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T. A. Minelli and F. Zardi: THE COUPLED CHANNEL PROBLEM IN A DYNAMICAL APPROACH TO THE RESONANCE THEORY. -

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SUMMARY. -

In this note we extend to the coupled channel problem a formalism we have developed in a previous paper for the potential scattering problem. We start from the Lippmann-Schwinger scattering theory and exploit directly the consequences of the short range of the nuclear interactions. As far as the parametrisation problem is concerned the results we obtain are equivalent to the Humblet and Rosenfeld theory; however some dynamical features which characterize the Lipmann-Schwinger theory are also included in our formalism.

INTRODUCTION. -

The aim of this paper and forthcoming ones is to propose a general formulation of the nuclear resonances problem, based on the scattering integral equation. The final goal is to obtain a general dynamic resonance theory, in which any model can be introduced in the final stage of the calculations, contrary to existing dynamical approaches, in which the model is assumed as the starting point.

Starting from Siegert's definition of the resonant state[1], a general theory of nuclear resonances has been developed by Humblet and Rosenfeld (H-R)[2, 3, 4]. The basic device they used is the Mittag-Leffler expansion of the scattering matrix, in the form obtained by the wave func

tion matching method. The H-R theory has not been designed to be a dynamic theory; there the emphasis is given to the best way of parametrising resonant cross section, while no practical prescription is to be sought for the theoretical calculation of the resonance parameters. We propose here an alternative formulation which is based on the Mittag-Leffler expansion of the scattering matrix in the integral representation given by the Lippmann-Schwinger theory[5].

The present approach, following the general trend of the H-R theory, allows one to take profit of several results which have been already obtained in the framework of that formalism.

In this paper the emphasis is given to the conceptual aspects of our approach. The results here displayed can be summarized as follows. As far as the parametrisation problem is concerned, our approach is equivalent to the H-R theory; however some dynamical features which characterize the Lipmann-Schwinger theory are also included in our formalism.

Further developments and applications, the nature of which can be easily understood, are reserved for forthcoming papers. The direction we are pointing to is a Mittag-Leffler expansion of the Green function. This opens the path for application to problems similar to those faced by Berggren and by Romo[6].

1.- THE POTENTIAL SCATTERING PROBLEM.-

In view of illustrating in the simplest manner how our method works, we give a short account of the potential scattering problem[7]. We work on a single sheet of the energy plane. The "physical energy plane" we shall consider has a cut from $E = 0$ to $E = -\infty$, and is the mapping of the right hand side of the k plane, according to the definition of Humblet and Rosenfeld(*)[3,4]. The interaction potential is assumed to be identically zero for $r \geq a$.

The Schroedinger equation for the ℓ partial wave is (units $\hbar^2/2M = 1$)

$$(1) \qquad \left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + V(r) \right] \psi_\ell(k,r) = k^2 \psi_\ell(k,r).$$

(*) - We note that this is not the definition generally used in analiticity property studies.

The physical solution which is regular at the origin, is normalized in such a way that its behaviour for $r \geq a$ is

$$\psi_\ell^{(+)}(k, r) = \frac{i}{2}\left[I_\ell(k, r) - 0_\ell(k, r) S_\ell(k) \right] \qquad r \geq a$$

where I_ℓ and 0_ℓ are the ingoing and outgoing functions respectively, defined in ref. (3)(*).

The solution $\psi_\ell^{(+)}$ can be expressed in the following integral form

$$(2) \qquad \psi_\ell^{(+)}(k, r) = u_\ell(k, r) + \int_0^\infty G_\ell^{(+)}(k; r, r')\, V(r')\, u_\ell(k, r')\, dr'$$

where u_ℓ is the regular spherical Bessel function normalized so that

$$\lim_{r \to \infty} u_\ell(k, r) = \operatorname{sen}(kr - \ell\pi/2)$$

and $G_\ell^{(+)}(k; r, r')$ is the Green function pertaining to the complete Hamiltonian. In writing the boundary conditions, it is useful to consider that we are dealing with a finite range potential. One has

$$G_\ell^{(+)}(k; r, r') = \begin{cases} -\dfrac{\varphi_\ell(k, r) f_\ell(k, r')}{W(f_\ell, \varphi_\ell)} & r \leq r' \leq a \\ -\dfrac{f_\ell(k, r)\, \varphi_\ell(k, r')}{W(f_\ell, \varphi_\ell)} & r' \leq r \leq a \end{cases}$$

where φ_ℓ is the regular solution of eq. (1) defined by the following conditions at the origin

$$\lim_{r \to 0} r^{-\ell-1}\, \varphi_\ell(k, r) = 1;$$

(*) - Here and in the following we mean that I_ℓ and 0_ℓ are solutions of the unperturbed Schroedinger equation in the whole space.

f_ℓ is the outgoing Jost solution, and satisfies the boundary conditions(*)

$$f'_\ell(k,a) - \mathcal{L}_\ell(k,a) f_\ell(k,a) = 0 \quad ; \qquad f_\ell(k,a) = 0_\ell(k,a)$$

where $\mathcal{L}_\ell(k,a) = 0'_\ell(k,a)/0_\ell(k,a)$; the energy function $W(f_\ell, \varphi_\ell) = f_\ell \varphi_\ell' - f'_\ell \varphi_\ell$ is the Wronskian of the two solutions.

