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E. Di Salvo and G.A. Viano: UNIQUENESS AND STABILITY IN THE INVERSE PROBLEM OF SCATTERING THEORY. -

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SUMMARY. -

In this paper we analyze the stability of the various methods which are used in the inverse problem of scattering theory. In fact, in many inversion procedures, even if the uniqueness of the reconstructed potential can be proved, nevertheless the solution does not depend continuously on the data; i.e., the solution is not stable. We show some methods for restoring the stability. Furthermore we discuss in detail the restored continuity, proving that, in some cases, it is so weak that any numerical computation of the solutions is practically excluded.

1. -INTRODUCTION. -

The inverse problem of scattering theory has been widely investigated and the list of papers of this subject is enormous (see, for instance, refs. (1) and (2)). As it is well-known, the authors considered essentially two different types of input information:

a) - the knowledge of one phase-shift at fixed angular momentum and for all energies (but the knowledge of the bound-state energies is also required);
b) - the knowledge of all the phase-shifts (or of the scattering amplitude) at fixed energy.

In the second case, which is also the more realistic from the physical point of view, the semiclassical inversion methods, which are essentially based on the short wavelenght approximations, have also been extensively discussed[2].

The quantities which can be directly measured are not the phase-shifts, but the cross sections. Therefore, one is primarily faced with the problem of determining the scattering amplitude for all angles from the measured differential cross sections.

One approach, which avoids this type of difficulty, is to measure the correlated counting rates of two detectors[3]; however the beam intensities actually available in particle physics make unrealistic this approach. Therefore, for energies at which only elastic scattering is possible, the more general procedure used is to apply the generalized unitarity theorem, which follows from the conservation of flux. This theorem gives a nonlinear integral equation for the phase function. Many papers[4, 5, 6, 7] have been devoted to this equation, and we do not intend to return on these questions. Hereafter we shall assume that the phase-shifts are known within a certain degree of accuracy.

Now, if we suppose that a certain type of input information is known, the questions which must be preliminarily solved are if it does exist a potential which generates these scattering data and if this potential is or is not unique. The questions of existence and uniqueness have been deeply investigated in relation to the different types of input information[1]. Nevertheless, this is not sufficient; in fact, quoting the Courant-Hilbert treatise (ref. (8), p. 227), we say that a mathematical problem which is to correspond to physical reality should satisfy the following basic requirements:

1) the solution must exist;
2) the solution should be uniquely determined;
3) the solution should depend continuously on the data (requirement of stability).

The third requirement is necessary if the mathematical formulation is to describe observable natural phenomena. Data in nature cannot possibly be conceived as rigidly fixed; the mere process of measuring them involves small errors. Therefore, a mathematical problem cannot be considered as realistically corresponding to physical phenomena, unless a variation of the given data in a sufficiently small range leads to an arbitrarily small change in the solution. This requirement of stability is not only essential for meaningful problems in mathematical physics, but also for approximation and numerical methods.

Now it is possible to give examples of problems where the first and the second condition are satisfied but not the third; such problems are called, after Hadamard, ill-posed or improperly-posed problems. The most famous of these examples was discovered by Hadamard (ref. (9), p. 33) and it is connected with the Cauchy problem for the Laplace equation (see Appendix A).

At the beginning, when Hadamard discovered such a pathology, he solved the question conjecturing that none of the physical problems, connected with the Laplace equation, is formulated analytically in Cauchy's way[9]. All of them lead to statements such as Dirichlet's, i. e. with only one numerical datum at every point of the boundary. It was discovered later that Hadamard's conclusion was erroneous. In fact a number of important problems of geophysics lead to the Cauchy problem for the Laplace equation (ref. (10), p. 2).

Following Tikonov[11], we call direct problems those which are oriented along a cause-effect sequence, and inverse problems those associated with the reversal of the chain of causally related effects. Tikonov suggested that the inverse problems, in many cases, present the Hadamard pathology.

Therefore, it becomes essential to discover how to restore the stability. This is usually done imposing a suitable qualitative a-priori restriction on the solutions admitted. Of course, the type of restriction, as well as the degree of restored stability, change from one case to another.

All these considerations make evident the necessity of analyzing the question of stability for the inverse problem of scattering theory, and this also in connection with the numerical solutions of the inversion procedures. In fact, while very solid results have been obtained for what concerns the existence and uniqueness, not enough attention has been paid to the question of the continuous dependence on the data, except for an example of instability given by Newton[12] and a paper of one of the authors[13], where some partial and preliminar results were obtained. In the present paper we analyze the problem of the stability for the various inversion procedures, i.e. semiclassical, at fixed angular momentum and at fixed energy. This analysis shall be done in the next Section. All the mathematical proofs, methods and examples shall be given in the Appendices, which contain results which have some interest by themselves.

2. - STABILITY IN THE INVERSE PROBLEM. -

2.1. - Semiclassical Inversion Methods. -

In these procedures it is assumed a-priori that the potential to be obtained is such that the JWKB approximation is valid for the phase-shift. Then it is possible to define a function $H(\lambda)$, from which the potential $V(r)$ can be derived, at least for large enough r, and the phase-shifts are related to $H(\lambda)$ through the following equation[14]:

$$\delta(\ell) = \int_{\ell+\frac{1}{2}}^{\infty} \frac{\lambda}{\sqrt{\lambda^2 - (\ell - \frac{1}{2})^2}} H(\lambda)\, d\lambda \tag{1}$$

where the notations are the usual. Conversely, if δ is known as a differentiable function of ℓ, the Abel equation (1) can be uniquely inversed and $H(\lambda)$ can be obtained from the phase-shifts, then $V(r)$ from $H(\lambda)$.

