \tag{2.4}$$

and represents the solution in the factorized form

$$\psi(\mathbf{r},\zeta) \;=\; e^{i\mathcal{K}z}\varphi(\mathbf{r},\zeta) \quad , \tag{2.5}$$

where the linear momentum operator

$$\mathcal{K} = \frac{1}{\hbar}\sqrt{2m(E - H_0)} \tag{2.6}$$

has been introduced. Through simple operator manipulations it is easy to verify that, disregarding again the term $\Delta\varphi(\mathbf{r},\zeta)$ which arises from developing $\Delta\psi(\mathbf{r},\zeta)$, the differential equation satisfied by $\varphi(\mathbf{r},\zeta)$ comes out to be

$$\frac{d}{dz}\varphi(\mathbf{r},\zeta) \;=\; -\frac{i}{\hbar v}\, e^{-i\mathcal{K}z}V(\mathbf{r},\zeta)e^{i\mathcal{K}z}\;\varphi(\mathbf{r},\zeta)\;\;. \tag{2.7}$$

The differential equation (2.7) can be written in the form

$$\frac{d}{dz}\varphi(\mathbf{r},\zeta) \;=\; -\frac{i}{\hbar v}\;\mathcal{V}(\mathbf{r},\zeta)\varphi(\mathbf{r},\zeta)\;\;, \tag{2.8}$$

where

$$\mathcal{V}(\mathbf{r},\zeta) \;=\; e^{-i\mathcal{K}}V(\mathbf{r},\zeta)e^{i\mathcal{K}} \qquad . \tag{2.9}$$

Note that Eq. (2.8) is similar in its structure to Eq. (2.3). The static potential $V(\mathbf{r},\zeta)$ has been substituted by the dynamical interaction $\mathcal{V}(\mathbf{r},\zeta)$ which, through the operators $\mathcal{K}$, takes into account the excitation of the target during the collision process. Alternative derivations of Eq. (2.8) can be found in refs. [7,8].

To give an example of the versatility of the formalism we are developing, we derive the Feshbach-Hüfner coupled-channel equations. On defining the eigenstates and eigenvalues of H_0 according to

$$H_0\phi_n = \epsilon_n\phi_n, \qquad \epsilon_0 = 0 \quad , \tag{2.10}$$

the simple relation

$$e^{i\mathcal{K}z}\phi_n \;\equiv\; \exp\left(\frac{i}{\hbar}\sqrt{2m(E-\epsilon_n)}\;z\right)\phi_n \;=\; e^{ik_nz}\phi_n \tag{2.11}$$

allows one to give at any stage an eikonalized coupled-channel representation of the formalism, also evidencing the way the excitation energy is included in the formalism. We now expand $\varphi(\mathbf{r},\zeta)$ on the $\phi_n(\zeta)$ states basis according to

$$\varphi(\mathbf{r},\zeta) \;=\; \sum_n \varphi_n(\mathbf{r})|\phi_n(\zeta)> . \tag{2.12}$$

Equation (2.8) can then be written in the form

$$\sum_n \frac{d\varphi_n(\mathbf{r})}{dz}\,|\phi_n> \;=\; -\frac{i}{\hbar v}\, e^{-i\mathcal{K}z}V(\mathbf{r},\zeta)e^{i\mathcal{K}z}\,\sum_n \varphi_n|\phi_n> \;; \tag{2.13}$$

on projecting onto the generic state $< \phi_m|$ one gets the first-order system of coupled equations

$$\frac{d}{dz}\varphi_m(\mathbf{r}) = -\frac{i}{\hbar v}\sum_n \mathcal{V}_{mn}(\mathbf{r})\varphi_n(\mathbf{r}) \tag{2.14}$$

where

$$\mathcal{V}_{mn}(\mathbf{r}) = e^{-i(k_m-k_n)z} < \phi_m V \phi_n > \quad . \tag{2.15}$$

Eq. (2.14) represents the Feshbach-Hüfner system of coupled equations.

B. General structure of the scattering amplitude

To deduce the scattering amplitude starting from Eq. (2.8) one can resort to the formalism developed to solve the time dependent Schroedinger equation in the interaction picture (see, e.g., refs. [16] and the Appendix). In the modified eikonal approximation developed above, the scattering amplitude from an initial state i to a final state f assumes the form

$$F_{fi}(\mathbf{k}_i,\mathbf{k}_f) = \frac{ik}{2\pi}\int d^2b\, ^{i\mathbf{q}\cdot\mathbf{b}} < \phi_f(\zeta)(1 - \mathcal{Z}e^{\frac{-i}{\hbar v}\int_{-\infty}^{+\infty} \mathcal{V}(\mathbf{b},z;\zeta)dz})\, \phi_i(\zeta) > \quad , \tag{2.16}$$

where $\mathbf{q} = \mathbf{k}_i - \mathbf{k}_f$. The fundamental merit of the above expression of the scattering amplitude is that it is governed by the dynamical evolution operator $\mathcal{V}(\mathbf{r},\zeta)$ which at any stage takes into account the possible excitation of the target, so overcoming the sudden approximation.

