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G. Cattapan and V. Vanzani : ON A GENERALIZED DISTORTED-WAVE APPROXIMATION FOR NUCLEAR REARRANGEMENT REACTIONS.

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SUMMARY. -

Starting from an exact three-body theory for the nuclear rearrangement reaction $a+(b+c) \rightarrow b+(a+c)$, we study a generalized distorted wave approximation in the momentum-space representation. Using suitable kinematic transformations we recast this representation in a form coinciding with the one obtained by means of the Feynman diagram summation method. The generalized potential responsible for the transition will be written in a compact form involving the resolvent operator for the a-b subsystem.

1. - INTRODUCTION. -

In recent years an increasing interest has been devoted to the three-body rearrangement scattering problem in the distorted-wave formalism. A generalized distorted-wave approximation (GDWA) has been formally derived from the Faddeev-Lovelace integral equations written in terms of symmetric transition operators(1, 2). This GDWA involves the full interactions in both initial and final channels, not only in one channel as in the conventional DWBA. The occuring generalized transition potential contains the contributions of the basic rearrangement mechanisms which can be described by a polar and a triangular diagram.

It has been also recognized that the GDWA is equivalent to the Feynman-diagram summation method (FDSM) based on the above polar and triangular graphs(2, 3). However an explicit GDWA momentum-space representation to be used as starting point for calculational purposes has never been investigated. The aim of the present paper is to study this explicit representation.

In performing this program one is faced by nontrivial difficulties arising from the complicate three-body kinematics. In order to deal with manageable formulae we shall use physical momenta and not the normalized ones[(4)]. The compact procedure we follow allows to clarify the physical meaning of the GDWA and to perform a direct comparison with the FDSM.

In Sect. 2 we shall give a general outline of the GDWA in a three-body context. Section 3 deals with some kinematic transformations suitable for working in the momentum-space. In Sects. 4 and 5 we shall be concerned with the momentum-space representation of the GDWA and we shall give a detailed comparison between the GDWA and the FDSM.

2. - GENERAL FORMALISM. -

In a three-body context a nuclear rearrangement reaction A(a, b)B can be represented schematically as

$$a + (b + c) \rightarrow b + (a + c), \tag{2.1}$$

where $A = b+c$, $B = a+c$ and a, b, c, are treated as inert entities interacting only via two-body interactions.

Let us introduce the transition operators $U_{\beta\alpha}(z)$ from the initial channel α to the final one β, in the symmetric form[(5)]

$$U_{\beta\alpha}(z) = \bar{\delta}_{\beta\alpha} G_\alpha^{-1}(z) + V_\beta + V_\beta G(z) V_\alpha , \tag{2.2a}$$

$$U_{\beta\alpha}(z) = \bar{\delta}_{\beta\alpha} G_\beta^{-1}(z) + V_\alpha + V_\beta G(z) V_\alpha , \tag{2.2b}$$

where $\bar{\delta}_{\beta\alpha} = 1 - \delta_{\beta\alpha}$, V_δ $(\delta = \alpha, \beta)$ are the channel interactions, G_δ the resolvents of the channel Hamiltonians and G(z) the resolvent of the total Hamiltonian. These operators are defined as in ref. (6).

The transition operators (2.2) satisfy the Faddeev-Lovelace-like integral equations

$$U_{\beta\alpha}(z) = \bar{\delta}_{\beta\alpha} G_0^{-1}(z) + \sum_{\gamma \neq \beta} t_\gamma(z) G_0(z) U_{\gamma\alpha}(z), \tag{2.3a}$$

$$U_{\beta\alpha}(z) = \bar{\delta}_{\beta\alpha} G_0^{-1}(z) + \sum_{\gamma \neq \alpha} U_{\beta\gamma}(z) G_0(z) t_\gamma(z), \tag{2.3b}$$

where $G_0(z)$ is the resolvent of the free Hamiltonian, $t_\alpha(z)$ is the two-body scattering operator for the β-γ subsystem acting in the three-body space.