In general we can say that the common mathematical tool for all the semiclassical inversion methods is the Abelian integral equation of the following type:

$$g(x) = \int_{x}^{\infty} (y-x)^{-1/2} f(y)\, dy \tag{2}$$

Now we want to show the pathology connected with the eq. (2). In fact let us consider two functions $f(y)$: a function $f_1(y)$ which is a solution of (2) and the function

$$f_2(y) = f_1(y) + C\, e^{-by} \sin(ny)$$

where C and b are constants and n is an arbitrarily large integer. Then the amplitude of the oscillations of the function $F(y) = f_2(y) - f_1(y)$, can be made arbitrarily large (for any finite and not too great value of y, such that the effect of the damping factor e^{-by} is small), by taking for C a sufficiently large value. On the other hand, the difference between the functions $g_2(x)$ and $g_1(x)$ (which are the left members of eq. (2) corresponding to $f_2(y)$ and $f_1(y)$ respectively) can be done arbitrarily small, at fixed C, by making n sufficiently large; in fact

$$\left| g_2(x) - g_1(x) \right| \xrightarrow[n \to \infty]{} 0 \tag{3}$$

for any finite value of x.

This example proves that in the eq. (2) the solutions do not depend continuously on the data, even if the requirement of uniqueness is satisfied. In fact, arbitrarily small perturbations of the data, such as generated by the noise of the experimental measurements, can produce arbitrarily large oscillations on the solutions. A way for restoring the stability in eq. (2) is to impose a-priori a prescribed global bound on the derivatives of f(y). This procedure shall be illustrated, in the specific case of the Abelian integral equation of the type (2), in the Appendix D.

The procedures based on eq. (1) require an interpolation of the phase-shifts from δ_ℓ to $\delta(\ell)$. The uniqueness problem for the interpolation is a very sublte one, and it shall be discussed subsequently. However, if one does not care too much about the uniqueness, then one can take for granted any smooth continuous fitting of the phase-shifts; nevertheless, in any case, one is faced with the question of stability of the solutions of eqs. (1) or (2).

2.2.- The Fixed Angular Momentum Inversion Problem.-

The method of Gel'fand-Levitan for reconstructing the potential, at fixed energy, requires the knowledge of the spectral function, which can be obtained by writing the completeness relation, which holds for the regular solutions $\varphi(E, r)$ of the Schrödinger equation. Then the spectral function is related to the bound-state normalization constants N_n (where n is the number of the bound-states) and to the Jost functions; the latter can be evaluated from the phase-shift and from the energies of the bound-states of the same angular momentum, via a dispersion relation. As it is well-known, for a given phase-shift and a given set of n bound-states, there is an n-parameter family of associated spectral functions and, thus, of potentials (phase-equivalent potentials). Therefore, as Bargmann(15) and Levinson(16) firstly remarked, a potential cannot be reconstructed uniquely from prescribed energylevels and a given phase-shift. Levinson(16) showed that this lack of uniqueness is related to the existence of a discrete spectrum; moreover he proved that the reconstructed potential is unique when there is no discrete spectrum(16). Otherwise, in the presence of a discrete spectrum, there is a natural possibility, thanks to a Newton's theorem(12), of selecting a unique potential out of the family of those equivalent with respect to both phase-shift and binding energies. In fact, Newton proved(12), that there is only one potential with given phase-shift and given binding energies E_n and with the property that

(4) $$\lim_{r \to \infty} V(r)\, e^{2Kr} = 0 \qquad K^2 = \max(2|E_n|M)$$

where M is the reduced mass. Now the condition (4) implies an S matrix which is analytic in a strip of width K above the real k-axis (k is the momentum), except for the bound-state poles on the imaginary axis. Therefore a necessary condition for the existence of a unique potential, in the sense of (4), is that the S matrix is in fact analytic in a strip above the real k axis which contains all the bound-states, except for the points corresponding to the latter[12].

For what concerns the inversion procedure, the method works as follows: one starts from an "auxiliary function" which can be explicitly constructed if the spectral function $\varrho(E)$ is known, i.e.

$$(5) \qquad f_V^{V_1}(r,r') = \int \left[d\varrho_1(E) - d\varrho(E) \right] \varphi_1(E,r)\, \varphi_1(E,r')$$

where $\varphi_1(E,r)$ is the known solution of the radial Schrödinger equation for an arbitrary comparison potential V_1, whose associated spectral function is $\varrho_1(E)$. Then from the auxiliary function $f_V^{V_1}(r,r')$ one can derive the so-called "transformation-kernel" $K_V^{V_1}(r,r')$, through the following integral equation:

$$(6) \qquad K_V^{V_1}(r,r') = f_V^{V_1}(r,r') - \int_0^r K_V^{V_1}(r',r'')\, f_V^{V_1}(r',r'')\, dr''$$

which, for fixed r, is a Fredholm integral equation of the second kind. Then it is straightforward to derive the potential from $K_V^{V_1}(r,r')$[1].