In the above equations the quantity $\mathcal{Z}$ is the z-ordering operator, necessary to give a univocal meaning to the exponential since the interaction $\mathcal{V}(\mathbf{b},z;\zeta)$ does not commute for different values of z when finite excitation energies are involved, even if we assume that $V(\mathbf{b},z;\zeta)$ does commute for different values of z. For later reference we recall the explicit meaning of the z-ordered power expansion of the scattering operator S,

$$S = \mathcal{Z}e^{\frac{-i}{\hbar v}\int_{-\infty}^{+\infty} \mathcal{V}(\mathbf{b},z;\zeta)dz} \quad . \tag{2.17}$$

The compact form of the above equation is a shorthand notation for the series

$$
\begin{aligned}
S &= \mathcal{Z}\exp[\frac{-i}{\hbar v}\int_{-\infty}^{+\infty} \mathcal{V}(\mathbf{b},z;\zeta)dz] = 1 + S_{(1)} + S_{(2)} + S_{(3)} + \cdots \\
&= 1 + (\frac{-i}{\hbar v})\int_{-\infty}^{+\infty} \mathcal{V}(z)dz + (\frac{-i}{\hbar v})^2\int_{-\infty}^{+\infty} \mathcal{V}(z)dz\int_{-\infty}^{z} \mathcal{V}(z')dz' \\
&\quad + (\frac{-i}{\hbar v})^3\int_{-\infty}^{+\infty} \mathcal{V}(z)dz\int_{-\infty}^{z} \mathcal{V}(z')dz'\int_{-\infty}^{z'} \mathcal{V}(z'')dz'' + \cdots
\end{aligned}
\tag{2.18}
$$

where $z \geq z' \geq z'' \cdots$ (to simplify the notation we omitted writing $\mathbf{b}$ and ζ explicitly.)

In the time-dependent scattering theory, an alternative representation of the scattering operator, S, uses the phase-shift operator [15]. In the present approach one introduces the phase-shift operator $W(\mathbf{b};z)$ through the definition

$$
\mathcal{Z}\; e^{\frac{-i}{\hbar v}\int_{-\infty}^{\infty} \mathcal{V}(\mathbf{b},z;\zeta)dz} = e^{W(\mathbf{b};\zeta)} \quad . \tag{2.19}
$$

This method, developed in ref. [15] by Bialynicki-Birula *et al.*, has been exploited in the theory of Coulomb excitation by Alder and Winther [3]. In the present paper we are mainly interested to develop this method in the framework of our generalized eikonal approximation. More precisely, in the following we shall disentangle the scattering amplitude starting from its representation

$$
F_{fi}(\mathbf{k}_i,\mathbf{k}_f) = \frac{ik}{2\pi}\int d^2b\; ^{i\mathbf{q}\cdot\mathbf{b}} \;< \phi_f(\zeta)(1 - e^{W(\mathbf{b};\zeta)})\phi_i(\zeta) > \tag{2.20}
$$

The S-operator representation given by Eqs. (2.17) and (2.18) will be considered for comparison purposes to illustrate the merits of the phase-shift approach.

Following the results obtained in the framework of the time-dependent approach in refs. [15,3], it can be shown that the operator W can be expressed as a series of terms of increasing order in the interaction:

$$
W = W_{(1)} + W_{(2)} + W_{(3)} + \cdots \quad ; \tag{2.21}
$$

the subscript (n) in $W_{(n)}$ gives directly the power at which $\mathcal{V}$ appears in this term. The explicit form of $W_{(n)}$ up to the third order is

$$
W_{(1)} = \frac{-i}{\hbar v}\int_{-\infty}^{\infty} dz\mathcal{V}(z) \tag{2.22}
$$

$$W_{(2)} = \frac{1}{2}(\frac{-i}{\hbar v})^2 \int_{-\infty}^{\infty} dz \int_{-\infty}^{z} dz'[\mathcal{V}(z), \mathcal{V}(z')] \qquad (2.23)$$

$$W_{(3)} = \frac{1}{6}(\frac{-i}{\hbar v})^3 \int_{-\infty}^{\infty} dz \int_{-\infty}^{z} dz' \int_{-\infty}^{z'} dz''\{[\mathcal{V}(z), [\mathcal{V}(z'), \mathcal{V}(z'')]] + [\mathcal{V}(z''), [\mathcal{V}(z'), \mathcal{V}(z)]]\} \qquad (2.24)$$

General formulae for any order of coupling can be found in refs. [15] and [3].

A useful expansion of the scattering operator S can be obtained in the framework of the W-formalism, collecting the terms of the same order in the coupling interaction

$$S = 1 + W_{(1)} + \left[\frac{1}{2}W_{(1)}^2 + W_{(2)}\right] + \left[\frac{1}{6}W_{(1)}^3 + \frac{1}{2}(W_{(1)}W_{(2)} + W_{(2)}W_{(1)}) + W_{(3)}\right] + \cdots \qquad (2.25)$$

As an example of the practical effectiveness of this representation, one immediately sees from Eq. (2.24) that if the lowest order commutator $W_{(2)}$ is zero, all the terms $W_{(n)}$ with $n > 2$ are also zero, and Eq. (2.25) reduces to an ordinary series of powers of $W_{(1)}$.

A comparison term-by-term of the two expansions (2.18) and (2.25) gives the relation between the two approaches. In the next sections we shall examine in particular the second order term

$$S_{(2)} = (\frac{-i}{\hbar v})^2 \int_{-\infty}^{+\infty} \mathcal{V}(z)dz \int_{-\infty}^{z} \mathcal{V}(z')dz' = \frac{1}{2}W_{(1)}^2 + W_{(2)} \ . \qquad (2.26)$$

III. THE SCATTERING AMPLITUDE IN THE C.C. REPRESENTATION

Although the two general approaches outlined in the previous section to describe the scattering amplitude are tightly connected, most approximation scheme are more easily formulated in the framework of the W-operator method. The relation with the z-ordering approach will be evidenced when possible and useful. The formalism is developed in the coupled-channel representation; other methods are briefly discussed in the final section.