By iterating eqs. (2.3) once and omitting the energy parameter z for the sake of simplicity, one gets

(2.4a) $$U_{\beta\alpha} = \bar{\delta}_{\beta\alpha} G_0^{-1} + \sum_{\gamma \neq \alpha,\beta} t_\gamma + \sum_{\substack{\gamma \neq \beta \\ \delta \neq \gamma}} t_\gamma G_0 t_\delta G_0 U_{\delta\alpha} ,$$

(2.4b) $$U_{\beta\alpha} = \bar{\delta}_{\beta\alpha} G_0^{-1} + \sum_{\gamma \neq \alpha,\beta} t_\gamma + \sum_{\substack{\gamma \neq \alpha \\ \delta \neq \gamma}} U_{\beta\delta} G_0 t_\delta G_0 t_\gamma .$$

The inhomogeneous term of integral equations (2.4a) or (2.4b), characterized by a compact kernel, is expressed only in terms of G_0^{-1} and of the two-body scattering operators. As far as the process (2.1) is concerned, the two simplest terms which appear in the equation for U_{ba} give, in the channel state representation, the amplitudes for the polar and triangular diagrams, corresponding to basic rearrangement mechanisms[2]. By using the relation

(2.5) $$G_c = G_0 + G_0 t_c G_0 ,$$

the above terms can be rewritten in the compact form $G_0^{-1} G_c G_0^{-1}$.

The integral equations (2.3) with compact squared kernel provide a rigorous mathematical basis for a quantitative formulation of the intuitive physical picture associated with the distorted-wave representation of the nuclear rearrangement amplitudes[1,2].

Following the procedure of ref. (2), taking account of eq. (2.5) and neglecting the coupling terms between elastic and rearrangement channels in the equations expressing U_{aa} and U_{bb} in terms of U_{ba}, one obtains

(2.6a) $$U_{ba} = G_0^{-1} G_c (G_0^{-1} + t_a G_0 U_{aa}) + t_c G_0 t_b G_0 U_{ba} ,$$

(2.6b) $$U_{aa} = t_c + t_c G_0 t_a G_0 U_{aa} ;$$

(2.7a) $$U_{ba} = (G_0^{-1} + U_{bb} G_0 t_b) G_c G_0^{-1} + U_{ba} G_0 t_a G_0 t_c ,$$

(2.7b) $$U_{bb} = t_c + U_{bb} G_0 t_b G_0 t_c .$$

By analogy with the arguments of ref. (2), one can derive from eqs. (2.6a) and (2.7b) or (2.6b) and (2.7a) the following compact expression for U_{ba} on-the-energy-shell

(2.8) $$U_{ba} = (1 + U_{bb} G_b) v_b G_c v_a (1 + G_a U_{aa}) .$$

In eq. (2.8) v_α denotes the interaction between β and γ $(\alpha \neq \beta \neq \gamma)$.

If the channel resolvent operators G_α (α=a, b) are approximated by their dominant separable part $|\psi_\alpha\rangle g_\alpha \langle \psi_\alpha|$, which corresponds to the channel bound state $|\psi_\alpha\rangle$ and appears in their spectral representation, one obtains the generalized distorted-wave approximation

$$T^{GDWA}_{ba} = \langle \Phi_b | U^{GDWA}_{ba} | \Phi_a \rangle =$$

(2. 9)

$$= \langle \vec{p}^{\,\prime}_b m'_b | (1+u_{bb} g_b) \langle \psi_b | v_b G_c v_a | \psi_a \rangle (1+g_a u_{aa}) | \vec{p}_a m_a \rangle$$

In eq. (2. 9) $|\Phi_\alpha\rangle = |\vec{p}_\alpha m_\alpha\rangle |\psi_\alpha\rangle$ are the channel states, $\vec{p}_a$ ($\vec{p}^{\,\prime}_b$) the initial (final) channel linear momentum, m_α the z-component of the spin s_α of the particle α. All operators are evaluated for $z = E_a + i\varepsilon = E_b + i\varepsilon$. The total channel energy E_α is given by ($\hbar = 1$)

$$E_\alpha = \frac{p^2_\alpha}{2\nu_\alpha} - \varepsilon_\alpha \ , \qquad (2.10)$$

where ε_α is the binding energy of the bound state in the channel α and ν_α is the reduced mass for the system consisting of α and $\beta+\gamma$ ($\alpha \neq \beta \neq \gamma$).

The quantities

$$u_{\alpha\alpha} = \langle \psi_\alpha | U_{\alpha\alpha} | \psi_\alpha \rangle \qquad (2.11)$$

are the optical scattering operators acting on the plane-wave states $|\vec{p}_\alpha m_\alpha\rangle$. They are constructed as expectation values of the actual scattering operators $U_{\alpha\alpha}$ in the subspace of the channel bound states. Then, the wave-operators $1+g_\alpha u_{\alpha\alpha}$ operating on $|\vec{p}_\alpha m_\alpha\rangle$ give the effective two-body distorted-wave states[1]. In eq. (2. 9) the operator $\langle \psi_b | v_b G_c v_a | \psi_a \rangle$, acting on channel distorted-wave states, has the meaning of a generalized transition potential. Its compact form explicitly exhibits the role of the two-body resolvent operator $G_c^{(*)}$.

