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SOME APPLICATIONS OF NON-HERMITIAN OPERATORS IN QUANTUM MECHANICS AND QUANTUM FIELD THEORY(*)

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ABSTRACT

Due to the possibility of rephrasing it in terms of Lie-admissible algebras, some work done in the past in collaboration with A. Agodi, M. Baldo and V. S. Olkhovsky is here reported. Such work led to the introduction of non-Hermitian operators in (classical and relativistic) quantum theory. We deal in particular with: (i) the association of unstable states (decaying "Resonances") with the eigenvectors of non-Hermitian Hamiltonians; (ii) the problem of the four-position operators for relativistic spin-zero particles.

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PART 1 - UNSTABLE STATES AND NON-HERMITIAN HAMILTONIANS

1.1. - INTRODUCTION

This first Part is based on work done in collaboration with A. Agodi and M. Baldo[1].

In quantum mechanics the "resonance" peaks are generally described as corresponding to unstable states (remember e.g. Schwinger's[2] approach). The present attempt proceeds as follows: (i) singling out one state $|\phi\rangle$ in the state space; (ii) finding out the effect of the (internal, virtual) state $|\phi\rangle$ on the transition-amplitude; (iii) finding, in particular, the necessary conditions for $|\phi\rangle$ to be connected with a Resonance in the cross-section. In this way we shall associate the "resonant states" with the eigenvectors of a non-Hermitian Hamiltonian (for simplicity, a "quasi self-adjoint" Hamiltonian), such eigenvectors being shown to decay in time correctly. We shall adopt the formalism introduced by Akhieser and Gladsman[3], by Lifshitz, by Galinsky and Migdal[4] and by Agodi et al.[5].

Chosen a state $|\phi\rangle$, let us define the projectors

$$P \equiv |\phi\rangle\langle\phi| ; \qquad Q \equiv \mathbb{1} - P. \tag{1}$$

1.2. - PRELIMINARY CASE: TIME-DEPENDENT DESCRIPTION OF POTENTIAL SCATTERING

Let us preliminarly consider the time-dependent description of potential scattering. Quantity V be the potential operator. In the limiting case of plane-waves, the scattering amplitude writes

$$T(\mathbf{k}, \mathbf{k}') = \langle\mathbf{k}'| V |\mathbf{k}\rangle + \langle\mathbf{k}'| VG(E^+)V |\mathbf{k}\rangle \tag{2a}$$

with

$$G(E^+) \equiv (E^+ - H)^{-1} ; \qquad E^{\pm} \equiv E \pm i\varepsilon. \tag{2b}$$

Chosen the exploring vector $|\phi\rangle$ and using definitions (1), we have

$$H = \overset{0}{H} + \overset{1}{H} ; \tag{3a}$$

$$\overset{0}{H} \equiv QHQ ; \qquad \overset{1}{H} \equiv PHP + PHQ + QHP. \tag{3b}$$

By introducing the scattering states $|\overset{0}{\psi}\rangle$ due to $\overset{0}{H}$

$$\left| \overset{o}{\psi}{}^{(\pm)}_{\mathbf{k}} \right> = \left[1 + \frac{1}{E^{\pm} - \overset{o}{H}} (\overset{o}{H} - E) \right] \left| \mathbf{k} \right> , \tag{4}$$

we obtain

$$\begin{aligned} S(\mathbf{k}, \mathbf{k}') \equiv \left< \psi^{(-)}_{\mathbf{k}'} \right| \left. \psi^{(+)}_{\mathbf{k}} \right> &= \left< \overset{o}{\psi}{}^{(-)}_{\mathbf{k}'} \right| \left. \overset{o}{\psi}{}^{(+)}_{\mathbf{k}} \right> - 2\pi i \cdot \delta(E_{k'} - E_k) \cdot \\ &\cdot \left< \overset{o}{\psi}{}^{(-)}_{\mathbf{k}'} \right| H \, \underbrace{PG(E_k^{+}) P} \, H \left| \overset{o}{\psi}{}^{(+)}_{\mathbf{k}} \right> , \end{aligned} \tag{5}$$