In the present formalism, $V(\mathbf{r}, \zeta)$ can be specialized both to microscopic and collective descriptions of the internal structure of the colliding nuclei, including electromagnetic interactions. In the latter case, ζ can be, e.g., the set of the parameters necessary to describe the shape of an oscillating system. Some features of the present approach arising in the microscopic descriptions are discussed in sec. IV. We assume that the interaction $V(\mathbf{r},\zeta)$ commute at different values of z, although more general interactions can be used (see e.g. ref. [17]). Commuting interactions are sufficient to describe the main features of heavy-ion scattering. The dynamical interaction $\mathcal{V}$ in general does not commute, owing to its structure (2.9) arising from the finite excitation energies of the levels.

A. General features

Let us recall the definition (2.20) of the scattering amplitude

$$F_{fi}(\mathbf{k}_i,\mathbf{k}_f) \;=\; \frac{ik}{2\pi}\int d^2b\; ^{i\mathbf{q}\cdot\mathbf{b}} \;<\phi_f(\zeta)(1-e^{W(\mathbf{b};\zeta)})\phi_i(\zeta)> \qquad (3.1)$$

The related scattering operator S can be expanded in the ordinary power series

$$S \;=\; 1 + W + W^2 + W^3 + \cdots \qquad (3.2)$$

To develop our formalism in the coupled-channel representation one defines the completeness of the states supposed to play an important role in the scattering process. For simplicity sake we assume that the projectile is always in its ground state; the formalism becomes a little more cumbersome when relaxing this hypothesis, but nothing changes conceptually. One writes

$$\mathbf{1} \;\simeq\; \sum_{n=0}^{N}\phi_n^T\phi_0^P><\phi_n^T\phi_0^P \;=\; \sum_{n=0}^{N}\phi_n><\phi_n \quad , \qquad (3.3)$$

and inserts this expression between each product of W's in the term of the r-th order in the expansion (3.2). The matrix element of the expression so obtained is then taken between an initial state i and a final state f; the result is

$$\frac{1}{r!}<\phi_f\underbrace{[W(\mathbf{r},\zeta)\;W(\mathbf{b},\zeta)W(\mathbf{b},\zeta)...]}_{r\ factors}\phi_i>$$

$$=\frac{1}{r!}\sum_{q',q'',..}\underbrace{W_{fq'}(\mathbf{b})W_{q'q''}(\mathbf{b})\;\cdot\;\cdot\;\cdot\;W_{q^{n-1}i}(\mathbf{b})}_{r\ factors} \;=\; \frac{1}{r!}[\mathbf{W}(\mathbf{b})]^r_{fi} \qquad (3.4)$$

where $W_{mn}(\mathbf{b})$ is the matrix element

$$W_{mn}(\mathbf{b}) \;=\; \langle\phi_m|W(\mathbf{b};\zeta)|\phi_n\rangle \tag{3.5}$$

From these equations and Eq. (3.1) one gets the scattering amplitude

$$F_{fi}(\mathbf{q}) = \frac{ik}{2\pi}\int d^2b\; e^{i\mathbf{q}\cdot\mathbf{b}} \sum_{r=0}^{\infty}\frac{1}{r!}\,[(\mathbf{W}(\mathbf{b}))^r]_{fi} \tag{3.6}$$

and, on re-summing the series,

$$F_{fi}(\mathbf{q}) = \frac{ik}{2\pi}\int d^2b\; e^{i\mathbf{q}\cdot\mathbf{b}}(\delta_{fi} - [e^{\mathbf{W}(\mathbf{b})}]_{fi}) \quad . \tag{3.7}$$

In general the multichannel scattering amplitude Eq. (3.7) cannot be further simplified and one must resort to the diagonalization of the matrix or to a series expansion. The diagonalization procedure can be cast in a quite general form considering the complete phase-shift operator. For this purpose one introduces the operator W' through the definition

$$W \;=\; iW' \qquad ; \tag{3.8}$$

this definition is convenient since in usual physical situations W' is a Hermitian operator. To evaluate the S-matrix element

$$S_{fi} \;=\; [e^{i\mathbf{W}'}]_{fi} \quad , \tag{3.9}$$

where $\mathbf{W}'$ is the matrix representation of the operator W' in the channel space, one considers a unitary transformation U on the operator W',

$$< n|U^{\dagger}W'U|m > \;=\; \lambda_n\delta_{nm} \quad ; \tag{3.10}$$

the eigenvalues λ_n are real numbers since W' is Hermitian. The same unitary transformation will also diagonalize the operator $e^{iW'}$, so that one can write

$$\begin{aligned} S_{fi} \;&=\; < f|U(U^{\dagger}e^{iW'}U)U^{\dagger}|i > \;=\; < f|Ue^{iU^{\dagger}W'U}U^{\dagger}|i > \\ &=\; \sum_n < f|U|n >< n|U^{\dagger}|i > e^{i\lambda_n} \qquad . \end{aligned} \tag{3.11}$$

This general procedure gives a (formally) exact solution of the scattering problem.

We shall now specialize this formalism to different levels of approximation.

B. The $W_{(1)}$ approximation

Let us consider the approximation

$$W(\mathbf{b};\zeta) \simeq W_{(1)}(\mathbf{b};\zeta) = -\frac{i}{\hbar v}\int_{-\infty}^{\infty} dz\ \mathcal{V}(\mathbf{r};\zeta). \tag{3.12}$$