The generalized distorted-wave approximation (2. 9) can be formulated in the language of the nonrelativistic Feynman diagrams. If the initial and final channel interactions are described by graphs involving the half-off-energy-shell optical scattering amplitudes

$$\langle \vec{p}^{\,\prime}_\alpha m_\alpha | u_{\alpha\alpha} | \vec{p}_\alpha m_\alpha \rangle,$$

one obtains the FDSM based on the polar and triangular diagram rearrangement mechanisms. We shall give a detailed and complete account of the GDWA-FDSM equivalence, by using complete sets of intermediate states and by resorting to the Feynman-diagram rules. The equivalence of the two methods has been already noticed in ref. (8) for a particular triangular graph amplitude.

The transition amplitudes T_{ba} introduced by us in the distorted-wave (Sect. 4) and Feynman diagram formalisms (Sect. 5) are related to the differential cross sections by the formula

$$(2.12) \qquad \frac{d\sigma_{ba}}{d\Omega} = (2\pi)^4 \, \nu_a \, \nu_b \, \frac{p'_b}{p_a} \, \frac{1}{(2s_a+1)(2s_A+1)} \sum_{\substack{m_a m_A \\ m'_b m_B}} \left| T_{ba} \right|^2$$

3. - KINEMATIC CONSIDERATIONS. -

As is well known, the three-body problem involves great complications arising from the number of kinematic variables. Then, it seems useful to give some kinematic transformations suitable for representing the GDWA in the momentum-space. Throughout this paper we shall use physical momenta, not the normalized ones as in ref. (4).

Let us denote by $\vec{p}_\alpha$ the momentum of particle α in the total center of mass system and by $\vec{k}_\alpha$ the relative momentum between the particles β and γ. By definition

$$(3.1) \qquad \vec{k}_\alpha = \frac{M_\gamma \vec{p}_\beta - M_\beta \vec{p}_\gamma}{M_\beta + M_\gamma},$$

with M_α the mass of particle α and (α, β, γ) a cyclic permutation of (a, b, c). As is well known, in the total center of mass system only two of the six variables $\vec{k}_a$, $\vec{k}_b$, $\vec{k}_c$, $\vec{p}_a$, $\vec{p}_b$, $\vec{p}_c$ are linearly independent. Then, one can choose a pair $(\vec{k}_\alpha, \vec{p}_\alpha)$ as independent momenta for characterizing a three-particle state.

In order to pass from any pair $(\vec{k}_\alpha, \vec{p}_\alpha)$ to any other pair $(\vec{k}_\delta, \vec{p}_\delta)$ $(\delta = \beta, \gamma)$, one may use eq. (3.1) and the relation

$$(3.2) \qquad \vec{p}_a + \vec{p}_b + \vec{p}_c = 0 .$$

One gets

$$(3.3a) \qquad \left\{ \begin{aligned} \vec{k}_\beta &= - \frac{M_\alpha}{M_\alpha + M_\gamma} \vec{k}_\alpha - \frac{M_\gamma M}{(M_\alpha + M_\gamma)(M_\beta + M_\gamma)} \vec{p}_\alpha \\ \vec{p}_\beta &= \vec{k}_\alpha - \frac{M_\beta}{M_\beta + M_\gamma} \vec{p}_\alpha \end{aligned} \right.$$

4. - THE MOMENTUM-SPACE REPRESENTATION OF THE GDWA. -

By inserting in eq. (2. 9) intermediate momentum integrations and magnetic sums over the complete sets of three-particle states

$$|\vec{k}''_a \vec{p}''_a m''_a m''_b m''_c \rangle \quad \text{and} \quad |\vec{k}'''_b \vec{p}'''_b m'''_a m'''_b m'''_c \rangle$$

one gets

$$T^{GDWA}_{ba} = \sum_{m'''_b m''_a} \int \chi^{(+)}_{\vec{p}'_b m'_b} (\vec{p}'''_b m'''_b)\, T_0(\vec{p}'''_b \vec{p}''_a m'''_b m''_a) \; \times \tag{4. 1}$$
$$\times \; \chi^{(+)}_{\vec{p}_a m_a} (\vec{p}''_a m''_a)\, d\vec{p}'''_b\, d\vec{p}''_a \; ,$$

with

$$T_0(\vec{p}'_b \vec{p}_a m'_b m_a) = \sum_{\substack{m'_a m'_c \\ m_b m_c}} \int F(\vec{k}'_b \vec{k}_a m'_a m'_c m_b m_c)\, d\vec{k}'_b\, d\vec{k}_a \; , \tag{4. 2}$$