where the first addendum in the r. h. s. of eq. (5) (let us call it A) is the contribution coming from processes developing entirely in the subspace onto which Q projects, whilst the second addendum (B) is contributed by processes going through the exploring state $|\phi>$ onto which P projects. In other words, the processes with $|\phi>$ as intermediate state correspond to the term

$$\left[\delta(E_{k'} - E_k) \right]^{-1} \cdot B = -2\pi i \frac{\left< \overset{o}{\psi}{}^{(-)}_{\mathbf{k}'} \right| H \left| \phi \right> \left< \phi \right| H \left| \overset{o}{\psi}{}^{(+)}_{\mathbf{k}} \right>}{E_k^{(+)} - \left< \phi \right| H \left| \phi \right> - \left< \phi \right| W^{\phi}(E_k^{+}) \left| \phi \right>} ; \tag{6a}$$

$$W^{\phi}(z) \equiv PHQ \frac{1}{z - QHQ} QHP. \tag{6b}$$

Our problem is : under what conditions one (or more) Resonances are actually associated with the chosen $|\phi>$?

Let us notice, in particular, that if $E_\phi \equiv <\phi|H|\phi> - \mathrm{Re}<\phi|W^{\phi}(E^{+})|\phi>$ and $\Gamma_\phi = \mathrm{Im}<\phi|W^{\phi}(E^{+})|\phi>$ are smooth functions of E, then B gets just the "Breit and Wigner" form :

$$B \simeq -2\pi i \frac{\left< \overset{o}{\psi}{}^{(-)}_{\mathbf{k}'} \right| HPH \left| \overset{o}{\psi}{}^{(+)}_{\mathbf{k}} \right>}{E - E_\phi + i\Gamma_\phi} .$$

1. 3. - CASE OF CENTRAL POTENTIAL AND SPIN-FREE PARTICLES

Let us choose the angular-momentum representation. If $|\phi>$ is assumed to be in particular invariant under $O(3)$, then both terms in which S was split are diagonal. If $\overset{o}{\delta}_\ell$ are the phase-shifts due to QHQ and μ is the reduced mass, then

$$S_\ell(k) \equiv \exp\left[2i\,\delta_\ell(k)\right] = \exp\left[2i\,\overset{o}{\delta}_\ell(k)\right] \cdot F_\ell(k) \tag{7a}$$

with

$$F_\ell(k) \equiv 1 - \frac{2\pi i \mu}{\hbar^2 k} \frac{\left| <\phi|H| \overset{o}{\psi}{}^{(+)}_{k\ell m} > \right|^2}{E^{+} - <\phi|H|\phi> - <\phi|W^{\phi}(E^{+})|\phi>} . \tag{7b}$$

Let us observe that the phase-shift of $F_\ell(k)$ crosses the value $\frac{1}{2}\pi$ (with positive slope) when:

$$F_\ell(k) = -1 . \tag{8}$$

The conditions for a Resonance to appear are particularly transparent for $\ell = 0$:

$$F_o(k) = \frac{E - E_\phi(k) - i\lambda_o(k)}{E - E_\phi(k) + i\lambda_o(k)} , \tag{9a}$$

when

$$\lambda_o(k) \equiv -\mathrm{Im} < \phi | W^\phi(E^+) | \phi > = \left| < \phi | H | \overset{0}{\psi}{}^{(+)}_{k00} > \right|^2 \tag{9b}$$

is positive-definite. Namely, the condition $F_o(k) = -1$ yields

$$\left| 1 - S_o(k) \right|^2 = 4\cos^2 \overset{0}{\delta}_o , \tag{8'}$$

with the supplementary conditions $\lambda_o(k) \neq 0$; $\cos \overset{0}{\delta}_o \neq 0$. When $\cos \overset{0}{\delta}_o \simeq 1$ the scattering due to QHQ is negligible, i. e. the scattering proceeds entirely via the intermediate formation of the (quasi-bound) state $|\phi>$; and the possible resonant effects are really related to $|\phi>$. Of course $\cos \overset{0}{\delta}_o \simeq 1$ when, at the resonance $\left[E = E_\phi ;\ F(k) = -1\right]$, it is $| \psi^{(\pm)}_{k\ell m} > \simeq | k\ell m >$.