By following the general procedure developed above one gets

$$\frac{1}{r!}(\frac{-i}{\hbar v})^r < \phi_f \underbrace{[\int_{-\infty}^{\infty} dz\mathcal{V}(\mathbf{r},\zeta) \int_{-\infty}^{\infty} dz\mathcal{V}(\mathbf{r},\zeta) \int_{-\infty}^{\infty} dz\mathcal{V}(\mathbf{r},\zeta)...]}_{r\ factors}\phi_i >$$

$$= \frac{1}{r!}\sum_{q',q'',..} \underbrace{C_{fq'}(\mathbf{b})C_{q'q''}(\mathbf{b}) \cdot \cdot \cdot C_{q^{n-1}i}(\mathbf{b})}_{r\ factors} = \frac{1}{r!}[\mathbf{C}(\mathbf{b})]^r_{fi} \tag{3.13}$$

where $C_{mn}(\mathbf{b})$ is the matrix element

$$C_{mn}(\mathbf{b}) = -\frac{i}{\hbar v}\langle\phi_m|\int \mathcal{V}(\mathbf{r};\zeta)dz|\phi_n\rangle = -\frac{i}{\hbar v}\int_{-\infty}^{\infty} dz\ e^{ik_z^{(hk)}z} < \phi_h V(b,z)\phi_k > \tag{3.14}$$

and $k_z^{(mn)} = k(_m)_z - k(_n)_z$. From these equations and Eq.(3.7) one gets the scattering amplitude

$$F_{fi}(\mathbf{q}) = \frac{ik}{2\pi}\int d^2b\ e^{i\mathbf{q}\cdot\mathbf{b}} \sum_{r=0}^{\infty}\frac{1}{r!}[(\mathbf{C}(\mathbf{b}))^r]_{fi} \tag{3.15}$$

and, on re-summing the series,

$$F_{fi}(\mathbf{q}) = \frac{ik}{2\pi}\int d^2b\ e^{i\mathbf{q}\cdot\mathbf{b}}(\delta_{fi} - [e^{(\mathbf{C}(\mathbf{b})}]_{fi}) \tag{3.16}$$

The above scattering amplitude has the same form it has in the Feshbach–Hüfner coupled-channel approach and in ref. [5].

It is useful to extract from the diagonal elements of the matrix a phase-shift $\chi(b)$ arising from the average central potential. With this purpose one re-defines the $\mathbf{C}$ matrix as follows:

–if $q' \neq q''$, it is still valid the definition (3.14)

–if $q' = q'' = q$ one defines the diagonal elements through the relation

$$C_{qq} = -\frac{i}{\hbar v}\langle\phi_q|\int \mathcal{V}(\mathbf{r};\zeta)dz|\phi_q\rangle - i\chi(b) \tag{3.17}$$

where

$$\chi(b) \;=\; -\frac{1}{\hbar v}\langle\phi_q|\int V^{central}(\mathbf{r};\zeta)dz|\phi_q\rangle \tag{3.18}$$

and $V^{central}$ is the central part of the interaction; it is currently accepted that $\chi(b)$ is channel-independent. The scattering amplitude is then written

$$F_{fi}(\Delta) = \frac{ik}{2\pi}\int d^2b\; e^{i\Delta\cdot\mathbf{b}}(\delta_{fi} - e^{i\chi(\mathbf{b})}[e^{\mathbf{C}(\mathbf{b})}]_{fi})\;\;. \tag{3.19}$$

To get an explicit representation of Eq. (3.19) one can use the diagonalization method or a series expansion; as an example, the second order inelastic scattering amplitude is obtained in the form

$$F^{(2)}_{fi}(\mathbf{q}) = -\frac{ik}{2\pi}\int d^2b\; e^{i\mathbf{q}\cdot\mathbf{b}}\; e^{i\chi(b)}\sum_q C_{fq}(\mathbf{b})C_{qi}(\mathbf{b}) \tag{3.20}$$

where the sum is extended to all the states assumed to play a relevant role in the process considered. The same result could be obtained from the S-matrix formalism by considering the $S_{(2)}$ approximation according to Eq. (2.26) and disregarding the commutator term.

C. The approximation $W \simeq W_{(1)} + W_{(2)}$

We consider now the approximation

$$S_{nm} \;=\; [e^{\mathbf{W}_{(1)}+\mathbf{W}_{(2)}}]_{nm} \tag{3.21}$$

One must evaluate therefore the matrix elements of the operator $W_{(2)}$, whose definition, given in Eq. (2.23), is here written in the form

$$W_{(2)} = \frac{1}{2}(\frac{-i}{\hbar v})^2\int_{-\infty}^{\infty}dz\int_{-\infty}^{z}dz'[\mathcal{V}(z),\mathcal{V}(z')] = \frac{1}{2}(\frac{-i}{\hbar v})^2\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dz'\epsilon(z-z')\mathcal{V}(z),\mathcal{V}(z') \tag{3.22}$$

where

$$\epsilon(z-z') \;=\; 2\theta(z-z') - 1 \tag{3.23}$$

In the spirit of the coupled-channel approach, to calculate the matrix element

$$[W_{(2)}]_{nm} \;=\; <\phi_n|W_{(2)}|\phi_m> \tag{3.24}$$

one inserts a completeness of the type (3.3) between the product of the dynamical interactions $\mathcal{V}$ and gets

$$[W_{(2)}]_{nm} \;=\; \frac{1}{2}(\frac{-i}{\hbar v})^2 \sum_q \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz' \epsilon(z-z') e^{ik_z^{(qn)} z} e^{ik_z^{(mq)} z'} <\phi_n|V(z)|\phi_q><\phi_q|V(z')|\phi_m> \tag{3.25}$$

From the above equation it appears that in this representation one should perform a double integral involving z-dependent matrix elements of the interaction; we shall next consider this representation to examine some conditions in which $W_{(2)}$ gives a negligible contribution. To evaluate matrix elements of $W_{(2)}$, however, it is preferable to exploit a general method which allows one to express matrix elements of $W_{(n)}$ in terms of integrals of a sort of off-shell matrix elements of lower m's [3]. This is obtained by representing the function ϵ in the form

$$\epsilon(z-z') \;=\; \frac{i}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\exp[i(z-z')\alpha]}{\alpha} d\alpha \tag{3.26}$$

where $\mathcal{P}$ denotes the principal part of the integral. By introducing this representation into Eq. (3.22) one gets

$$[W_{(2)}]_{nm} \;=\; \frac{i}{2\pi}\sum_q \mathcal{P}\int_{-\infty}^{\infty} \frac{d\alpha}{\alpha} C_{n,q}(\mathbf{b}; k_z^{(n,q)}-\alpha) C_{q,m}(\mathbf{b}; k_z^{(q,m)}+\alpha)] \tag{3.27}$$