$$F(\vec{k}'_b \vec{k}_a m'_a m'_c m_b m_c) = f^*_b(\vec{k}'_b m'_a m'_c) \; \times \tag{4. 3}$$
$$\times \; \langle \vec{k}'_b \vec{p}'_b m'_a m'_b m'_c | G_c | \vec{k}_a \vec{p}_a m_a m_b m_c \rangle\, f_a(\vec{k}_a m_b m_c) \, ,$$

$$\chi^{(+)}_{\vec{p}_\alpha m_\alpha} (\vec{p}'_\alpha m'_\alpha) = \langle \vec{p}'_\alpha m'_\alpha | (1 + g_\alpha u_{\alpha\alpha}) | \vec{p}_\alpha m_\alpha \rangle \, , \tag{4. 4}$$

$$f_\alpha (\vec{k}_\alpha m_\beta m_\gamma) = \langle \vec{k}_\alpha m_\beta m_\gamma | v_\alpha | \psi_\alpha \rangle \qquad (\alpha \neq \beta \neq \gamma) \, . \tag{4. 5}$$

From the definition of g_α it follows

$$g_\alpha (E_\alpha + i\varepsilon) | \vec{p}'_\alpha m'_\alpha \rangle = \left(\frac{p^2_\alpha}{2\nu_\alpha} - \frac{p'^2_\alpha}{2\nu_\alpha} + i\varepsilon \right)^{-1} | \vec{p}'_\alpha m'_\alpha \rangle \, . \tag{4. 6}$$

Then, the channel distorted-wave states can be rewritten, in the momentum-space representation, in the explicit form

$$(4.7)\qquad \chi^{(+)}_{\vec{p}_\alpha m_\alpha}(\vec{p}'_\alpha m'_\alpha) = \delta(\vec{p}'_\alpha - \vec{p}_\alpha)\delta_{m'_\alpha m_\alpha} + \frac{u_{\alpha\alpha}(\vec{p}'_\alpha \vec{p}_\alpha m'_\alpha m_\alpha)}{\frac{p_\alpha^2}{2\nu_\alpha} - \frac{p'^2_\alpha}{2\nu_\alpha} + i\varepsilon} .$$

Equation (4. 5) gives the usual two-body form factors corresponding to the channel bound states $|\psi_\alpha\rangle$. By means of standard angular momentum expansions[6, 9, 10], one may give an explicit momentum-space representation of f_α, involving spectroscopic factors and single-particle or single-cluster reduced widths.

In order to simplify the expression (4. 2) for T_0, let us consider the relations (3. 8) and change the integration variables $\vec{k}'_b$ and $\vec{k}_a$ in $\vec{p}'_c$ and $\vec{p}_c$, respectively (see eq. (3. 4a) with $(\alpha, \beta, \gamma) = (b, c, a)$ and eq. (3. 4b) with $(\alpha, \beta, \gamma) = (a, b, c)$). Taking account of the property (3. 9) for the matrix elements of G_c and performing the integration over $\vec{p}'_c$, one gets

$$(4.8)\qquad \begin{aligned} T_0(\vec{p}'_b \vec{p}_a m'_b m_a) &= \sum_{m'_a m_b m_c} \int f^*_b(\vec{q}'_b m'_a m_c) \times \\ &\times \langle \vec{q}'_c m'_a m'_b | \hat{G}_c(S_c + i\varepsilon) | \vec{q}_c m_a m_b \rangle f_a(\vec{q}_a m_b m_c)\, d\vec{p}_c , \end{aligned}$$

where

$$(4.9)\qquad S_c = E_a - \frac{p_c^2}{2\nu_c}$$

and $\vec{q}_a$, $\vec{q}_c$, $\vec{q}'_b$, $\vec{q}'_c$ stand for the momenta $\vec{k}_a$, $\vec{k}_c$, $\vec{k}'_b$, $\vec{k}'_c$ respectively, evaluated for $\vec{p}'_c = \vec{p}_c$. Therefore, they are connected to the integration variables by the relations

$$(4.10a)\qquad \vec{q}_a = -\frac{M_c}{M_b + M_c}\vec{p}_a - \vec{p}_c , \qquad \vec{q}_c = \frac{M_a}{M_a + M_b}\vec{p}_c + \vec{p}_a ;$$

$$(4.10b)\qquad \vec{q}'_b = \frac{M_c}{M_a + M_c}\vec{p}'_b + \vec{p}_c , \qquad \vec{q}'_c = -\frac{M_b}{M_a + M_b}\vec{p}_c - \vec{p}'_b .$$