We can say that the fundamental mathematical tool, used in this inversion procedure, is an equation like (6). Now, for showing that a unique solution of eq. (6) exists, it suffices to prove that the homogeneous equation has only a trivial solution. This proof can be done (see for instance, ref. (1)).

Moreover the solution given by the Fredholm integral equation of the second kind depends continuously on the data, as it shall be proved in the Appendix A. Therefore we can conclude that if there is no discrete spectrum, or if one is looking for a potential with a prescribed asymptotic behaviour (in the sense specified by (4)), then a unique potential can be reconstructed, and the solution depends continuously on the data. Unfortunately this very nice result is rather academic. In fact the input information required is, at least, the knowledge of the phase-shift for all energies, from zero up to infinity; of course this information cannot be obtained by direct experimental measurements and therefore one is obliged to extrapolate the experimental phase-shifts, and this fact introduces large ambiguities. Now one may ask: does knowledge of a phase-shift over any finite range of energy tell us if the potential decreases asymptotically as an exponential or less rapidly? The answer is "No". In fact, as we said above, the analyticity of s in a strip of finite width along the real k axis is a necessary condition for exponential decrease of the potential, and no finite piece of δ_ℓ can reveal it. A small "kink" in the phase-shift at high energies is sufficient to alter radically the asymptotic behaviour of the potential(x) (see refs. (1), (12), (17) and Appendix A).

2.3.- The fixed energy Inversion Problem.-

The fixed energy inversion problem requires, as input information, the knowledge of all the phase-shifts (or of the scattering amplitude) at one energy. This type of inversion has been tried with three different methods, which we analyze separately.

2.3.1.- Martin - Targonski method[18].-

In this procedure, the potential to be obtained is supposed to belong to the following class:

$$V(r) = \frac{1}{r} \int_r^\infty C(\alpha)\, e^{-\alpha r}\, d\alpha \tag{7}$$

where $\mu > 0$ and $C(\alpha) \in L_1(\mu, \infty)$. If V(r) is given by (7), then the scattering amplitude T(t) (where $t = -2\, k^2 (1-\cos\theta)$ is the negative square of the momentum transfer, k is the momentum and θ is the scattering angle), is analytic in the whole complex t-plane, except along a cut on the real axis from $t = \mu^2$ to $t = +\infty$(o). Now the region where T(t) is approximately known (i.e. the physical region) belongs to the real axis and it is given by: $-4k^2 \leq t \leq 0$. Furthermore the following dispersion formula hods (or a suitably substracted form):

$$T(t) = \int_{\mu^2}^{\infty} D(t')\, (t'-t')^{-1}\, dt' \tag{8}$$

(x) - This analysis of Newton[12] can be considered the first example of the pathologies and instabilities of the inverse problem in scattering theory.

(o) - More precisely the point $t = \mu^2$ is the first of an infinite sequence of branch-points, as we shall explain below.

where $2\pi i D(t)$ is the discontinuity across the cut. Equation (8) is an integral Fredholm equation of the first kind. Therefore the inversion procedure works as follows: one must firstly determine D(t) through the eq. (8), i.e. extrapolating the approximately known values of T(t) from the physical region up to the cut. Then one can determine the potential from the discontinuity across the cut. The uniqueness of the reconstructed potential is guaranteed by the uniqueness theorem of the analytic continuation. Nevertheless, we must remark once more that, even if the solution is unique, it does not depend continuously on the data, in the sense that small changes in the data can produce large effects in the solutions (see Appendix A). Moreover, in this inversion procedure, one must extrapolate the approximate values of T(t) from the physical region up to the cut. More precisely, the scattering amplitude T(t) can be expanded as follows:

$$T(t) = T_1(t) + T_2(t) + \ldots + T_n(t) + \ldots \tag{9}$$

where $T_n(t)$ is the n-th Born term and has a cut starting at $t = n^2\mu^2$. It follows that T(t) presents an infinite sequence of cuts.

Therefore, if we map the cut t-plane into the unit disc (see Appendix C), then all these branch-points are mapped on the unit circle. Now, even if the unit circle is not the natural boundary of the scattering amplitude, since the singular points are not everywhere dense on it, however it can be taken, from the practical and numerical point of view, as the boundary of the domain where the function is holomorphic.

Moreover the analytic functions are extremely smooth and well behaved deep inside their domain of holomporphy, but may grow rough and oscillatory as we approach the boundary of this domain. In such a situation, even if one restores the stabilitity of the complex analytic continuation imposing a-priori some suitable stabilizing constraint, nevertheless the restored stability remains extremely poor, i.e. of logarithmic type. This fact shall be proved in the Appendix C. Therefore, we can conclude that this inversion procedure cannot be attempted, since the logarithmic continuity practically excludes the numerical computation of the solutions.

2.3.2. - Regge-Loeffel method[17]. -

This and the subsequent method are essentially the analog of the Gel'fand-Levitan formalism. Of course, also for the fixed energy case, one needs an "auxiliary-function" $f^{V}_{V1}(r, r')$ (where V_1 is again a well behaved, comparison potential) which allows us to write explicitly an integral equation for the transformation kernel $K_V^{V1}(r,r')$:

$$K_V^{V1}(r,r') = f_V^{V1}(r,r') - \int_0^r dr''\, r''^{-2}\, K_V^{V1}(r,r'')\, f_V^{V1}(r'',r') \tag{10}$$

which, for fixed r, is an integral Fredholm equation of the second kind.