The S-matrix element is obtained from Eq. (3.21) by matrix diagonalization as discussed in sec.III.A and/or by performing a perturbative expansion. As an example, the second-order scattering amplitude, derived from Eq. (3.21), is written in the form

$$F_{fi}^{(2)}(\mathbf{q}) = -\frac{ik}{2\pi}\int d^2b\, e^{i\mathbf{q}\cdot\mathbf{b}}\, e^{i\chi(b)} \Big[\sum_q C_{fq}(\mathbf{b}) C_{qi}(\mathbf{b}) + \frac{i}{2\pi}\sum_q \mathcal{P}\int_{-\infty}^{\infty}\frac{d\alpha}{\alpha} C_{f,q}(\mathbf{b}; k_z^{f,q}-\alpha) C_{q,i}(\mathbf{b}; k_z^{q,i}+\alpha)\Big]. \tag{3.28}$$

The same approximation can be obtained by inserting the completeness of the coupled states in the second order term

$$S_{(2)} \;=\; (\frac{-i}{\hbar v})^2 \int_{-\infty}^{+\infty} \mathcal{V}(z) dz \int_{-\infty}^{z} \mathcal{V}(z') dz' \;=\; \frac{1}{2} W_{(1)}^2 + W_{(2)} \tag{3.29}$$

of the scattering operator.

When the factorized form (3.34) of the matrix **C** is used to give an approximate description of Q-value effects (see next section), the integral on the variable α results in a simple integral over elementary transcendental functions.

D. Q-value effects in the $W_{(1)}$ and $W_{(2)}$ approximations

1. The sudden approximation for $W_{(1)}$.

Let us consider the sudden approximation for $W_{(1)}$. In this approximation it is assumed that all the excitation energies of the inelastic channels are zero. In this case the matrix of the z-integrated form factor has elements

$$D_{q'q''}(\mathbf{b}) \;=\; -\frac{i}{\hbar v}\langle\phi_{q'}|\int V(\mathbf{r};\zeta)dz|\phi_{q''}\rangle \tag{3.30}$$

and the scattering amplitude is written

$$F_{fi}(\mathbf{q}) = \frac{ik}{2\pi}\int d^2b\; \mathrm{e}^{i\mathbf{q}\cdot\mathbf{b}}(\delta_{fi} - \mathrm{e}^{i\chi(\mathbf{b})}[e^{\mathbf{D}(\mathbf{b})}]_{fi}) \tag{3.31}$$

In the sudden approximation this relation gives the complete solution of the scattering problem, since all the $W_{(n)}$ with $n \geq 2$ are identically zero when all the levels have the same energy.

We show now that the relation between the matrices **C** and **D** can be described through a suitable adiabaticity factor. Since in the Coulomb excitation Q-value effects can be accounted for exactly [5], the method here proposed is particularly suited in dealing with nuclear interactions. To exemplify this method let us consider the simple case of a transition from a state with initial angular momentum zero to a state with angular momentum and magnetic quantum number LM. The method we developed in ref. [5] and summarize here, amounts to approximating C_{LM} in the form

$$C_{LM}(b) = \kappa_{(LM)}(b)D_{LM}(b) \tag{3.32}$$

(the parentheses in the subscript LM are related to the fact that in different approximations the dependence on L and/or M could be absent). To find an approximate expression for κ_{LM} one observes that it is reasonable to assume that the form factor $V_{0,LM}(\mathbf{r})$ has the behaviour

$$V_{0,LM}(\mathbf{r}) = V_L \, e^{-r/a} \, e^{-iM\phi} \, Y^*_{LM}(\theta,0) \tag{3.33}$$

in the decaying tail, i.e. in the region where the excitation process occurs without being absorbed; a is typically the decaying constant of the densities and transition densities. A simple analytical form for $C_{LM}(b)$ and $D_{LM}(b)$ can be obtained in the widely used approximation $Y_{LM}(\theta,0) \simeq Y_{LM}(\pi/2,0)$ which amounts to approximating the angular dependence of the interaction by the value it has at the point of maximum approach. In ref. [5] it has been shown that a good approximation for the adiabaticity factor is then given by the relation

$$\kappa(b) \simeq \left(1 - \frac{3}{4}(a\delta k)^2\right) e^{\frac{-b}{2a}(a\delta k)^2}. \tag{3.34}$$

It is worth noting that the dependence on Q-value, given by the Gaussian function, is the same as the one derived in the framework of the semiclassical time dependent description of the inelastic scattering (cf. ref. [2]).

2. Q-value effects in the $W_{(2)}$ approximation.

We considered previously the Q-value effects on the matrix elements of the first-order interaction $W_{(1)}$. It is now interesting to examine these effects on the second-order term, in particular to study the conditions in which $W_{(2)}$ can be neglected. If this is the case the complete phase-shift operator coincides with $W_{(1)}$, as discussed above.