By expressing $\hat{G}_c$ in terms of $\hat{t}_c$ and $\hat{G}_0$ (see eq. (2. 5) in the hat notation), one gets

$$\langle \vec{q}\,'_c m'_a m'_b | \hat{G}_c(S_c + i\varepsilon) | \vec{q}_c m_a m_b \rangle =$$

(4.11)
$$= \frac{\psi_c^{(+)}(\vec{q}\,'_c \vec{q}_c m'_a m'_b m_a m_b ; S_c + i\varepsilon)}{S_c - \frac{q'^2_c}{2\mu_c} + i\varepsilon} ,$$

where

$$\psi_c^{(+)}(\vec{q}\,'_c \vec{q}_c m'_a m'_b m_a m_b ; S_c + i\varepsilon) = \delta(\vec{q}\,'_c - \vec{q}_c)\, \delta_{m'_a m_a}\, \delta_{m'_b m_b} +$$

(4.12)
$$+ \frac{\langle \vec{q}\,'_c m'_a m'_b | \hat{t}_c(S_c + i\varepsilon) | \vec{q}_c m_a m_b \rangle}{S_c - \frac{q^2_c}{2\mu_c} + i\varepsilon}$$

is the momentum-space representation of the two-body scattering state for the a-b subsystem. Recalling the above kinematic transformations and the procedure followed in deriving eqs. (4.8), one may write

(4.13)
$$| \vec{q}_c \vec{p}_c \rangle = | \vec{q}_a \vec{p}_a \rangle , \qquad | \vec{q}\,'_c \vec{p}_c \rangle = | \vec{q}\,'_b \vec{p}\,'_b \rangle .$$

Then from (3.7a) it follows

(4.14)
$$S_c - \frac{q^2_c}{2\mu_c} = E_a - \left(\frac{q^2_a}{2\mu_a} + \frac{p^2_a}{2\nu_a}\right) , \qquad S_c - \frac{q'^2_c}{2\mu_c} = E_b - \left(\frac{q'^2_b}{2\mu_b} + \frac{p'^2_b}{2\nu_b}\right) .$$

By inserting in (4.1) the expressions (4.8) and (4.11) with a correct number of primes for the intermediate momentum and magnetic variables, one obtains an explicit momentum-space representation of the GDWA in terms of two different types of two-body scattering waves: the initial and final channel distorted-waves $\chi_\alpha^{(+)}$ and the intermediate scattering wavefunction $\psi_c^{(+)}$.

From eqs. (4.1), (4.11), (4.12) and (4.14) it is immediately seen that, owing to the presence of the half-off-energy-shell optical scattering amplitudes ($\vec{p}\,''_a \neq \vec{p}_a$, $\vec{p}\,'''_b \neq \vec{p}\,'_b$), the GDWA amplitude has in p'^2_b (or in p^2_a) two three-particle cuts running to the right of the normal three-particle threshold $(p'^2_b)_0 = 2\nu_b \varepsilon_b$ (or $(p^2_a)_0 = 2\nu_a \varepsilon_a$).

If only the plane-wave term $\delta(\vec{p}\,'_\alpha - \vec{p}_\alpha)\,\delta_{m'_\alpha m_\alpha}$ is retained in the expression (4.7) for the distorted-waves χ_α^{+}, eq. (4.1) reduces to the generalized plane-wave approximation (GPWA)

$$T^{GPWA} = T_0(\vec{p}_b'\vec{p}_a m_b' m_a) \ , \tag{4.15}$$

where now the variables $\vec{p}_a$, $\vec{p}_b'$, m_a, m_b' are channel physical variables, not the intermediate state ones. In this particular case the three-particle cuts disappear and, in virtue of eqs. (2.10), (4.14), one gets

$$T^{GPWA} =$$

$$= - \sum_{m_b m_c m_a'} \int \frac{f_b^{*}(\vec{q}_b' m_b' m_c)\, \psi_c^{(+)}(\vec{q}_c'\vec{q}_c m_a' m_b' m_a m_b; S_c + i\varepsilon) f_a(\vec{q}_a m_b m_c)}{\varepsilon_b + \dfrac{q_b'^2}{2\mu_b}} d\vec{p}_c \tag{4.16}$$