Notice that with every fixed $|\phi>$ a series of Resonances (also for different values of ℓ) may be a priori associated, if they are not destroyed by the $\overset{0}{\delta}_o$ behaviour.

1. 4. - RESONANCE DEFINITION

It is essential to recognize that the "resonance condition" F (k) = - 1 may be written(1)

$$1 - \alpha(k,\ell) < \phi_\ell | G(E^+) | \phi_\ell > = 0 \tag{10a}$$

with

$$\alpha(k,\ell) \equiv \frac{i\pi\mu}{\hbar^2 k} \left| < \phi_\ell | H | \overset{0}{\psi}{}^{(+)}_{k\ell m} > \right|^2 .$$

Let us now study the more general equation

$$\left\{ \begin{array}{l} \boxed{1 - \lambda < \phi_\ell | G(z) | \phi_\ell > = 0 ,} \\ \text{with } z, \lambda \text{ complex numbers.} \end{array} \right. \tag{11}$$

Of course, a Resonance will appear at $\sim \mathrm{Re}\, z$ if z is near the real axis and if

$$\lambda \simeq \alpha(k,\ell)\ ,$$

both satisfying eq. (11).

If we introduce now the non-Hermitian Hamiltonian-operator

$$\boxed{\mathcal{H} \equiv H_+ \lambda P};\quad \lambda \text{ complex}\ , \tag{12}$$

whose "resolvent operator" is

$$\mathcal{G}(z) \equiv \frac{1}{z - \mathcal{H}}\ , \tag{12'}$$

then eq. (11) becomes

$$\frac{<\phi_\ell\,|G(z)|\,\phi_\ell>}{<\phi_\ell\,|\mathcal{G}(z)|\,\phi_\ell>}\ ; \tag{13}$$

in other words, studying the (necessary) conditions for Resonance-appearing is just equivalent to find out the poles in the diagonal elements of the "resolvent" $\mathcal{G}$-matrix, i. e. the eigenvalues of the quasi self-adjoint operator $\mathcal{H}$. Notice that, since

$$\mathcal{G} = G + G\,\frac{\lambda P}{1 - \lambda<\phi_\ell\,|\,G\,|\,\phi_\ell>}\,G\ ,\qquad (\mathrm{Im}\,\lambda > 0)$$

the difference between the spectra of H and $\mathcal{H}$ is just the presence of complex eigenvalues (corresponding to the solution of our "condition-equation" (13)).

Therefore, in our framework the "resonant (decaying) state" $|\Psi>$ is expected to be an eigenvector of $\mathcal{H}$ (notice that it does not coincide with the state $|\phi>$ which is not unstable!), corresponding to the complex energy $\mathcal{E}$.

1. 5. - APPLICATIONS

Let us confine ourselves to the case $\ell = 0$, and rewrite the non-Hermitian (quasi self-adjoint) Hamiltonian as

$$\mathcal{H} \equiv H_+ i\,\alpha_k\,|\phi><\phi|\ ;\qquad \alpha_k \equiv -\,i\,\alpha(k,\,0) \tag{14a}$$

when

$$V_\phi \equiv i\,\alpha_k\,|\phi><\phi| \tag{14b}$$

is anti-Hermitian. We shall therefore write

$$(H - \mathscr{E})|\psi> = - V_{\phi}|\psi> \equiv -|\phi> i\alpha_k <\phi|\psi> \quad , \tag{15}$$

which immediately yields for the eigenvalues the "dispersion-type relation" $\left[\mathscr{E} \equiv \mathscr{E}_{\phi}\right]$:

$$1 + i <\phi| \frac{1}{H - \mathscr{E}} |\phi> \alpha_k = 0 , \tag{16}$$

and for the eigenvectors the explicit expression

$$|\psi> = - <\phi|\psi> i\alpha_k \frac{1}{H - \mathscr{E}} |\phi> , \tag{17}$$

where $<\phi|\psi>$ is a normalization constant. Notice that to solve eq. (16) we do not need knowing α_k, i.e. the scattering states due to QHQ, since fortunately <u>at the resonances</u> it is $\left[E \equiv E_R\right]$:

$$\alpha_k \propto \left| <\phi|H|\overset{0}{\psi}{}^{(+)}_{k00}>\right|^2 = \left| <\phi|\psi^{(+)}_E> - <\phi|k00>\right|^{-2} .$$

Notice moreover that the present approach, a priori, allows distinguishing between true resonances and other effects.