In the double-integral representation of the matrix elements of $W_{(2)}$ (see Eq. (3.25)) we consider for simplicity sake only one intermediate state, so that the equation can be written

$$[W_{(2)}]_{nm} = \frac{1}{2}\left(\frac{-i}{\hbar v}\right)^2 \sum_q \int_{-\infty}^{\infty} dz \int_{-\infty}^{z} dz' \; \Big[e^{k_z^{(qn)}z} e^{k_z^{(mq)}z'} < \phi_n|V(z)|\phi_q >< \phi_q|V(z')|\phi_m > - e^{k_z^{(qn)}z'} e^{k_z^{(mq)}z} < \phi_n|V(z')|\phi_q >< \phi_q|V(z)|\phi_m > \Big]. \tag{3.35}$$

For our purposes it is useful to particularize the above formula to the cases

i)Second order elastic scattering.

ii)Second order inelastic scattering in a harmonic-oscillator-like problem.

In both cases we shall assume that the nuclear states are described by real functions of their variables (see the discussion in ref. [3]).

In the first case, by putting $m = n = 0$ one obtains for the integrand the expression

$$< \phi_0 V(b,z)\phi_q >< \phi_q V(b,z')\phi_0 > [e^{-ik_z^{(0q)}(z-z')} - e^{ik_z^{(0q)}(z-z')}] \quad . \tag{3.36}$$

Since

$$[e^{-ik_z^{(0q)}(z-z')} - e^{ik_z^{(0q)}(z-z')}] = -2i\sin[k_z^{(0q)}(z-z')] \simeq -2i\sin[\omega^{(0q)}(z-z')/v] \tag{3.37}$$

where $k_z^{(0q)} \simeq \omega^{(0q)}/v$ at high collision energies, one sees that in the regime $\omega/v \simeq 0$ the contribution of $W_{(2)}$ is vanishing. Note, however, that this does not means that we are coming back to the sudden approximation. For example, in the Coulomb excitation the matrix elements C_{mn} have a strong Q-value dependence which can be retained also in the cases in which the second-order correction can be neglected.

In the second-order inelastic scattering we consider a harmonic-oscillator like situation; more precisely, we assume that $E_q - E_i = (1/2)(E_f - E_i)$ and that the form factor for the transition from the state i to the state q is the same as the one from q to f. In this case it is easy to verify that the two addend of the integral in Eq. (3.35) are identical and therefore $[\mathbf{W}_{(2)}]_{fi} \equiv 0$. It follows that in a multichannel problem of this type one has $\mathbf{W} = \mathbf{W}_{(1)}$.

IV. SOME FEATURES OF THE MICROSCOPIC APPROACH

The aim of this section is to show how the approach so far developed can be specialized when a microscopic description of the collision process is assumed, and to compare the results we obtain with the results of the standard Glauber approach. Some important differences are discussed.

According to the Kerman, McManus and Thaler theory [18], the Hamiltonian for two interacting nuclei can be written [6,1]

$$H = H_0 + \mathcal{N} \sum_{\alpha A} \tau_{\alpha A} \tag{4.1}$$

where $\tau_{\alpha A}$ is the effective operator describing the interaction of the α-th nucleon of the projectile with the A-th nucleon of the target and $\mathcal{N} = (A_P - 1)(A_T - 1)/A_P A_T$ where $H_0 = H_P + H_T$. We introduce the further assumption that $\tau_{\alpha A}$ can be described in the impulse approximation, i.e.

$$\tau_{\alpha A} \simeq t_{\alpha A}(|\mathbf{r} - \mathbf{r}_\alpha - \mathbf{r}_A|) \ , \tag{4.2}$$

where t is a local and central effective interaction describing the nucleon-nucleon scattering. For heavy-ion scattering this description is sufficient in most of the cases; no difficulty arises from the inclusion of iso-spin degrees of freedom [19]. The interaction $V(\mathbf{r}, \zeta)$ is then written

$$V(\mathbf{r}, \zeta) = \mathcal{N} \sum t_{\alpha A}(|\mathbf{r} - \mathbf{r}_\alpha - \mathbf{r}_A|) \tag{4.3}$$

If we put this interaction into the sudden approximation scheme developed in sec.III.D we obtain

$$F_{fi}(\mathbf{q}) = \frac{ik}{2\pi} \int d^2b \ e^{i\mathbf{q}\cdot\mathbf{b}} (\delta_{fi} - e^{i\chi(\mathbf{b})}[e^{\mathbf{D}(\mathbf{b})}]_{fi}) \tag{4.4}$$

where now

$$\begin{aligned} D_{nm}(\mathbf{b}) &= -\frac{i}{\hbar v} \langle \phi_n | \int V(\mathbf{r}; \zeta) dz | \phi_m \rangle \\ &= -\frac{i\mathcal{N}}{\hbar v} \int \rho_0^P(\mathbf{r}_P) \delta\rho_{nm}^T(\mathbf{r}_T) \ t_{NN}(|\mathbf{r} - \mathbf{r}_P + \mathbf{r}_T|) d\mathbf{r}_P d\mathbf{r}_T dz \ . \end{aligned} \tag{4.5}$$

In the momentum representation the above quantity is written

$$D_{nm}(\mathbf{b}) = -\frac{i\mathcal{N}}{\hbar v} \int d\mathbf{q} dz \ e^{i\mathbf{q}\cdot\mathbf{r}} \ f_{NN}(q) \hat{\rho}_0^P(\mathbf{q}) \delta\hat{\rho}_{nm}^T(\mathbf{q}) \ , \tag{4.6}$$

where the hat accent denotes Fourier transformation; the free scattering $N - N$ amplitude is related to the Fourier transform of the t_{NN} operator,

$$f_{NN}(q) = -\frac{m}{2\pi\hbar^2}\int d\mathbf{r}\ e^{i\mathbf{q}\cdot\mathbf{r}}\ t_{NN}(r) \quad . \tag{4.7}$$

The elements of the matrix $\mathbf{D}$ are so expressed in terms of the elementary amplitudes f_{NN} directly through a Fourier transform of their space representation (4.7) avoiding the introduction of the elementary profiles γ_{NN}. (We recall the definition $\gamma_{NN}(\mathbf{b}) = 2\pi/(ik)\int d\mathbf{q} f(\mathbf{q})\exp(i\mathbf{q}\cdot\mathbf{b})$).