Taking account of eq. (4.12) one may split the GPWA amplitude (4.16) in the following form

$$T^{GPWA} = T^{PWA(o)} + T^{PWA(t)} \tag{4.17}$$

with

$$T^{PWA(o)} = - \sum_{m_c} \frac{f_b^{*}(\vec{\varkappa}_b' m_a m_c)\, f_a(\vec{\varkappa}_a m_b' m_c)}{\varepsilon_b + \dfrac{\varkappa_b'^2}{2\mu_b}} \ , \tag{4.18}$$

$$T^{PWA(t)} =$$

$$= \sum_{m_b m_c m_a'} \int \frac{f_b^{*}(\vec{q}_b' m_a' m_c) \langle \vec{q}_c' m_a' m_b' | \hat{t}_c(S_c + i\ ) | \vec{q}_c m_a m_b \rangle f_a(\vec{q}_a m_b m_c)}{(\varepsilon_b + \dfrac{q_b'^2}{2\mu_b})(\varepsilon_a + \dfrac{q_a^2}{2\mu_a})} d\vec{p}_c \tag{4.19}$$

The amplitude $T^{PWA(o)}$ arises from the plane-wave term of eq. (4.12). In eq. (4.18) $\vec{\varkappa}_a$ and $\vec{\varkappa}_b'$ stand for the momenta $\vec{q}_a$ and $\vec{q}_b'$, respectively, evaluated for $\vec{q}_c = \vec{q}_c'$. They are given in terms of the channel physical momenta by the relations

$$\vec{\varkappa}_a = \frac{M_b}{M_b + M_c}\vec{p}_a + \vec{p}_b' \ , \qquad \vec{\varkappa}_b' = - \frac{M_a}{M_a + M_c}\vec{p}_b' - \vec{p}_a \ . \tag{4.20}$$

From (4.14) it follows for $\vec{q}_c = \vec{q}_c'$

$$\varepsilon_a + \frac{\varkappa_a^2}{2\mu_a} = \varepsilon_b + \frac{\varkappa_b'^2}{2\mu_b} . \tag{4.21}$$

Obviously the amplitudes $T^{PWA(o)}$ and $T^{PWA(t)}$ can be directly obtained by starting from the formal expressions $\langle \Phi_b | G_0^{-1} | \Phi_a \rangle$ and $\langle \Phi_b | t_c | \Phi_a \rangle$, respectively.

It is worthwhile noticing that in the amplitudes (4.8), (4.16), (4.18) and (4.19) there are no mass-dependent moltiplicative factors, because we have used physical momenta and not the normalized ones (see, on the contrary, formula (3.19) or ref. (4)).

5. - THE GDWA IN THE FEYNMAN DIAGRAM LANGUAGE. -

The FDSM amplitude, based on the polar and triangular diagrams, can be written schematically as[3]

$$T_{ba}^{FDSM} = T_0 + T_{0a} + T_{b0} + T_{b0a} , \tag{5.1}$$

where T_0 is the sum of the basic contributions (the polar and triangular amplitudes), T_{0a} (T_{b0}) involves the initial (final) channel interactions, besides the basic contributions, and T_{b0a} involves both the initial and the final channel interactions (see Fig. 2 of ref. (10) with M_{fi} replaced by T_{ba}^{FDSM}).

In order to derive transition amplitudes which are directly comparable with those of Sect. 4, we start from nonrelativistic Feynman-diagram rules written in a form slightly different from the usual one[11]. We shall introduce the following factors:

a) a factor $-2M_\alpha i(p_\alpha^2 - 2M_\alpha e_\alpha - i\varepsilon)^{-1} \sum_{m_\alpha}$ for each virtual particle characterized by the four-momentum $(\vec{p}_\alpha, e_\alpha)$, $(\alpha = a, b, c, A, B)$;

b) a vertex amplitude T_v for each vertex v;

c) an integration $\int d\vec{p}_\zeta \, de_\zeta$ for each independent four-momentum $(\vec{p}_\zeta, e_\zeta)$;

d) a general moltiplicative factor $(-1)^n i^{n+1} (2\pi)^{-1}$ where n is the number of vertices and l the number of independent four-momenta.

By using these rules one gets the following expression for T_{b0a} [(xx)]

$$T_{b0a} = -\frac{4}{\pi^2} M_a M_b M_A M_B \times$$
$$\times \sum_{m_b''' m_a''} \int A(\vec{p}_b''' e_b''' \vec{p}_a'' e_a'' m_b''' m_a'') \, d\vec{p}_b''' de_b''' d\vec{p}_a'' de_a'' , \tag{5.2}$$

with

$$A(\vec{p}_b'''e_b'''\vec{p}_a''e_a''m_b'''m_a'') =$$

(5.3)
$$= \frac{u_{bb}(\vec{p}_b'\vec{p}_b'''m_b'm_b''')T_0(\vec{p}_b'''e_b'''\vec{p}_a''e_a''m_b'''m_a'')u_{aa}(\vec{p}_a''\vec{p}_a m_a''m_a)}{(p_b'''^2-2M_b e_b'''-i\varepsilon)(p_b'''^2-2M_B e_B-i\varepsilon)(p_a''^2-2M_A e_A-i\varepsilon)(p_a''^2-2M_a e_a-i\varepsilon)}$$

(5.4)
$$e_A = \frac{p_a^2}{2\nu_a} - e_a'' , \qquad e_B = \frac{p_b'^2}{2\nu_b} - e_b''' .$$

According to the notation of Sect. 4, we have denoted by u_{aa}, u_{bb} and T_0 the three four-ray vertex amplitudes appearing in the two-loop graph described by T_{b0a}.