At this point one must relate the input information to the "auxiliary - function" $f_V^{V1}(r,r')$.

In the fixed - angular - momentum case, this result was achieved through the completeness and orthogonality of the set of radial functions of all energies. In the present contect, this cannot be done. Two different approaches were elaborated in order to overcome this difficulty. One of them, the Regge - Leoffel method, makes use of the complex angular momentum interpolation. More precisely, Loeffel[17] writes for $f_V^1(x,y)$ an expansion of the following type:

$$f_V^1(x,y) = (2\pi)^{-1} \int_{-\infty}^{+\infty} d\tau\, \gamma(\tau)\, (xy)^{i\tau - \frac{1}{2}} + \sum_{k=1}^{\infty} d_k\, (xy)^{\nu_k - \frac{1}{2}} \tag{11}$$

where the continuous function $\gamma(\tau)$ and the sequences $\{d_k\}$ and $\{\nu_k\}$ are the so called "spectral-data", and are unique for a given potential. Moreover the series, in formula (11), converges uniformly for (x, y) in any compact contained in $R_+ \times R_+$. Then, using this expansion, Loeffel can prove a uniqueness - theorem for a class W of real potentials which satisfy the conditions

$$\int_1^{+\infty} |V(r)|\, dr < +\infty \tag{12a}$$

$$\int_0^1 r^{1-2\epsilon}\, |V(r) - 1|\, dr < +\infty, \quad \epsilon > 0 \tag{12b}$$

The Loeffel uniqueness theorem reads as follows.

Theorem (Loeffel): - Let $\sigma(\ell)$ be the interpolation of $S_\ell = e^{2i\delta_\ell}$ (the so-called Regge interpolation); let the potentials V_1 and V_2 belong to the class W; if the corresponding Regge interpolations σ_1 and σ_2 satisfy

$$\sigma_1(\nu) = \sigma_2(\nu), \qquad \nu = \ell + \frac{1}{2} \tag{13}$$

for all ν, with Re $\nu > 0$, where both the interpolations are holomporphic, then

$$V_1(r) = V_2(r) \tag{14}$$

for almost all positive r.

Thanks to the above theorem, one can realize that any non-uniqueness in the correspondence between the sequence $\{S_\ell\}$ and the potential can be interpreted as a non-uniqueness in the step from $\{S_\ell\}$ to its interpolation σ. Now the theorems which guarantee the uniqueness of the interpolations are essentially the Carlson[20] and the Lagrange-Valiron[13] theorems, which can be used only for particurlar classes of potentials. For instance, the Carlson' theorem can be applied to the class of Yukawa-like potentials. Moreover the Loeffel' method does not furnish a constructive procedure for evaluating the "spectral-data" from the sequence $\{S_\ell\}$, indeed a computation of the "spectral-data" requires the knowledge of the Jost functions at the physical points. Finally, for what concerns the continuity, one must ask: how the "spectral-data" are stable with respect to small changes in the phase-shifts? It is possible to answer to this question, analyzing the stability of the Regge interpolation $\sigma(\nu)$ in all the domain where $\sigma(\nu)$ is holomorphic and, necessarily, up to the boundary of this domain. We shall discuss this point in the Appendix C and we shall see that, even if the continuity can be guaranteed by imposing suitable stabilizing constraints, nevertheless it remains very poor. Therefore we come to a conclusion similar to that illustrated above, concerning the Martin - Targonski method.

2.3.3. - Newton - Sabatier method. -

In this method one starts from the Gel'fand - Levitan integral equation (10), but instead of using the formula (11) for the "auxiliary function", one represents f(r, r') through an expansion over the integer values of the angular momentum ℓ[21]. This choice has the advantage that the relationship between the phase-shifts δ_ℓ and the coefficients C_ℓ which enter in the expansion for f(r, r') does not involve any complex analytic continuation, and it is given by an infinite system of linear equations. The key to the procedure is the inversion of some infinite matrices. However, for what concerns the uniqueness of the solution, the answer is negative. In other words, there exists, at every energy, at least a one-parameter family of potentials which produce no scattering whatever (i.e. "transparent-potentials"). Sabatier[22] has proved that these "transparent-potentials" asymptotically oscillate and decrease as $r^{-3/2}$; therefore, the uniqueness is obtained if and only if we restrict ourselves to potentials that decrease faster than $r^{-3/2}$. Furthermore the almost uniqueness, encountered in this case, corresponds to very special properties of the potentials[22], and is due to the restriction of ℓ to the integer values on the real axis. In other words, the potentials, built through the Newton method, form a narrow class, such that it does not include, for instance, the Yukawián potentials. Therefore Sabatier[23, 24] generalized the Newton method, by allowing ℓ to take any real value larger than $-1/2$. This yields a much larger class of equivalent potentials. Successively Sabatier[25, 26] elaborated other methods, but, in any case, the answer concerning the uniqueness is negative. Therefore we shall not discuss further these procedure, in spite of their great relevance.