Starting from Eq. (4.6), the integral over the z-coordinate is now performed in the following steps

$$\begin{aligned} D_{nm} &= \frac{-i\mathcal{N}}{\hbar v}\int d\mathbf{q}_b\ dq_z\ e^{i\mathbf{q}_b\cdot\mathbf{b}}\ f_{NN}(q)\ \hat{\rho}_0^P(\mathbf{q})\ \delta\hat{\rho}_{nm}^T(\mathbf{q})\ \delta(q_z) \\ &\equiv \frac{-i\mathcal{N}}{\hbar v}\int d\mathbf{q}\ e^{i\mathbf{q}\cdot\mathbf{b}}\ f_{NN}(q)\ \hat{\rho}_0^P(\mathbf{q})\ \delta\hat{\rho}_{nm}^T(\mathbf{q})\ . \end{aligned} \tag{4.8}$$

The last form of the integral is justified by the fact that the range of integration over the $\mathbf{q}$-variable is restricted by the product $\mathbf{q}\cdot\mathbf{b}$ to the plane of the $\mathbf{b}$-variable, as demanded by the $\delta(q_z)$ in the preceding line.

A notable feature of the above derivation is that the phase shift and the coupling matrix elements are defined only in terms of phenomenological quantities such as $N-N$ scattering amplitudes and nuclear densities; the γ_{NN} profiles did not appear either as subsidiary or intermediate quantities. *This is a consequence of a basic feature of the present formalism: the eikonal propagation is imposed to the nucleus-nucleus motion and not to the microscopic collisions.* (In the Glauber approach the microscopic eikonal propagation is implicit in the use of microscopic profile functions).

Another significant difference with respect to the Glauber approximation derives from the fact that we insert the completeness of the coupled states inside each product of two operators. In this way, as in the ordinary coupled-channel approach, all the nuclear structure problems are expressed in terms of the knowledge of one-particle densities and transition densities. Conversely, in the Glauber method, matrix elements of products of microscopic scattering operators are considered, and so one needs higher and higher order correlation functions.

A common way of overcoming this point is based on the independent particle description of the nuclear states, an assumption which allows one to express all the correlation functions in terms of nuclear densities. Note that the assumption of uncorrelated wave functions gives rise to a center-of-mass correction factor depending on the transferred momentum and the masses of the colliding nuclei (cf. refs. [20], [21] and [22]). In our approach the densities and transition densities are not related to independent-particle descriptions and they can be taken directly from phenomenological data, and no correction factor appears.

By the same steps performed in sec.III.D one can obtain the non-sudden form of the scattering amplitude

$$F_{fi}(\mathbf{q}) = \frac{ik}{2\pi} \int d^2b\, e^{i\mathbf{q}\cdot\mathbf{b}}(\delta_{fi} - e^{i\chi}[e^{\mathbf{C}(b)}]_{fi}) \tag{4.9}$$

where the matrix $\mathbf{C}$ is defined by the relation

$$\begin{aligned} C_{mn}(\mathbf{b}) &= -\frac{i}{\hbar v}\langle\phi_m| \int \mathcal{V}(\mathbf{r})dz|\phi_n\rangle \\ &= -\frac{i\mathcal{N}}{\hbar v} \int dz e^{i\delta k_z^{(mn)} z} \rho_0^P(\mathbf{r}_P)\delta\rho_{mn}^T(\mathbf{r}_T)\, t_{NN}(|\mathbf{r} - \mathbf{r}_P + \mathbf{r}_T|)d\mathbf{r}_P d\mathbf{r}_T \quad . \end{aligned} \tag{4.10}$$

The above scattering amplitude has the same form it has in the Feshbach–Hüfner approach and in ref. [5].

In the reasonable assumption that the tail of the microscopic form factor has the same behaviour it has in phenomenological approaches (see Eq. (3.33)), the same factor κ (see Eq.(3.34)) can be assumed to relate the $\mathbf{C}$ and the $\mathbf{D}$ matrices.

V. SUMMARY AND PERSPECTIVES

To show the unifying power of the method developed in the previous sections we recall here several results, obtained in different context, that can be seen as particular cases of the procedure here presented. The first-order pertubative approach accounting for the central Coulomb potential and nuclear interactions has been accounted for by eikonal methods in the sudden approximation by Ahmad [23], Chauvin *et al.* [24], Alexander and Rinat [25], and by Lenzi, Vitturi and Zardi [26,27,19]. First-order Coulomb excitation in the framework of the electromagnetic retarded potentials has been used by Bertulani and Baur [4]. In ref. [5] the interference of nuclear and Coulomb excitation in the first-order approximation has been described without resorting to the sudden approximation.

The eikonal coupled-channel method has been developed by Feshbach and Hüfner in ref. [6] where a general formalism based on z-ordering of the form-factor matrices has been presented. This formalism in its exp[$\mathbf{C}(\mathbf{b})$] approximation has been applied in ref. [6] and [28]. A similar multichannel formalism has been derived in ref. [5] by re-summing the eikonal Lippmann-Schwinger series and has been applied to a few schematic cases (see refs. [5] and [29]).

One can wonder whether in some cases the multichannel scattering problem can be solved in a compact form avoiding the construction and diagonalization of the channel coupling matrix. This has been obtained by describing the scattering process in the framework of the Interacting Boson Model [30] in the sudden approximation. The methods developed by Ginocchio *et al.* [31] succeeds in describing the S-matrix element

$$S_{fi} \;=\; <\phi_f(\zeta)|e^{i\int dz V(\mathbf{r},\zeta)}|\phi_i(\zeta)>$$

in terms of elementary transcendental functions and hypergeometric functions.