The integration over e_a'' and e_b''' in (5.2) can be performed in the complex e_a''- and e_b'''-plane. By writing explicitly the amplitude T_0 in terms of polar and triangular amplitudes, one can see that T_0 has no singularities in the lower half-plane of variables e_a'' and e_b'''. Then, by means of the residue method and of some straightforward manipulations one obtains

(5.5)
$$T_{b0a} = \sum_{m_b''' m_a''} \times$$
$$\times \int \frac{u_{bb}(\vec{p}_b'\vec{p}_b'''m_b'm_b''')T_0(\vec{p}_b'''\vec{p}_a''m_b'''m_a'')u_{aa}(\vec{p}_a''\vec{p}_a m_a''m_a)}{\left(\frac{p_b'^2}{2\nu_b} - \frac{p_b'''^2}{2\nu_b} + i\varepsilon\right)\left(\frac{p_a^2}{2\nu_a} - \frac{p_a''^2}{2\nu_a} + i\varepsilon\right)} d\vec{p}_b'''d\vec{p}_a''$$

where $T_0(\vec{p}_b'''\vec{p}_a''m_b'''m_a'')$ stands for $T_0(\vec{p}_b'''e_b'''\vec{p}_a''e_a''m_b'''m_a'')$ evaluated for $e_a'' = p_a''^2/2M_a$ and $e_b''' = p_b'''^2/2M_b$.

Let us now consider the amplitude T_0 appearing in (5.5). By definition it can be written in the form

(5.6)
$$T_0(\vec{p}_b'\vec{p}_a m_b'm_a) = T^{(o)}(\vec{p}_b'\vec{p}_a m_b'm_a) + T^{(t)}(\vec{p}_b'\vec{p}_a m_b'm_a)$$

with

$$T^{(o)}(\vec{p}_b'\vec{p}_a m_b'm_a) =$$

(5.7)
$$= -2M_c \sum_{m_c} \frac{f_b^*(\vec{\varkappa}_b' m_a m_c) f_a(\vec{\varkappa}_a m_b' m_c)}{(\vec{p}_a+\vec{p}_b')^2 - 2M_c\left(E_a - \frac{p_a^2}{2M_a} - \frac{p_b'^2}{2M_b}\right) - i\varepsilon} ,$$

$$T^{(t)}(\vec{p}_b^{\,\prime}\vec{p}_a m_b^{\prime} m_a) =$$

(5.8)

$$= -\frac{4i}{\pi} M_a M_b M_c \sum_{m_b m_c m_a^{\prime}} \int B(\vec{p}_b^{\,\prime}\vec{p}_c e_c \vec{p}_a m_b^{\prime} m_b m_c m_a^{\prime} m_a)\, d\vec{p}_c de_c \,,$$

$$B(\vec{p}_b^{\,\prime}\vec{p}_c e_c \vec{p}_a m_b^{\prime} m_b m_c m_a^{\prime} m_a) =$$

(5.9)

$$= \frac{f_b^{*}(\vec{q}_b^{\,\prime} m_a^{\prime} m_c) t_c (\vec{q}_c^{\,\prime}\vec{q}_c m_c^{\prime} m_b^{\prime} m_a m_b ; E+i\varepsilon) f_a(\vec{q}_a m_b m_c)}{\left[(\vec{p}_b^{\,\prime}+\vec{p}_c)^2 - 2M_a e_a - i\varepsilon\right](p_c^2 - 2M_c e_c - i\varepsilon)\left[(\vec{p}_a+\vec{p}_c)^2 - 2M_b e_b - i\varepsilon\right]}$$

(5.10)

$$E = E_a - e_c - \frac{p_c^2}{2(M_a+M_b)} \,,$$

$$e_a = E_b - \frac{p_b^{\prime 2}}{2M_b} - e_c \,, \qquad e_b = E_a - \frac{p_a^2}{2M_a} - e_c \,.$$

In eqs. (5.9) and (5.7) t_c stands for the four-ray vertex amplitude and f_a, f_b^{*} are the two three-ray vertex functions appearing in the basic diagrams. The relative momenta at the vertices $\vec{\varkappa}_a$, $\vec{\varkappa}_b^{\,\prime}$ and $\vec{q}_a$, $\vec{q}_c$, $\vec{q}_b^{\,\prime}$, $\vec{q}_c^{\,\prime}$ are given in terms of the intermediate momenta $\vec{p}_a$, $\vec{p}_c$, $\vec{p}_b^{\,\prime}$ by relations formally identical to (4.20) and (4.10), respectively.