APPENDIX A. -

Let us start from the classical Hadamard example (ref. (9), p. 33) of a problem which is improperly-posed. Consider the Laplace two-dimensional equation

(A.1) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with the following Cauchy's data:

(A.2) $$u(x,0) = 0$$

(A.3) $$\frac{\partial u}{\partial y}(x,0) = \frac{1}{n}\sin(nx)$$

n being a very large number. These data differ from zero as little as can be wished. Nevertheless, such a Cauchy problem has for its solution

(A.4) $$u = \frac{1}{n^2}\sin(nx)\sinh(ny)$$

which is very large for any determinate value of y different from zero. The presence of the factor sin(nx) produces a "fluting", and we see that this fluting, however imperceptible on the x-axis, becomes enormous at any given distance of it however small, provided the fluting be taken sufficiently thin by taking n sufficiently great.

The Hadamard example can be easily translated to the case of the analytic continuation of complex-valued functions. More precisely, the knowledge of $u(x,0)$ and $(\partial u/\partial y)(x,0)$ is equivalent, thanks to the Cauchy-Riemann equations, to the knowledge on the real axis of the real part $u(x,0)$ and the imaginary part $v(x,0)$ of a complex valued analytic function. In fact one obtains:

(A.5) $$u(x,0) = 0$$

(A.6) $$v(x,0) = -\frac{1}{n^2}\cos(nx) + C$$

For the sake of simplicity, let us take $C = 0$. Then the analytic functions $f_n(z)$ $(z = x + iy)$, which on the real axis assume the values given by (A.5) and (A.6), can be written as follows:

$$\text{(A.7)} \qquad f_n(z) = -\frac{1}{n^2}\left\{\frac{e^{ny}-e^{-ny}}{2}\sin(nx)+i\frac{e^{ny}+e^{-ny}}{2}\cos(nx)\right\}$$

Now it is easy to show, be means of the functions (A.7), that the complex analytic continuation of functions which are approximately known on a segment Γ of the real axis is completely unstable.

In fact let us suppose that the entire function f(z) (which is assumed to be real on the real axis), is approximately known on Γ within an accuracy ε. Then it is impossible to discriminate, from the numerical and practical point of view, between two approximations $g_1(x)$ and $g_2(x)$, whose difference is given by

$$|g_1(z)-g_2(z)|_{z\in\Gamma} = |f_n(z)|_{z\in\Gamma}$$

with n sufficiently large. Nevertheless the difference between $g_1(z)$ and $g_2(z)$ becomes enormous at any given distance, arbitrarily small, from the real axis.

Of course one comes to the same conclusions also in those cases where the physical problems present different geometry. If one, for instance, considers the Martin - Targonski inversion method, then the scattering amplitude is approximately known on the physical region $-4K^2 \leq t \leq 0$ and it must be analytically continued up to the cut. Also in this case one can produce pathological examples, which show that the solutions of the equation (8) are unstable.

Next we consider the Dirichlet problem, for a plane region D, with a piecewise smooth boundary ∂D; it consists in finding a solution of the bidimensional Laplace equation $\Delta_2 u = 0$, which is continuous in $D+\partial D$ and regular in D, and assumes on ∂D prescribed boundary values. As it is well-known, the existence and uniqueness for the solution of this problem can be proved in a quite general setting(8); moreover it can also be proved that the solutions depend continuously on the boundary values. In fact the difference of two solutions, whose prescribed boundary data differ everywhere by an amount less than ε in absolute value, is again an harmonic function and cannot have an absolute value greater than ε in the inferior of D, because it assumes its maximum and minimum on ∂D. Therefore the Dirichlet problem is properly posed(8). Furthermore, as it is well-known, the Dirichlet problem for certain special regions can be reduced to a Fredholm integral equation of the second kind. Therefore we can conjecture that the solutions of the Fredholm integral equations of the second kind depend continuously on the data.

APPENDIX B. -

In this Appendix we analyze the bounds, which are necessary for restoring the stability in the analytic continuation of complex-valued functions. We recall that the problem is to approximately determine by analytic continuation certain values of a function f(z), which is holomorphic in a domain Ω, but where measurements for f(z) are possible only at data points in the segment Γ of the real axis, which segment is called the physical region. We denote with Y the data space, i.e. a certain metric space of functions on Γ. Then we denote with X the solution space, i.e. a certain meteric space of functions holomorphic in Ω.

Now we recall the following theorem on compactness.

Theorem (ref. (27); Let σ be a continuous map on a compact topological space into a Hausdorff topological space; if σ is 1-1, then its inverse map σ^{-1} is continuous.

From this theorem it follows that the compactness of the solution space is sufficient to guarantee the continuity of σ^{-1}, since the uniqueness theorem of analytic continuation guarantees that σ is 1-1.

Now suppose that Ω is the unit disc $|z| \leq 1$; in this case a bound of the following type:

$$\sup_{z \in \partial\Omega} |f(z)| \leq E \tag{B.1}$$

(where $\partial\Omega$ is the boundary of Ω and E is a constant) is sufficient to restore the stability of the analytic continuation from the physical region Γ to any compact subdomain of Ω.