In Ref. (1) the application was considered to the case of scattering by a spherical-well potential $U(r) = U_o\, \Theta\,(a - r)$, and as exploring states the class was adopted of the normalized Laurentian wave-packets (good for low energies):

$$<k00|\phi> = \sqrt{2b}\, \frac{1}{k^2 + b^2} \Longleftrightarrow <r|\phi> = \sqrt{\frac{b}{2\pi}}\, \frac{\exp[-br]}{r} .$$

By integration, for low entering energies ($k^2 \ll 2mU_o$) one gets one equation, whose real and imaginary parts forward a system of two equations. The latters individuate $|\phi>$, i.e. the parameter b, for which a series of (true) Resonances arises. These Resonances are expected to appear for $\left[k^2 = 2mE;\ K^2 = 2m(E + V_o)\right]$:

$$\cos Ka = 0 \Rightarrow Ka = (n + \tfrac{1}{2})\pi .$$

The system of equations is rather complicated (even when the resonance width is $\gamma < k_o$). But the first equation does not contain γ and yields b. For instance, for $n = 0$ one gets a unique solution ($ab \simeq 0.69$).

<u>1. 6. - DECAY OF THE UNSTABLE STATE</u>

We are more interested in the decay in time of the unstable state $|\psi>$:

$$<\psi|\psi_t> \equiv <\psi|U_t|\psi> \equiv <\psi|\exp\left[-i\mathcal{O}t\right]|\psi>. \tag{18}$$

If we assume, as usual, $\mathcal{O} = H$, then

$$<\psi|\psi_t> \simeq \int_0^\infty dE \left|<\psi|\psi_E^{(+)}>\right|^2 \exp\left[-iEt\right] \tag{19}$$

since the bound-states do not contribute for large t. Moreover, let us remember that

$$|\psi> = -i\,\alpha_k <\phi|\psi> \frac{1}{H - \mathscr{E}} |\phi>.$$

Therefore

$$\left|<\psi_E^{(+)}|\psi>\right|^2 = \frac{|\alpha_k|^2}{(\mathrm{Re}\mathscr{E} - E)^2 - (\mathrm{Im}\mathscr{E})^2} C; \quad C \equiv \left|<\psi_E^{(+)}|P|\psi>\right|^2.$$

The integral (19) can be evaluated following Ref. (4). The expression C contains denominators that - analytically extended - produce one pole in $E = \mathscr{E}$. If in the strip $\mathrm{Im}\mathscr{E} < \mathrm{Im}E < 0$ no other singularities arise from the remaining factors, then we obtain the exponential-type decay

$$<\psi|\psi_t> = (C + Dt)\exp\left[-(iE_o t + \gamma_o t)\right] \tag{20}$$

with $E_o \equiv \mathrm{Re}\mathscr{E}$; $\gamma_o \equiv \mathrm{Im}\mathscr{E}$; C and D constants.

More interesting appears, however, the assumption

$$\mathcal{O} = \mathscr{H} , \tag{21}$$

since in this case our approach does surely possess a "Lie-admissible" structure(6) (due to the fact that the time-evolution operator with $\mathscr{H}$ is not unitary). In such a case one would simply get

$$<\psi|\psi_t> = \bar{K}\exp\left[iE_o t + \gamma_o t\right] \tag{22}$$

with $\bar{K} \equiv <\psi|\psi>$. But in this case the whole approach ought to be carefully rephrased in "Lie-admissible" terms (otherwise, e. g., all states would seem to be decaying).