By passing from the form (3.7b) to the form (3.7a) for the total kinetic energy expressed in terms of $\vec{\varkappa}$- and $\vec{p}$-type variables, one gets

$$(5.11) \qquad T^{(o)}(\vec{p}_b^{\,\prime}\vec{p}_a m_b^{\prime} m_a) = \sum_{m_c} \frac{f_b^{*}(\vec{\varkappa}_b^{\,\prime} m_a m_c) f_a(\vec{\varkappa}_a m_b^{\prime} m_c)}{E_a - \left(\frac{\varkappa_a^2}{2\mu_a} + \frac{p_a^2}{2\nu_a}\right) + i\varepsilon} \,.$$

Since for $\vec{q}_c = \vec{q}_c^{\,\prime}$ the relations (4.14) coincide, the propagator in (5.11) can also be written in the form $\left[E_b - (\varkappa_b^{\prime 2}/2\mu_b) - p_b^{\prime 2}/2\nu_b + i\varepsilon\right]^{-1}$.

After suitable kinematic transformations the denominator D in eq. (5.9) takes the form

$$D(p'_b p_c p_a E) = \frac{2M_c(M_a + M_b)}{\mu_c} \times \tag{5.12}$$
$$\times \; (q'^2_c - 2\mu_c E - i\varepsilon)(E - S_c - i\varepsilon)(q^2_c - 2\mu_c E - i\varepsilon)$$

with S_c defined by (4.9). By changing in eq. (5.8) the energy integration variable from e_c to E, we may perform the integration over E in the upper half-plane by means of the method used in ref. (10). Notice that the bound state poles of the amplitude t_c lie on the lower half-plane. Then, taking account of the kinematic relations (4.14), we obtain

$$T^{(t)}(\vec{p}'_b \vec{p}_a m'_b m_a) = \sum_{m_b m_c m'_a} \times \tag{5.13}$$
$$\times \int \frac{f^{\star}_b(\vec{q}'_b m'_a m_c)\, t_c(\vec{q}'_c \vec{q}_c m'_a m'_b m_a m_b; S_c + i\varepsilon) f_a(\vec{q}_a m_b m_c)}{\left[E_b - \left(\frac{q'^2_b}{2\mu_b} + \frac{p'^2_b}{2\nu_b}\right) + i\varepsilon\right]\left[E_a - \left(\frac{q^2_a}{2\mu_a} + \frac{p^2_a}{2\nu_a}\right) + i\varepsilon\right]} d\vec{p}_c$$

By summing up the amplitudes (5.11) and (5.13) and taking into account that $\vec{q}_a$ and $\vec{q}'_b$ reduce to $\vec{\varkappa}_a$ and $\vec{\varkappa}'_b$, respectively, for $\vec{q}_c = \vec{q}'_c$, one gets eq. (4.8) with (4.11), (4.12) and (4.14).

If the variables $\vec{p}_a$, $\vec{p}'_b$, m_a, m'_b are interpreted as channels physical variables, the propagators appearing in (5.11) and (5.13) simplify. In this case eqs. (5.11) and (5.13) reduce to (4.18) and (4.19) respectively; they give the polar and triangular amplitudes without channel interactions. By summing them one gets the first term in (5.1).

The terms T_{0a} and T_{b0} can be evaluated in a similar way. Performing the sum (5.1) and taking account of (4.7), one obtains just the T^{GDWA}_{ba} amplitude considered in Sect. 4. Therefore

$$T^{FDSM}_{ba} = T^{GDWA}_{ba} . \tag{5.14}$$

In conclusion, the results of Sects. 4 and 5 explicitly exhibit the equivalence between the FDSM based on the polar and triangular diagram mechanisms and the GDWA momentum-space representation.

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(x) - Notice that the use of two-body resolvents leads to formulae more compact with respect to those obtained in ref. (2). The slightly different method of ref. (1) leads to a similar generalized transition potential. The method of ref. (7), developed in the context of the static limit model, appears to be rather different, because it is based on a different off-shell continuation of the transition operators.

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(xx) - As usual, the channel interactions are assumed to be independent of the spins of A and B[9].