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THE SPECTRAL DISTRIBUTION METHOD IN BOSON SPACE

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SUMMARY.

The spectral distribution method is extended to a system of monopole (L = 0) and quadrupole (L = 2) bosons. For non-interacting L = 0 bosons, analytical expressions for the first four moments are given.

The main conclusion of this paper is that, in the case of bosons, the presence of many single-particle levels seems to be essential in generating a normal level density, the number of particles playing a minor role.

1. - INTRODUCTION.

The spectral distribution method (SDM), introduced by Moszkowski(1) in problems of atomic spectroscopy, and developed by French and co-workers(2), allows one to obtain information on the energies and wave functions of systems requiring extremely large spectroscopic spaces.

The basic assumption in this method is, roughly speaking, that the level density tends to a Gaussian form, when the number of particles increases(3); therefore, the low-order moments of the Hamiltonian operator carry the most important spectroscopic information. This assumption was checked and found satisfactory for fermion systems(4).

The aim of this paper is to discuss the validity of this hypothesis in the bosons space. It should be noted that, in this case, one may consider the number of bosons going to infinity, without increasing the number of single-particle levels (dense limit)(5).

2. - MONOPOLE BOSONS[6, 7, 8].

Let S(N, M) be the spectroscopic space of M bosons distributed over N single-particle levels. The dimensionality of S(N, M) will be denoted by d(N, M) and amounts to

$$d(N, M) = \binom{N+M-1}{M} . \tag{1}$$

Let 0(t) be a t-body operator ($t \leq N$) acting on S(N, M). Since only the diagonal part of 0(t) subsists in the trace calculation, we shall write 0(t) as follows:

$$0(t) = \sum_{\alpha_1 <} \cdots \sum_{< \alpha_t} \sum_{(p)} \langle \alpha_1^{p_1} \cdots \alpha_t^{p_t} | 0(t) | \alpha_1^{p_1} \cdots \alpha_t^{p_t} \rangle \cdot$$

$$\cdot \, a_{\alpha_1}^{+p_1} \cdots a_{\alpha_t}^{+p_t} a_{\alpha_1}^{p_1} \cdots a_{\alpha_t}^{p_t} + \text{traceless operators} , \tag{2}$$

where $a^+(a)$ is a creation (destruction) operator and $(p) = (p_1 \ldots p_t)$ any partition of integer t. Thus the trace of 0(t) over S(N, M), denoted by $\langle\langle 0(t) \rangle\rangle^M$, is given by

$$\langle\langle 0(t) \rangle\rangle^M = \sum_{\alpha_1 <} \cdots \sum_{< \alpha_t} \sum_{(p)} \langle \alpha_1^{p_1} \cdots \alpha_t^{p_t} | 0(t) | \alpha_1^{p_1} \cdots \alpha_t^{p_t} \rangle \cdot$$

$$\cdot \sum_{m_1} \cdots \sum_{m_N} \langle m_1 \ldots m_N | a_{\alpha_1}^{+p_1} \cdots a_{\alpha_t}^{+p_t} a_{\alpha_1}^{p_1} \cdots a_{\alpha_t}^{p_t} | m_1 \ldots m_N \rangle , \tag{3}$$

$$(m_1 + \ldots m_N = M) .$$

Furthermore, for a fixed succession of levels and for a given partition (p), one has

$$\sum_{m_1} \cdots \sum_{m_N} \langle m_1 \ldots m_N | a_{\alpha_1}^{+p_1} \cdots a_{\alpha_t}^{+p_t} a_{\alpha_1}^{p_1} \cdots a_{\alpha_t}^{p_t} | m_1 \ldots m_N \rangle =$$

$$= \sum_{k_1=0}^{M} \cdots \sum_{k_j=0}^{M} p_1! \binom{k_1}{p_1} \cdots p_j! \binom{k_j}{p_j} d(N-t+j, M-k_1 \ldots k_j) , \tag{4}$$