Let us denote with $\mathcal{F}$ the family of functions f(z) which are holomorphic in Ω and satisfy the condition (B.1). It can be easily proved that also the first derivatives of these functions are locally uniformly bounded in any compact subdomain of Ω. To show this, we draw a circle C about the point z, such that C is entirely contained in Ω. Hence, if z_o is a point within C, we have:

$$|f'(z_o)| = \left| \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(\zeta - z_o)^2} \right| < \frac{ER}{d^2} \tag{B.2}$$

where R is the radius of C and d is the smallest distance of z, from C. If z is in a neighborhood of z_o which is contained within C and whose distance from C is at least d_1, we have in this neighborhood $|f'(z)| < E R d_1^{-2}$, which shows that $f'(z)$ is indeed locally uniformly bounded[28].

From these considerations it follows that from any sequence of functions belonging to $\mathcal{F}$ it is possible to extract a subsequence which converges uniformly in any compact subdomain of Ω. Moreover all the limit functions belong also to $\mathcal{F}$, and therefore we can conclude that F is compact. This fact, plus the uniqueness of the analytic continuation, restores the continuity to the problem of the analytic continuation from the physical region to any compact subdomain of Ω. Of course, the condition (B.1) is not sufficient if we want to perform a continuation up to the boundary of Ω. In fact, the relation (B.2) does not work for the points on the boundary of Ω. Therefore, if we want to restore the continuity up to the boundary, we must require an a-priori quantitative bound on the first derivative, at least, i.e.

$$\text{(B.3)} \qquad \sup_{z \in \partial\Omega} \left| \frac{df}{dz} \right| \leq E$$

However we shall return on this point with more details in the Appendix C.

Now we can explicitly write a bound which shows the type of stability for the analytic continuation. Recall that f(z) is analytic in the unit disc Ω, continuous on $\bar{\Omega}$ and satisfies the bound (B.1); moreover we assume that it is given approximately as g(z) at the points of the physical region Γ, i.e.

$$\text{(B.4)} \qquad |f(z) - g(z)| \leq \varepsilon , \qquad z \in \Gamma$$

where ε is the data accuracy. If $\varphi(z)$ is the difference of any two analytic functions which satisfy (B.1) and (B.4), it is analytic in Ω and continuous on $\bar{\Omega}$; moreover, writing $F(z) = (1/2)\ \varphi(z)$, we have:

$$\begin{cases} |F(z)| \leq \varepsilon & z \in \Gamma \qquad \text{(B.5)} \\ |F(z)| \leq E & z \in \partial\Omega \qquad \text{(B.6)} \end{cases}$$

Then the Carlemann inequality gives[29]:

$$\text{(B.7)} \qquad |F(z)| \leq \varepsilon^{\omega(z)}\, E^{(1-\omega(z))} , \qquad z \in \Omega$$

where $\omega(z)$ is the harmonic function on $\Omega-\Gamma$, which is continuous on $\bar{\Omega}$ and equals 1 and 0 on Γ and $\partial\Omega$ respectively. The bound (B.7) shows that the continuation to points well within the domain of analyticity has a fairly satisfactory ε^{λ} Hölder type stability $(0<\lambda<1)$. On the other hand, at the boundary $\partial\Omega$, the inequality (B.7) reduces to the bound (B.6), i.e. it does not give any dependence on $\mathcal{E}$. We can effectively say that the bound (B.1) is not sufficient for restoring the stability of the continuation up to the boundary.

APPENDIX C. -

In this Appendix we want to discuss the type of stability which can be restored in the analytic continuation up to the boundary of the analyticity domain. Let us first consider the inversion method of Martin-Targonski[18]. These authors conformally map the cut t-plane into the unit disc, through the following formula:

$$t = \frac{4\mu^2 z}{(z+1)^2} \tag{C.1}$$

In this map the upper (lower) half of the unit circle corresponds to the upper (lower) lip of the cut. Therefore, the continuation up to the cut corresponds to the continuation up to the unit circle $|z|=1$. Therefore, as we have seen in the Appendix B, in order to stabilize this continuation, it is necessary a bound on the first derivative, like (B.3) (supposing that the physics of the problem and the type of singularities allow us to use such a condition). Nevertheless the restored continuity remains very poor, i.e. of logarithmic type. In fact, following John[30], it is possible to prove that, if F(z) satisfies the following conditions:

$$\left\{\begin{array}{lll} (C.2) & |F(z)| \leq \mathcal{E} & |z| \leq a \\ (C.3) & \left|\dfrac{dF(z)}{dz}\right| \leq E & |z| \leq 1 \end{array}\right.$$

(where the physical region Γ is supposed to be a circle of radius a, $0<a<1$), then the following bound holds:

$$|F(z)| < 2E\,\frac{\log\left(\frac{2}{a}\right)}{\log\left(\frac{E}{\varepsilon}\right)}\,, \qquad |z|\leq 1 \tag{C.4}$$

which shows that the dependence on the data accuracy ε, is of logarithmic type; i.e. as $[\log(E/\varepsilon)]^{-1}$.