PART 2 - ON FOUR-POSITION OPERATORS IN Q. F. T.

2.1. - THE KLEIN-GORDON CASE: THREE-POSITION OPERATORS

The usual position-operators, being Hermitian, are known to possess real eigenvalues: i. e., they yield a point-like localization. J. M. Jauch showed, however, that a point-like localization would be in contrast with "unimodularity". In the relativistic case, moreover, phenomena so as the pair production forbid a localization with precision better than one Compton wave-length. The eigenvalues of a realistic position-operator $\hat{\mathbf{z}}$ are therefore expected to represent space regions, rather than points. This can be obtained only making recourse to non-Hermitian position-operators $\hat{\mathbf{z}}$ (a priori, one can make recourse either to non-normal operators with commuting components, or to normal operators with non-commuting components[7]). Following the spirit of Refs. (7), we are going to show that the mean values of the Hermitian part of $\hat{\mathbf{z}}$ will yield a mean (point-like) position[8], while the mean values of the anti-Hermitian part of $\hat{\mathbf{z}}$ will yield the sizes of the localization region[9].

Let us consider e. g. the case of relativistic spin-zero particles, in natural units and with the metric (+---). The position operator, $i\nabla_{\mathbf{p}}$, is known to be actually non-Hermitian, and may be in itself a good candidate for an extended-type position operator. To show this, we want to split[8] it into its Hermitian and anti-Hermitian parts.

Consider, then, a vector space V of complex differentiable functions on a 3-dimensional phase-space equipped with an inner product defined by $\left[p_o \equiv \sqrt{\mathbf{p}^2+m_o^2}\right]$:

$$(\Psi, \Phi) = \int \frac{d^3\mathbf{p}}{p_o} \Psi^*(\mathbf{p})\Phi(\mathbf{p}) . \tag{23}$$

Let the functions in V further satisfy a condition

$$\lim_{R\to\infty} \int_{S_R} \frac{dS}{p_o} \Psi^*(\mathbf{p})\Psi(\mathbf{p}) = 0 , \tag{24}$$

where the integral is taken over the surface of a sphere of radius R. If $\mathscr{D}: V \to V$ is a differential operator of degree one, condition (24) allows a definition of the transpose $\mathscr{D}^T$ by

$$(\mathscr{D}^T\Psi, \Phi) = (\Psi, \mathscr{D}\Phi) \qquad \text{for all } \Phi, \Psi \in V , \tag{25}$$

where $\mathscr{D}$ is changed into $\mathscr{D}^T$, or vice-versa, by means of integration by parts.

This allows further to introduce a dual representation $(\mathscr{D}_1, \mathscr{D}_2)$ of a single operator $\mathscr{D}_1^T + \mathscr{D}_2$ by

$$(\mathscr{D}_1\psi, \phi) + (\psi, \mathscr{D}_2\phi) = (\psi, (\mathscr{D}_1^T + \mathscr{D}_2)\phi). \tag{26}$$

With such a dual representation it is easy to split any operator into its Hermitian and anti-Hermitian (or skew-Hermitian) parts

$$(\psi, \mathscr{D}\phi) = \frac{1}{2}\left[(\psi, \mathscr{D}\phi) + (\mathscr{D}^*\psi, \phi)\right] + \frac{1}{2}\left[(\psi, \mathscr{D}\phi) - (\mathscr{D}^*\psi, \phi)\right]. \tag{27}$$

Here the pair

$$\frac{1}{2}(\mathscr{D}^*, \mathscr{D}) \equiv \overleftrightarrow{\mathscr{D}}_h \tag{28a}$$

corresponding to $\frac{1}{2}(\mathscr{D} + \mathscr{D}^{*T})$, represents the Hermitian part, while

$$\frac{1}{2}(-\mathscr{D}^*, \mathscr{D}) \equiv \overleftrightarrow{\mathscr{D}}_a \tag{28b}$$

represents the anti-Hermitian part.