$$(m_1 + \ldots m_N = M) ,$$

where j is the total number of levels among t occupied for the partition (p), i.e. the number of p_i's $(i = 1, \ldots, t)$ which are different from zero. Taking (1) into account and using some elementary combinatorial identities[9] there is no difficulty in showing that

$$\sum_{m_1} \cdots \sum_{m_N} \langle m_1 \ldots m_N | a_{\alpha_1}^{+p_1} \cdots a_{\alpha_t}^{+p_t} a_{\alpha_1}^{p_1} \cdots a_{\alpha_t}^{p_t} | m_1 \ldots m_N \rangle =$$

$$= p_1! \ldots p_t! \binom{M+N-1}{M-t} , \qquad (m_1 + \ldots m_N = M) . \tag{5}$$

It follows therefore that the needed trace is given by

$$\langle\langle 0(t)\rangle\rangle^M = \binom{M+N-1}{M-t} \sum_{\alpha_1<} \cdots \sum_{<\alpha_t} \sum_{(p)} p_1! \dots p_t! \langle \alpha_1^{p_1} \dots \alpha_t^{p_t} | 0(t) | \alpha_1^{p_1} \dots \alpha_t^{p_t} \rangle . \quad (6)$$

Expression (6) exhibits the characteristic features of a trace propagation, similar in some respects to the one derived for fermions[2]. The trace of 0(t) over S(N, t) propagates forward to S(N, M) by a binomial coefficient. However, as one cannot define a boson plenum state, the symmetry particle-hole does not hold and the "backward" propagation, typical for the fermion averages, does not occurs (see also Ref. (10)).

As an example, we analyse the case of M noninteracting bosons. The Hamiltonian operator is written as follows:

$$H = \sum_{i=1}^{N} \varepsilon_i a_i^+ a_i . \quad (7)$$

We calculate the first four cumulants of the level density, defined in the usual way as polynomials of the distribution moments $\mu_n(N, M)$

$$\mu_n(N, M) = \langle\langle (H - \varepsilon(N, M))^n \rangle\rangle^M / d(N, M) , \quad (6a)$$

where $\varepsilon(N, M)$ is the centroid of the eigenvalue distribution

$$\varepsilon(N, M) = \langle\langle H \rangle\rangle^M / d(N, M) .$$

The analytical expressions of these cumulates are

$$k_1(N, M) = \varepsilon(N, M) = \frac{M}{N} \sum_i \varepsilon_i , \quad (9a)$$

$$k_2(N, M) = \mu_2(N, M) = \frac{M(M+N)}{N^2(N+1)} \left[(N-1) \sum_i \varepsilon_i^2 - 2 \sum_{i<j} \varepsilon_i \varepsilon_j \right] , \quad (9b)$$

$$k_3(N, M) = \mu_3(N, M) = \frac{M(M+N)(2M+N)}{N^3(N+1)(N+2)} \left[(N-1)(N-2) \sum_i \varepsilon_i^3 - 3(N-2) \sum_{i<j} (\varepsilon_i^2 \varepsilon_j + \varepsilon_i \varepsilon_j^2) + \right.$$
$$\left. + 12 \sum_{i<j<k} \varepsilon_i \varepsilon_j \varepsilon_k \right] , \quad (9c)$$

$$k_4(N, M) = \mu_4(N, M) - 3(\mu_2(N, M))^2 = \frac{M(M+N)}{N^4(N+1)^2(N+2)(N+3)} \left\{ \left[N^2(N+1)(N-1)(N-6) + \right.\right.$$
$$\left. + 6M(M+N)(N^3-4N^2-N+6) \right] \left[(N-1) \sum_i \varepsilon_i^4 - 4 \sum_{i<j} (\varepsilon_i^3 \varepsilon_j + \varepsilon_i \varepsilon_j^3) \right] - \left[N^2(N+1)(N^2-3N+6) + \right.$$
$$\left. + 2M(M+N)(2N^3-7N^2+3N+18) \right] 6 \sum_{i<j} \varepsilon_i^2 \varepsilon_j^2 \left[2N^2(N+1)+24M(M+N)(5N+6) \right] \cdot \quad (9d)$$
$$\cdot \left[(N-3) \sum_{i<j<k} (\varepsilon_i^2 \varepsilon_j \varepsilon_k + \varepsilon_i \varepsilon_j^2 \varepsilon_k + \varepsilon_i \varepsilon_j \varepsilon_k^2) - 6 \sum_{i<j<k<l} \varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l \right] .$$