Conversely, the harmonic oscillator collective model can be also algebrized when the finite excitation energy of the levels is taken into account. In this connection one must observe that the form (2.9) of the dynamical interaction between two composite colliding systems is not always well suited for algebraic manipulations, involving both the Hamiltonian H_0 and the interaction $V(\mathbf{r},\zeta)$. To facilitate this procedure it is possible to expand $\mathcal{K}$ and to take into account only the leading term in H_0. One has $\mathcal{K} \;=\; (1/\hbar)\sqrt{2m(E-H_0)} \;\simeq\; k - H_0/(\hbar v)$; this approximation becomes more and more justified with increasing collision energy. This approximation implies that the dynamical interaction (2.9) is equal to

$$\mathcal{V}(\mathbf{r},\zeta) \;=\; e^{\frac{i}{\hbar v}H_0 z}V(\mathbf{r},\zeta)e^{\frac{-i}{\hbar v}H_0 z} \quad . \tag{5.1}$$

It has been shown in ref. [9] that using this approximation leads to a closed form of the S-matrix. This is in close similarity to the results obtained by Alder and Winther [3] in the semiclassical time-dependent approach.

A systematic calculation of intermediate-energy heavy-ion Coulomb excitation by ordinary and retarded electromagnetic interactions in the framework of the C-matrix approximation has been planned [32].

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APPENDIX A

In order to deduce the scattering amplitude in the implemented eikonal approximation let us consider the general definition [16]

$$F_{fi}(\mathbf{k}_i, \mathbf{k}_f) = \frac{ik}{2\pi} \int d^3r \; e^{i\mathbf{q}\cdot\mathbf{r}} \int d\zeta \;\; < \phi_f(\zeta)(1-S) \, \psi_i^{(+)}(\mathbf{r};\zeta) > . \tag{A1}$$

In the eikonal approximation the above equation is written

$$F_{fi}(\mathbf{k}_i, \mathbf{k}_f) = \frac{ik}{2\pi} \int d^2b \; e^{i\mathbf{q}\cdot\mathbf{b}} \;\; < \phi_f(\zeta)(1-S^{eik}) \, \varphi_i^{(+)}(\mathbf{r};\zeta) > . \tag{A2}$$

To make explicit $\varphi_i^{(+)}(\mathbf{r};\zeta)$ let us start from Eq. (2.9),

$$\frac{d}{dz}\varphi_i^{(+)}(\mathbf{b}, z; \zeta) = -\frac{i}{\hbar v} \, \mathcal{V}(\mathbf{b}, z; \zeta)\varphi_i^{(+)}(\mathbf{b}, z; \zeta) \; . \tag{A3}$$

To solve this equation one exploits its formal similarity with the interaction picture of the time-evolution equation [16]. In this connection one introduces the z-evolution operator U through the equation

$$\varphi_i^{(+)}(\mathbf{b}, z; \zeta) = U(\mathbf{b}, z, z'; \zeta) \, \varphi_i^{(+)}(\mathbf{b}, z'; \zeta) \qquad . \tag{A4}$$

Since we are interested to study the evolution of $\varphi_i^{(+)}$ from $z' = -\infty$ where the boundary condition is

$$\varphi_i^{(+)}(\mathbf{b}, z' = -\infty; \zeta) = \phi_i(\zeta) \quad , \tag{A5}$$

the above equation can be written in the form

$$\varphi_i^{(+)}(\mathbf{b}, z; \zeta) = U(\mathbf{b}, z; \zeta)\, \phi_i(\zeta) \quad , \tag{A6}$$

with U satisfying the initial condition

$$U(\mathbf{b}, z = -\infty; \zeta) = 1 \quad . \tag{A7}$$

It is immediate to verify that $U(\mathbf{b}, z; \zeta)$ fulfills the same equation as φ, i. e.

$$\frac{d}{dz} U(\mathbf{b}, z; \zeta) = -\frac{i}{\hbar v} \mathcal{V}(\mathbf{b}, z; \zeta) U(\mathbf{b}, z; \zeta) \tag{A8}$$

or the equivalent integral equation

$$U(\mathbf{b}, z; \zeta) = 1 - \frac{i}{\hbar v} \int_{-\infty}^{z} \mathcal{V}(\mathbf{b}, z; \zeta) U(\mathbf{b}, z'; \zeta) dz'. \tag{A9}$$

By standard methods this equation is solved in the recursive form

$$U(\mathbf{b}, z; \zeta) = 1 + (\frac{-i}{\hbar v}) \int_{-\infty}^{z} \mathcal{V}(\mathbf{b}, z'; \zeta) dz' + (\frac{-i}{\hbar v})^2 \int_{-\infty}^{z} \mathcal{V}(\mathbf{b}, z'; \zeta) dz' \int_{-\infty}^{z'} \mathcal{V}(\mathbf{b}, z''; \zeta) dz'' + \cdots \tag{A10}$$

with $z \geq z' \geq z'' \cdots$. By introducing the z-ordering operator $\mathcal{Z}$ the above equation is written

$$U(\mathbf{b}, z; \zeta) = \mathcal{Z} e^{-\frac{i}{\hbar v} \int_{-\infty}^{z} \mathcal{V}(\mathbf{b}, z'; \zeta) dz'}. \tag{A11}$$

Since

$$S^{eik} = U(\infty, -\infty) \tag{A12}$$

the final result is

$$S^{eik} = \mathcal{Z} e^{\frac{-i}{\hbar v} \int_{-\infty}^{+\infty} \mathcal{V}(\mathbf{b}, z; \zeta) dz}. \tag{A13}$$

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