Let us apply what precedes to the case of the Klein-Gordon position-operator $\hat{\mathbf{z}} = i\nabla_{\mathbf{p}}$. When

$$\mathscr{D} = i\frac{\partial}{\partial p_j} \tag{29}$$

we have[9,10]

$$\frac{1}{2}(\mathscr{D}^*, \mathscr{D}) = \frac{1}{2}(-i\frac{\partial}{\partial p_j}, i\frac{\partial}{\partial p_j}) \equiv \frac{i}{2}\frac{\overleftrightarrow{\partial(-)}}{\partial p_j} \equiv \frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial p_j}, \tag{30a}$$

$$\frac{1}{2}(-\mathscr{D}^*, \mathscr{D}) = \frac{1}{2}(i\frac{\partial}{\partial p_j}, i\frac{\partial}{\partial p_j}) \equiv \frac{i}{2}\frac{\overleftrightarrow{\partial(+)}}{\partial p_j}. \tag{30b}$$

And the corresponding single operators turn out to be

$$\frac{1}{2}(\mathscr{D} + \mathscr{D}^{*T}) = i\frac{\partial}{\partial p_j} - \frac{i}{2}\frac{p_j}{\mathbf{p}^2 + m_o^2} \tag{31a}$$

and

$$\frac{1}{2}(\mathscr{D} - \mathscr{D}^{*T}) = \frac{i}{2}\frac{p_j}{\mathbf{p}^2 + m_o^2}. \tag{31b}$$

It is noteworthy[10,9] that operator (31a) is nothing but the usual Newton-Wigner operator, while (31b) has been interpreted[7,9] as yielding the sizes of the localization-region (an ellipsoid) by means of its average values over the considered wave-packet.

Let us underline that the previous treatment justifies from the mathematical point of view the formalism used in Refs. (8-10): We want to report it briefly here, due to its immediate legibility (its significance being now mathematically clarified by the preceding approach). In Ref. (8) we split the operator $\hat{\mathbf{z}}$ as follows:

$$\hat{\mathbf{z}} \equiv i\nabla_{\mathbf{p}} = \frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial \mathbf{p}} + \frac{1}{2}\frac{\overleftrightarrow{\partial(+)}}{\partial \mathbf{p}} , \tag{32}$$

where $\psi^{*}\frac{\overleftrightarrow{\partial(+)}}{\partial \mathbf{p}}\phi \equiv \psi^{*}\frac{\partial\phi}{\partial \mathbf{p}} + \phi\frac{\partial\psi^{*}}{\partial \mathbf{p}}$, and where we always referred to a suitable space of wave-packets[10,9]. Its Hermitian part[9,10]

$$\hat{\mathbf{x}} \equiv \frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial \mathbf{p}} , \tag{33}$$

which was expected to yield an (ordinary) point-like localization, was derived also by writing explicitly

$$(\psi, \mathbf{x}\phi) = i\int\frac{d^3\mathbf{p}}{p_o}\,\Psi(\mathbf{p})\,\nabla_{\mathbf{p}}\,\Phi(\mathbf{p})$$

and imposing Hermiticity, i. e. the reality of the diagonal elements. The calculation yielded

$$\mathrm{Re}(\phi, \mathbf{x}\phi) = \frac{i}{2}\int\frac{d^3\mathbf{p}}{p_o}\,\Phi^{*}(\mathbf{p})\,\frac{\overleftrightarrow{\partial}}{\partial \mathbf{p}}\,\Phi(\mathbf{p}) ,$$

just suggesting to adopt the Lorentz-invariant quantity (33) as Hermitian position operator. Then, integrating by parts (and due to the vanishing of the surface integral) we verified that (23) is equivalent to the ordinary Newton-Wigner operator N-W:

$$\frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial \mathbf{p}} \equiv i\nabla_{\mathbf{p}} - \frac{i}{2}\frac{\mathbf{p}}{\mathbf{p}^2 + m_o^2} \equiv \text{N-W} . \tag{34}$$

We were left with the anti-Hermitian part

$$\hat{\mathbf{y}} \equiv \frac{1}{2}\frac{\overleftrightarrow{\partial(+)}}{\partial \mathbf{p}} \tag{35}$$

whose average values over the considered state (wave-packet) were regarded as yielding[7,9] the sizes of an ellipsoidal localization-region.