Though we restricted ourselves to the first four cumulants, it is generally believed that the knowledge of the reduced cumulants $\gamma_1 = k_3/k_2^{3/2}$, $\gamma_2 = k_4/k_2^2$ is sufficient for deciding whether the level density is approximately Gaussian or not.

Expressions (9) show that at a fixed number of levels and for a large number of particles the form parameters γ_1 and γ_2 vary slowly with M ; this result had already been noted by the authors of Ref. (6), who made a numerical analysis of the non interacting boson spectra. If M goes to infinity γ_1 and γ_2 are not necessarily zero, their numerical values depending on the number of levels and on the form of the single-particle spectrum assumed. The level density may therefore show a noticeable departure from the Gaussian shape. For $M \to \infty$, one obtains for instance $\gamma_1 = 0$, $\gamma_2 = -1.2$ if $N = 2$.

Except for pathological cases, the asymptotic behaviour of the level density is almost Gaussian if the number of levels increases. For instance, if the single-particle levels are equidistant (or nearly equidistant), $\gamma_1 \simeq 0$ and γ_2 approaches zero very quickly for large N (we obtain $\gamma_2 = -0.14$ for 10 equidistant levels and $M \to \infty$). This trend, as noted above, is mostly indipendent of the number of particles M, if M is large enough ($M \gtrsim 20$) (see Figs. 1 and 2).

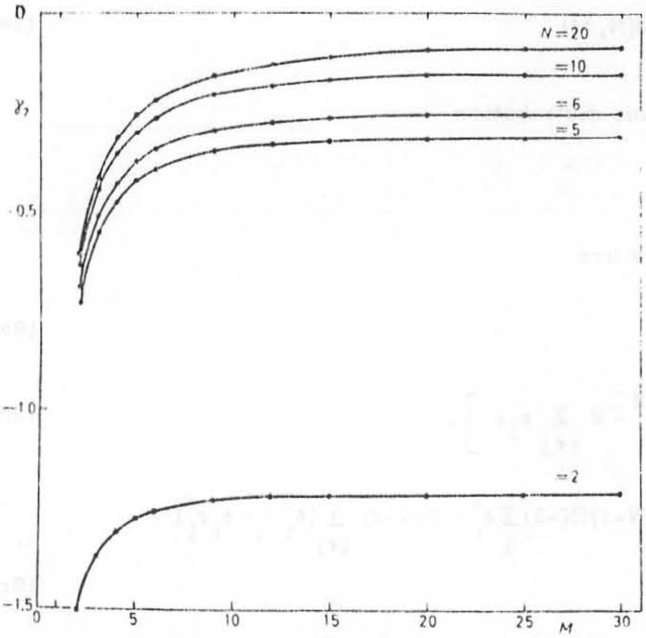


FIG. 1 - The excess γ_2 vs M.

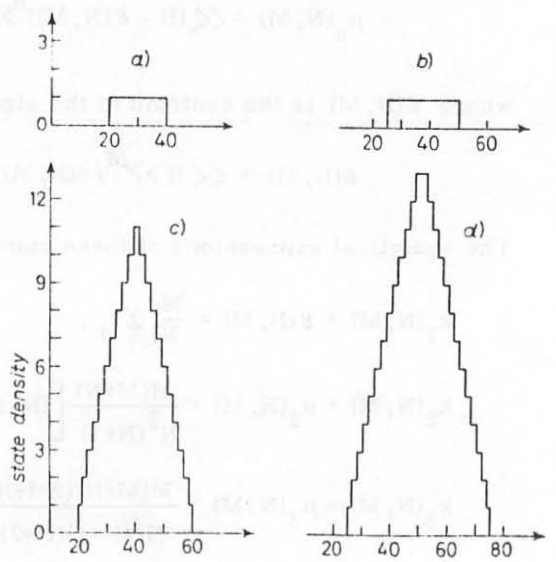


FIG. 2 - Exact state densities for M non-interacting bosons distributed among N equidistant levels :