We come to similar conclusions considering the Regge Loeffel inversion method. In this case the domain is the right half-plane $\mathrm{Re}\,\ell \geq -1/2$(*), where ℓ is the angular momentum and the data points are $l = 0, 12....$ This domain Ω can be mapped into the unit disc by the conformal mapping formula:

$$z = \frac{\ell}{\ell + 1} \tag{C.5}$$

and the data points are then given by

$$\alpha_j = \frac{j}{j+1} \quad (j = 0, 1, 2, \ldots.) \tag{C.6}$$

The boundary line $\mathrm{Re}\,\ell = -1/2$ is mapped into the unit circle. Therefore, one is faced with the following continuation problem: to determine, on the boundary $z = 1$, the values of a function $f(z)$ which is approximately known on a finite discrete set of data points $\alpha_j (j = 0, 1, 2, \ldots n)$. First of all, we observe that, also in this case, a bound on the function (let's say like (B.1) is not sufficient for restoring the stability to the continuation up to the unit circle, but it is required a bound on the first derivative at least. However, for the sake of semplicity, we limit ourselves to show that the restored stability is already extremely poor for the continuation toward and near to the boundary line $\mathrm{Re}\,\ell = -1/2$, even without reaching it. Furthermore the exponential Carlson bound[20] (which is usually invoked in this type of problem) is not suitable for a numerical evaluation of stability, and therefore it is convenient to use a more restrictive uniform bound, which can be normalized to 1. Finally the inequality (E.7) cannot be used, since in this case the physical region Γ is a finite set of discrete points (the Carlemann inequality requires a positive capacity for Γ). In conclusion we have the following conditions for $F(z)$: it is supposed to be holomporphic in the unit disc and to satisfy the following inequalities

$$|F(z)| \leq 1 \qquad |z| \leq 1 \tag{C.7}$$

$$|F(\alpha_j)| \leq \varepsilon \qquad 0 \leq \varepsilon \leq 1, \quad j = 0, 1, 2, \ldots. n \tag{C.8}$$

(*) - More generally one has $\mathrm{Re}\,\ell \geq L$ (where L is a constant); however the changes are irrelevant.

where $\alpha_0, \alpha_1, \alpha_2 \ldots, \alpha_n$ are the data points. Then it is possible to prove the following bound for $F(z)$, $|z| < 1$

$$(C.9) \qquad |F(z)| \leq \varepsilon \sum_{j=0}^{n} \frac{|B_j(z)|}{|B_j(\alpha_j)|} + \left[1 + \varepsilon \sum_{j=0}^{n} \frac{1}{|B_j(\alpha_j)|}\right] |B(z)|$$

where

$$(C.10) \qquad P_j(z) = \frac{z - \alpha_j}{1 - \bar{\alpha}_j z}\ ; \quad B(z) = \prod_{j=0}^{n} P_j(z); \qquad B_j(z) = \frac{B(z)}{P_j(z)}$$

and $B(z)$ is the so-called Blaschke-product. For a proof of the inequality (C.9) and of the consequent stability of the analytic continuation see ref. (31).

Now, in order to show how poor is the restored stability, one can construct pathological examples of functions which are quite small at the data set and yet surprisingly large at the desired points, such as $\ell = i$ or $2i$ etc., which are sufficiently near to the boundary line $\mathrm{Re}\,\ell = -1/2$. One can, for instance, take the function $B_j(z)$, then assume $B_j(\alpha_j) \simeq \varepsilon$ and thereafter evaluate $B_j(z)$ at the desired points. These numerical evaluations have been done in a previous work (see ref. (31)); here we limit ourselves to recall the main conclusions.

One must distinguish between two different cases.

i) Continuation to the left and upwards the imaginary axis. -

In this case the data points are $\ell = 1, 2, \ldots, n$, and one wants to continue to points like $\ell = 0, i, 2i, \ldots$ In this case the continuity is extremely poor.

ii) Prediction of an intermediate value or of a data value to the right.

In this case one tries to predict the value at the intermediate point in terms of data value given at $l = 0, 1, 2, \ldots k-1, k+1, \ldots, n$; or one wants to predict the value at $l = n$ or $n+1$ in terms of data given at $\ell = 0, 1, 2, \ldots, n$.

In these cases the stability is quite satisfactory.

These conclusions do not change even if we suppose that the functions of physical interest decrease exponentially as $|\ell| \to \infty$; or if we add further restrictions assuming that a few of the derivatives of $F(z)$ also satisfy prescribed bounds. In fact, even with these additional assumptions, one can easily construct pathological examples involving Blaschke products, which prove the conclusions of (i).

Therefore, we can say that small changes in the experimental data can produce large effects in the corresponding "spectral-data" and therefore in the corresponding potential. On the contrary, from the conclusions of (ii) we can deduce that it is possible to interpolate the partial-waves at different angular momenta (and at fixed energy), or to predict the values of some of them in terms of the others, provided that one does not attempt to use numerical methods which involve continuation up to the imaginary axis. This second point could be relevant in such phenomenological analyses where a large number of phase-shifts is involved.

APPENDIX D.-

In this appendix we outline a method for restoring the stability of the numerical solutions of the Abel equations (2), which can be re-written as follows:

(D. 1) $$\int_x^{+\infty} f(y)\, k(x,y)\, dy = g(x)$$

where $k(x,y) = (y-x)^{-1/2}$. The function $g(x)$ is not exactly known; therefore we shall denote with $h(x)$ the function actually measured and with $\delta(x) = g(x) - h(x)$ the error function, where $\delta(x)$ is an arbitrary function except for some condition on its size.