After this digression (eqs. (32)-(35)), let us go back to our present formalism (represented by eqs. (23)-(31)).

In general, the extended-type position operator $\hat{\mathbf{z}}$ will give

$$<\psi|\hat{z}|\psi> = (\vec{\alpha} + \Delta\vec{\alpha}) + i(\vec{\beta} + \Delta\vec{\beta}), \tag{36}$$

where $\Delta\vec{\alpha}$ and $\Delta\vec{\beta}$ are the mean-errors encountered when measuring the point-like position and the sizes of the localization-region, respectively. It is interesting to evaluate the commutators $[i, j = 1, 2, 3]$:

$$\left[\frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial p^i}, \frac{1}{2}\frac{\overleftrightarrow{\partial(+)}}{\partial p^j}\right] = \frac{i}{2p_o^2}\left(\delta_{ij} - \frac{2p_i p_j}{p_o^2}\right), \tag{37}$$

wherefrom the noticeable "uncertainty correlations" follow:

$$\Delta\alpha_i\,\Delta\beta_j \geqslant \frac{1}{4}\left|<\frac{1}{p_o^2}\left(\delta_{ij} - \frac{2p_i p_j}{p_o^2}\right)>_\psi\right| . \tag{38}$$

2.2. - FOUR-POSITION OPERATORS

It is tempting to propose as four-position operator the quantity $\hat{z}^\mu = \hat{x}^\mu + i\hat{y}^\mu$, whose Hermitian (Lorentz-covariant) part can be written:

$$\hat{x}^\mu \equiv -\frac{1}{2}\frac{\overleftrightarrow{\partial}}{\partial p_\mu} , \tag{39}$$

to be associated with its corresponding "operator" in four-momentum space:

$$\hat{p}^\mu \equiv +\frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial x_\mu}. \tag{40}$$

Let us recall the proportionality between the 4-momentum operator and the 4-current density operator in the chronotopical space, and underline then the canonical correspondence (in the 4-position and 4-momentum spaces, respectively) between the "operators" (cf. Sect. 2.1)

$$\text{(a)}\quad m_o\hat{\varrho} \equiv \hat{p}_o = \frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial t}; \qquad \text{(c)}\quad \hat{t} = -\frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial p_o};$$

$$\text{(b)}\quad m_o\hat{\mathbf{j}} \equiv \hat{\mathbf{p}} = -\frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial \mathbf{r}}; \qquad \text{(d)}\quad \hat{\mathbf{x}} = \frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial \mathbf{p}}, \tag{41}$$

where the four-position "operator" (41c, d) can be regarded as a 4-current density operator in the energy-impulse space[9]. Analogous considerations can be carried on for the anti-Hermitian parts[9].

2.3. - ON THE TIME-OPERATOR

Let us fix our attention only on the operator for time in the case of (non-relativistic) quantum mechanics. Time, as well as 3-position, sometimes is a parameter, but sometimes is an observable to be represented by an operator. We have shown elsewhere that in Q. M. the "operator" (41c) - cf. Sect. 2.1 - can be replaced with the "operator"

$$\hat{t} \equiv -i \frac{\partial}{\partial E} \tag{42}$$

provided that a suitable, subsidiary boundary-condition is imposed on the considered wave-packets[10].

In Q. M., however, the wave-packet space is a space of functions defined only over the interval $0 \leq E \leq \infty$, and not over the whole E-axis. As a consequnce, $\hat{t}$ is Hermitian (and symmetric) but not self-adjoint, and does not allow the identity resolution. In Q. M., therefore, one has to use non-selfadjoint operators[11] even for the observable Time. However, even if $\hat{t}$ does not admit true eigenfunctions, nevertheless one succeeds in calculating the average values of $\hat{t}$ over our wave-packets. And this is enough to evaluate the packet time-coordinate, the flight-times, the interaction-durations, the (mean) life-times of metastable states, and so on[8-10, 12].

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