a) N = 2, M = 20, d = 21;
b) N = 2, M = 25, d = 26;
c) N = 3, M = 20, d = 231;
d) N = 3, M = 25, d = 351;
e) N = 4, M = 20, d = 1771;
f) N = 4, M = 25, d = 3276.

Expressions (9) are drastically simplified if traceless single-particle energies $\tilde{\varepsilon}_i = \varepsilon_i - (\sum_{i=1}^{N} \varepsilon_i)/N$ are used[5]. In the dense limit the second central moments and the shape parameters are given by[5]

$$\sigma^2(M) = \frac{M^2}{N+1}\,\sigma^2(1)\ , \tag{10}$$

$$\gamma_1(M) = 2\gamma_1(1)\,\frac{(N+1)^{1/2}}{N+2}\ , \tag{11}$$

$$\gamma_2(M) = \frac{6\left\{\left[\gamma_2(1)+3\right](N+1)-(2N+3)\right\}}{(N+2)(N+3)}\ . \tag{12}$$

In order to test numerically the effect of a two-body residual interaction on the level-density shape of non interacting bosons, we chose a number-conserving Hamiltonian[8]

$$H = \sum_i \varepsilon_i a_i^+ a_i + \frac{1}{2}\sum_{i,j,k,l} \langle ij|V|kl\rangle a_i^+ a_j^+ a_k a_l\ , \tag{13}$$

where the two-body matrix elements V_{ijkl} are taken as random numbers uniformly distributed over the (- 0.1, 0.1) interval. We used different sets of single-particle values, with constant spacing $\Delta\varepsilon$ and centroid $\mathcal{E}(M=1)=0$ and twenty different sets of V_{ijkl}, for each value of $\Delta\varepsilon$.

Starting from the "exact" eigenvalues $E_1, E_2, \ldots$, obtained by a diagonalization procedure of the Hamiltonian (13), the centroid $\mathcal{E}$, the width σ, the skewness γ_1 and the excess γ_2, have been calculated

$$\mathcal{E} = \frac{\sum_j E_j}{d(N,M)}\ , \qquad \mu_n \quad \frac{\sum_j (E_j - \mathcal{E})^n}{d(N,M)} \qquad \sigma \quad \sqrt{\mu_2} \tag{14}$$

In tables I and II, $\mathcal{E}$, σ, γ_1 and γ_2 are shown, for two values of $\Delta\varepsilon$, as a function of the different sets of V_{ijkl}. The values corresponding to noninteracting bosons ($V_{ijkl}=0$) are also indicated.

The knowledge of the first moments μ_n makes it possible to construct approximate frequency functions, having the first moments in common with the "exact" one. A four-moment approximate frequency function is given by[11]

$$f(\chi) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\chi^2\right]\left\{1+\frac{\gamma_1}{6}(\chi^3 - 3\chi)+\frac{\gamma_2}{24}(\chi^4 - 6\chi^2+3)\right\}\ , \tag{15}$$

where

$$\chi = \frac{E-\mathcal{E}}{\sigma}\ . \tag{16}$$

As is well known, there is no unique way of reproducing a discrete spectrum from a continuous distribution. In this paper we adopt the Ratcliff prescription[12], solving the following equation:

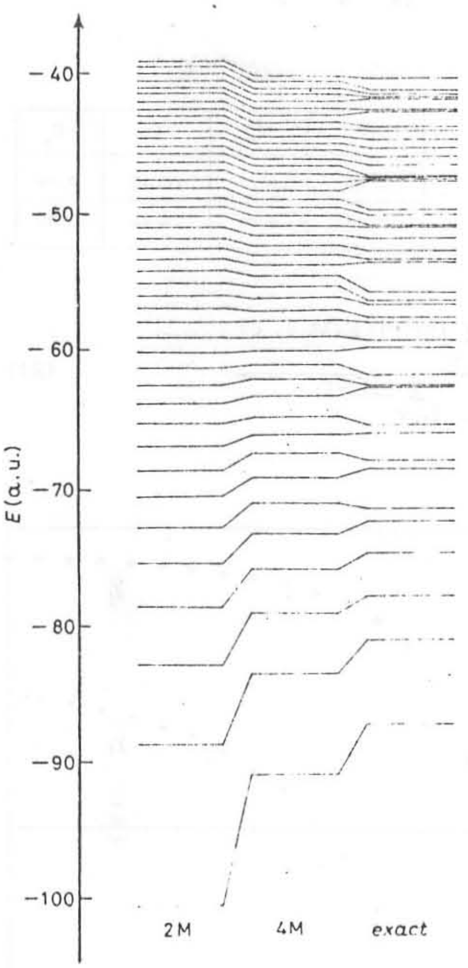
TABLE I

	M = 11	N = 4	d = 364	$\Delta\mathcal{E} = 1$ (a.u.)
set	$\mathcal{E}$(a.u.)	σ(a.u.)	γ_1	γ_2
$V_{ijkl} \neq 0$ 1	-0.77	9.30	0.15	-0.34
2	-0.04	6.55	-0.29	-0.28
3	1.92	9.20	0.02	-0.48
4	0.03	9.14	-0.01	-0.43
5	-2.51	9.12	-0.16	-0.34
6	1.98	7.66	0.01	-0.11
7	-0.19	8.64	0.20	-0.36
8	-1.91	9.38	-0.11	-0.44
9	-0.05	8.51	-0.21	0.06
10	-0.08	8.94	0.11	-0.71
11	1.19	7.52	0.22	-0.33
12	0.83	9.36	-0.20	-0.19
13	-0.07	8.01	0.08	-0.28
14	0.20	8.18	-0.25	0.09
15	0.85	8.73	-0.06	-0.49
16	0.38	7.71	-0.05	-0.48
17	2.39	8.67	0.14	-0.51
18	0.08	8.09	-0.21	-0.45
19	-1.05	9.28	-0.19	-0.39
20	0.68	7.46	0.47	0.22
	$\bar{\mathcal{E}} = 0.19 \pm 1.18$	$\bar{\sigma} = 8.47 \pm 0.77$	$\bar{\gamma}_1 = -0.02 \pm 0.19$	$\bar{\gamma}_2 = -0.34 \pm 0.17$
$V_{ijkl} = 0$	0	6.42	0	-0.43

TABLE II

	M = 11	N = 4	d = 364	$\Delta\mathcal{E} = 2$ (a.u.)
set	$\mathcal{E}$(a.u.)	σ(a.u.)	γ_1	γ_2
$V_{ijkl} \neq 0$ 1	-0.77	14.74	0.19	-0.35
2	-0.04	12.89	-0.11	-0.35
3	1.92	14.97	-0.02	-0.46
4	0.03	14.87	-0.02	-0.41
5	-2.51	14.84	-0.09	-0.41
6	1.98	13.40	-0.02	-0.26
7	-0.19	14.27	0.09	-0.46
8	-1.91	14.90	-0.08	-0.42
9	-0.05	14.00	-0.18	-0.17
10	-0.08	14.51	0.10	-0.55
11	1.19	13.02	0.20	-0.36
12	0.83	14.83	-0.18	-0.35
13	-0.07	13.41	0.03	-0.40
14	0.20	13.97	-0.16	-0.22
15	0.85	14.24	-0.07	-0.40
16	0.38	13.18	0.01	-0.39
17	2.39	14.22	0.11	-0.48
18	0.08	13.78	-0.14	-0.53
19	-1.05	14.63	-0.06	-0.45
20	0.68	13.13	0.27	-0.32
	$\bar{\mathcal{E}} = 0.19 \pm 1.18$	$\bar{\sigma} = 14.05 \pm 0.77$	$\bar{\gamma}_1 = -0.01 \pm 0.13$	$\bar{\gamma}_2 = -0.39 \pm 0.09$
$V_{ijkl} = 0$	0	12.84	0	-0.43