Next we develop a matrix approximation for the eq. (D. 1). At this purpose, it is necessary to make explicit our assumption that $k(x,y)\, f(y) \xrightarrow[y\to\infty]{} 0$, which we prefer to make somewhat stronger supposing that $f(y) \xrightarrow[y\to\infty]{} 0$. Therefore we can approximate eq. (D. 1) considering a finite integration interval instead of an infinite one; moreover we can neglect the error due to this approximation, since it can be done arbitrarily small. Now if $[a,b]$ is the interval where $g(x)$ is approximately known, then we subdivide this interval into n parts by the uniformly spaced points $a = x_0 < x_1 < x_2 < \dots < x_n = b$; then we shall denote with $g_i = g(x_i)$, $\delta_i = \delta(x_i)$ and $h_i = h(x_i)$ the values of $g(x)$, $\delta(x)$ and $h(x)$ at the points x_i. Furthermore we shall denote with $f_i = f(y_i)$ the values of $f(y)$ at the points y_i, where the set of uniformly spaced points $\{y_i\}$, $(y_0 < y_1 < {}_2 < \dots < y_n)$ subdivide the largest integration interval (i. e. for $y = x = x_0$), into n parts. From the computational point of view, it is convenient to introduce a small positive quantity η, such that $y_i = x_i + \eta$ $(i = 0, 1, 2, \dots, n)$, because, on the line $y = x$, $K(x,y) \to \infty$. For the sake of simplicity, we have supposed that it is possible to subdivide uniformly both the largest integration interval and the data interval [a, b] into n parts. Finally, let us denote with f, g, δ and h the column matrices whose elements are given by f_i, g_i, δ_i and h_i respectively.

Now we can replace the integral equation (D. 1) by the following matrix equation

$$Af = g \tag{D.2}$$

where A is a trinagular matrix whose elements are given by $(A)_{ij} = W_i(y_i - x_j)^{-1/2}$ and W_i are weight factors, whose values depend on the quadrature formula used.

Next we introduce the Hilbert space ℓ_2^{n+1}, equipped with the usual inner product of two vectors. Therefore, from (D. 2), we obtain:

$$\| Af - h \|^2_{\ell_2^{n+1}} = \| \delta \|^2_{\ell_2^{n+1}} = \sum_{i=0}^{n} \delta_i^2 \leq \varepsilon \tag{D.3}$$

where ε is the error bound.

At this point we must introduce suitable smoothness conditions, which are necessary in order to guarantee the stability. At first sight the stabilization constraint could be given by a prescribed uniform bound on the functions f(y) (for $y_0 \leq y \leq y_n$); this condition is necessary but it is not sufficient. In fact we must look for the solution in a compact family of functions f(y), as it is suggested by the compactness argument introduced in the Appendix B. Now the condition of uniform boundedness is necessary but not sufficient for the compactness of f(y); in fact, as the Ascoli-Arzela' theorem states, the equicontinuity of the functions is also required. Therefore we need a prescribed uniform bound on the first derivative at least. However, from the computational point of view, it turns out that it is convenient to reinforce the smoothness condition, requiring a prescribed uniform norm on the second derivative (see also refs. (32), (33), which can be expressed, in the discretized form, as follows[33]:

$$\sum_{i=0}^{n} (f_{i+1} - 2f_i + f_{i-1})^2 \leq E^2 \tag{D.4}$$

with the convention that $f_{-1} = f_{n+1} = 0$.

Now, instead of dealing with the two constraints (D. 3) and (D. 4) separately, we combine them as follows:

(D.5) $$\sum_i \delta_i^2 + \left(\frac{\varepsilon}{E}\right)^2 \sum_i (f_{i+1} - 2f_i + f_{i-1})^2 \leq 2\,\varepsilon^2$$

In fact any vector which satisfies (D.3) and (D.4) separately, satisfies also (D.5); conversely any vector which satisfies (D.5), satisfies also (D.3) and (D.4) except for factors of at most 2.

Next we shall find that vector which minimizes:

(D.6) $$\sum_i \delta_i^2 + \lambda^2 \sum_i (f_{i+1} - 2\,f_i + f_{i-1})^2$$

where $\lambda = \varepsilon/E$ is the Lagrange multiplier. This minimization yields the following matrix equation:

(D.7) $$A^{*} A f - A^{*} h + \lambda^2 H f = 0$$

where A^{*} in the transpose of A and H is the matrix which is obtained differentiating the second term of (D.6). For the explicit evaluation of H see ref. (33), where it is also given the expression of H if in the constraint (D.4) one uses the third differences instead of the second. Finally, from (D.7) one obtains:

(D.8) $$f = \frac{A^{*} h}{A^{*} A + \lambda^2 H}$$

Up to now we have just outlined a method which must be tested with explicit numerical calculations. At this purpose it should have some interest to know the geometrical properties of the set ξ of all the pairs of numbers (ε, E) which are permissible (we call a pair of numbers (ε, E) permissible if there exists a vector f satisfying (D.3) and (D.4)). In fact, in such a case, one could probably elaborate a strategy which requires a knowledge of only one of these numbers (see, for instance ref. (29)). We think, however, that this question requires a deeper analysis and especially numerical investigations and we hope to return on this point elsewhere.

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