$$\int_{-\infty}^{E} f(\chi)\,d\chi = \frac{j+\frac{1}{2}}{d(N,M)}\,, \qquad j = 0, 1, \ldots\ldots, d(N,M) - 1\,. \tag{17}$$



In a typical case, as represented by Fig. 3 a comparison is made between an "exact" spectrum and a two-moment and four-moment approximated one. The agreement between the "exact" spectrum and its four-moment approximation is quite striking.

Keeping in mind the results of Tables I and II, we may conclude that, in our simple model, with the exception of case $\Delta\varepsilon = 0$, the gross structure of the level density is unmodified by the introduction of a two-body residual interaction.

FIG. 3 - A typical comparison between an "exact" spectrum and its two-moment (2M) and four-moment (4M) approximation. The total number of states is 364. In the figure only the first 45 states, starting from the ground state, are shown.

3. - QUADRUPOLE BOSONS[6, 13].

In order to study the state density of interacting quadrupole bosons, we chose the Arima and Iachello model in its simplest formulation[14] (d^M configurations only are introduced).

The Hamiltonian is

$$H = \varepsilon \sum_m a_m^+ a_m + \sum_{L=0,2,4} C_L \left\{ (a^+ a^+)_L (aa)_L \right\}_0 \,, \tag{18}$$

where

$$C_L = \langle d^2 L\mu | V | d^2 L\mu \rangle \,.$$

The expectation value of H on the basis $|MLv\rangle$ is given by[14]

$$E(M,L,v) = \varepsilon M + \alpha \frac{M(M-1)}{2} + \beta(M-v)(M+v+3) + \gamma\left[L(L+1) - 6M\right]\,. \tag{19}$$

The quantities α, β, γ are related to the C_L parameters by

$$\alpha = \frac{1}{14}(6C_4 + 8C_2)\ , \qquad \beta = \frac{1}{10}(C_0 - \alpha + 12\gamma)\ , \qquad \gamma = \frac{1}{14}(C_4 - C_2)\ . \tag{20}$$

Two sets of values (taken from Ref. (14)) were used for the parameters ε and C_L (see Table III) and the energies were calculated up to $M = 18$. Figs. 4 and 5 show the skewness $\gamma_1 = \mu_3/\mu_2^{3/2}$ and the excess $\gamma_2 = \mu_4/\mu_2^2 - 3$ as a function of M. The moments μ_n are connected with the energies $E(L, M, v)$ by the relation

TABLE III

ε	C_0	C_2	C_4
0.60	- 0.25	- 0.10	0.09
0.35	- 0.09	- 0.10	0.05

$$\mu_n(M) = \frac{\sum_{L,v} (2L+1)\left[E(M,L,v) - \varepsilon(M)\right]^n}{\sum_{L,v}(2L+1)}\ , \qquad \varepsilon(M) = \frac{\sum_{L,v}(2L+1)\,E(M,L,v)}{\sum_{L,v}(2L+1)} \tag{21}$$

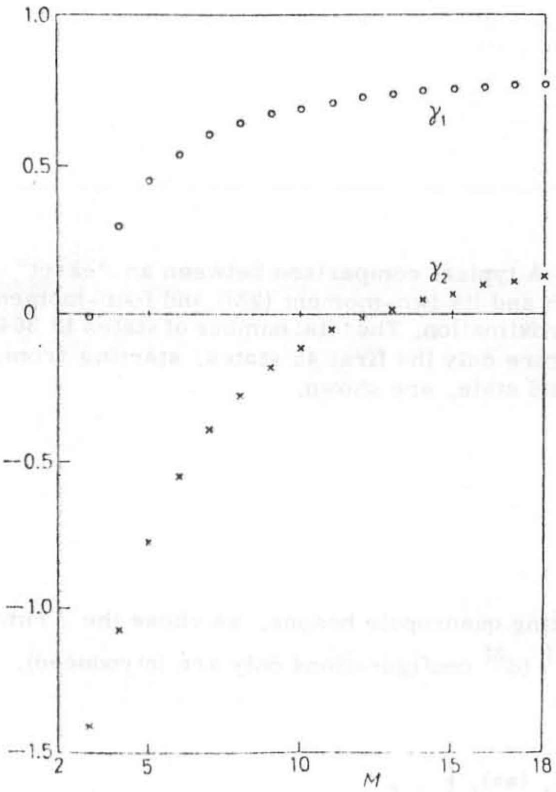


FIG. 4 - γ_1 and γ_2 vs M.

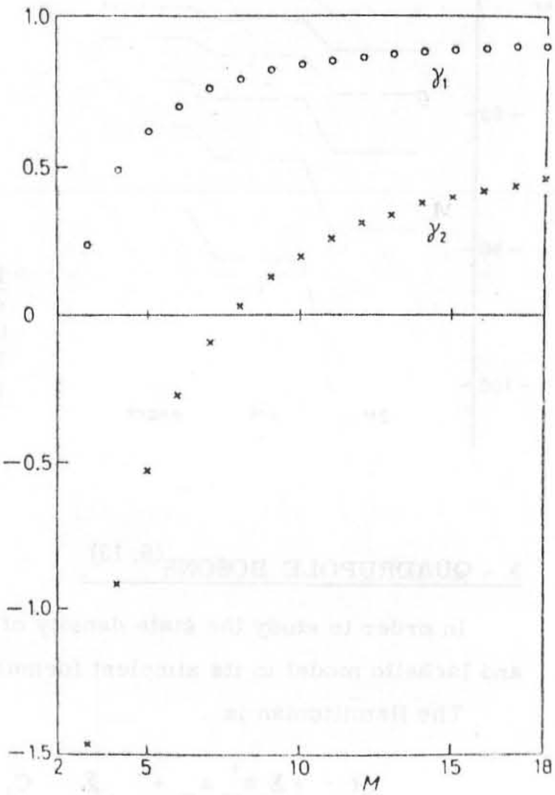


FIG. 5 - γ_1 and γ_2 vs M.

The quantities γ_1 and γ_2 do not decrease as the boson number M increases but tend to a definite large value (see also Ref. (5)).

In Figs. 6, 7 and 8 the "exact" state densities are shown for different numbers of bosons. Clearly the normality assumption for state density is not satisfied.

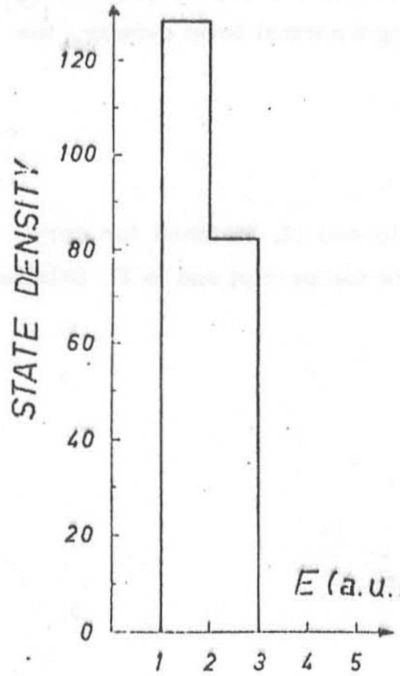


FIG. 6 - "Exact" state density for M = 6.

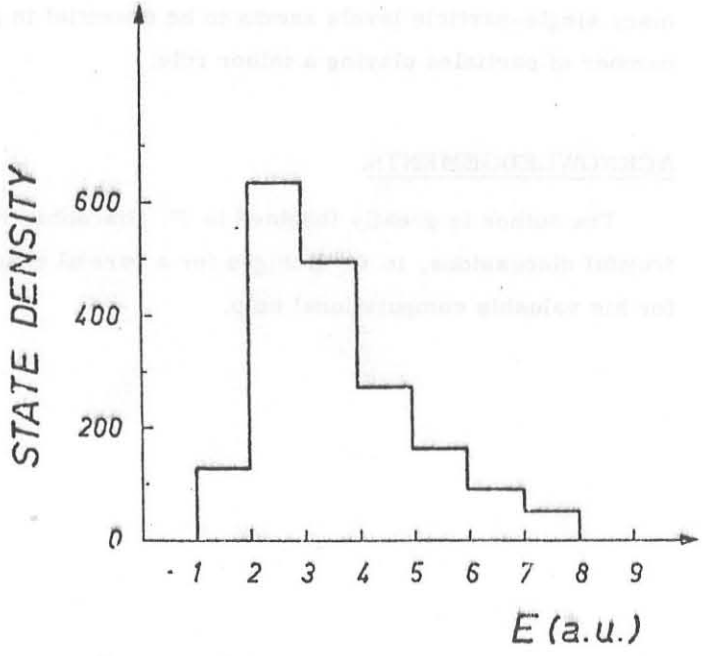


FIG. 7 - "Exact" state density for M = 12.

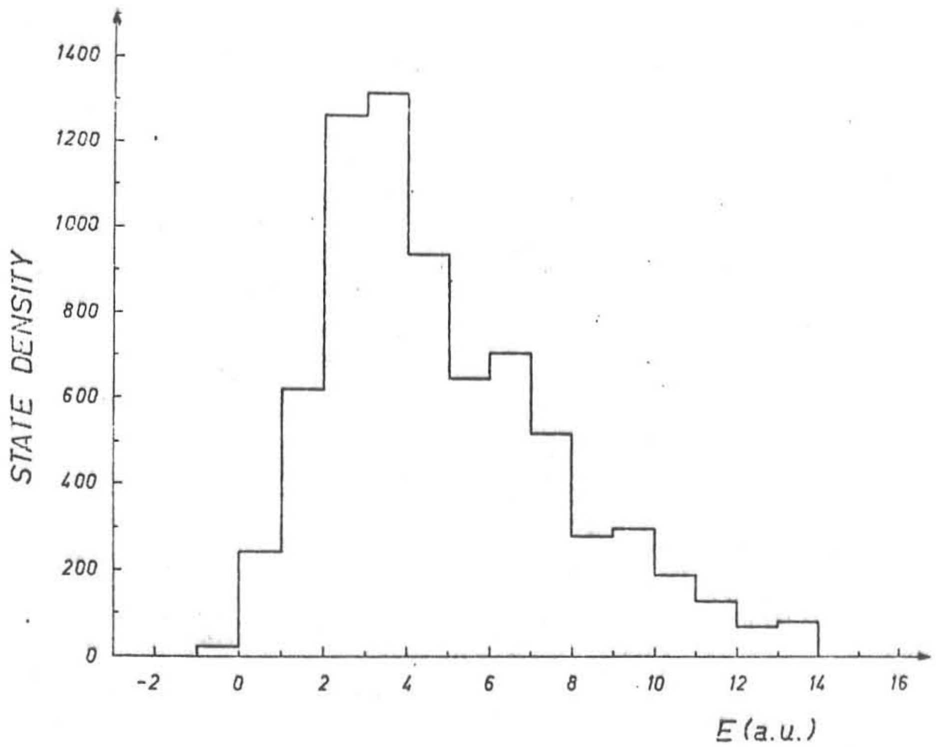


FIG. 8 - "Exact" state density for M = 18.

4. - CONCLUSIONS.

Keeping in mind Figs. 1 and 2 we may conclude that, in the boson case, the presence of many single-particle levels seems to be essential in generating a normal level density, the number of particles playing a minor role.

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