The S matrix element can be written

$$S_\ell(k) = 1 - \frac{2i}{k} \int_0^a u_\ell(k,r)\, V(r)\, \psi_\ell^{(+)}(k,r)\, dr; \tag{3}$$

by use of eq. (2) we can write eq. (3) in the form

$$S_\ell(k) = 1 - B_\ell(k) + kR_\ell(k) \tag{4}$$

where $B_\ell(k)$ is the Born integral

$$B_\ell(k) = \frac{2i}{k} \int_0^a u_\ell(k,r)\, V(r)\, u_\ell(k,r)\, dr$$

and $R_\ell(k)$ is the resonant part of S(k),

$$R_\ell(k) = \frac{2i}{k^2} \frac{1}{W(f_\ell, \varphi_\ell)} \int_0^a u_\ell(k,r)\, V(r) \left\{ \int_0^r \varphi_\ell(k,r')\, f_\ell(k,r) + \right.$$

$$\left. + \int_r^a f_\ell(k,r')\, \varphi_\ell(k,r) \right\} V(r')\, u_\ell(k,r')\, dr\, dr'.$$

Let us consider the analytic continuation of S on the "physical E-plane". The poles of S can be originated only by the zero of W, because the radial functions f_ℓ and φ_ℓ are analytical in the whole cut E-plane, and the integrals cover the finite interval $0 \le r \le a$. The pole's location is the same as in ref.(2, 3), because they have the same definition (the case in which $k = 0$ is a pole will be excluded). At the zeros of W (poles of

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(*) - By primed functions we intend derivative with respect r; in the following, by the dot we shall intend energy derivatives.

S_ℓ), the functions φ_ℓ and f_ℓ are proportional, and we pose $f_\ell(k_n, r) = g_{\ell n}\, \varphi_\ell(k_n, r)$, where k_n is the pole. The integral which appears in $R_\ell(k)$ takes then at $k = k_n$ the simple form

$$g_{\ell n} \left[\int_0^a u_\ell(k_n, r)\, V(r)\, \varphi_\ell(k_n, r)\, dr \right]^2 .$$

The Wronskian derivative at $E = E_n$ ($E_n = k_n^2$) can be easily calculated(*):

$$(5) \qquad \frac{d}{dE} W(f_\ell, \varphi_\ell)\Big|_{E=E_n} = -g_{\ell n} \left[\int_0^a \varphi_\ell^2(k_n, r)\, dr + \dot{\mathcal{L}}_\ell(k_n, a)\, \varphi^2(k_n, a) \right] .$$

By the position

$$J_{\ell n} = \int_0^a u_\ell(k_n, r)\, V(r)\, \varphi_\ell(k_n, r)\, dr$$

the residue of $R_\ell(k)$ takes the form

$$(6) \qquad r_{\ell n} = -\frac{2i}{k_n^2} \frac{J_{\ell n}^2}{\int_0^a \varphi_\ell^2(k_n, r)\, dr + \dot{\mathcal{L}}_\ell(k_n, a)\, \varphi_\ell^2(k_n, a)}$$

In eq. (6) the resonant state appears both in the integral expression and, esplicity evaluated, in the interaction boundary. To obtain a more homogeneous expression we use the following identity

$$-k_n \frac{\varphi_\ell(k_n, a)}{0_\ell(k_n, a)} \equiv J_{\ell n}$$

which can be easily obtained by part integration of the left-hand member of the identity

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(*) - Eq. (5) can be viewed as a particularization of the analog expression (18), derived in appendix A in connection with the coupled channel problem.

$$\int_0^a \varphi_\ell(k_n, r)\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+V(r)-k_n^2\right]u_\ell(k_n, r)\,dr \equiv$$

$$\equiv \int_0^a \varphi_\ell(k_n, r)\,V(r)\,u_\ell(k_n, r)\,dr.$$

The residue can then be given the alternative expressions

$$r_{\ell n} = -\frac{2i\left[\varphi_\ell(k_n, a)/0_\ell(k_n, a)\right]^2}{\int_0^a \varphi_\ell^2(k_n, r)\,dr + \dot{\mathcal{L}}_\ell(k_n, a)\;\varphi_\ell^2(k_n, a)}$$

$$r_{\ell n} = -\frac{2i\,J_{\ell n}^2}{k_n^2\int_0^a \varphi_\ell^2(k_n, r)dr + \dot{\mathcal{L}}_\ell(k_n, a)\,0_\ell^2(k_n, a)\,J_{\ell n}^2}$$

In any case S_ℓ can be put in the form

$$S_\ell = Q_\ell(E) + k\sum_n \frac{r_{\ell n}}{E - E_{\ell n}}$$

where $Q_\ell(E)$ is the following entire function:

(7) $$Q_\ell(E) = 1 - B_\ell(E) + q_\ell(E);$$

in eq. (7) $q_\ell(E)$ is the entire part of $kR_\ell(k)$.

2. - THE COUPLED CHANNEL PROBLEM. -

a) - Definition of the problem. -

In this section we consider the scattering of a structureless particle by a target which we assume to admit only a finite number of excited states and no breaking possibility. A suitable schematization of this situation gives rise to the mathematical coupled channel problem, which has

been analysed by various authors[5, 8-12] owing to its remarkable heuristic interest. We refer here to the Newton's formulation[5]; the symbols we use are the same as in Newton's book, with minor modifications: in any case each quantity will be explicitly defined. According to the previous section we shall make the assumption that the interaction vanishes identically for $r \geq a$.

The total angular momentum J is a good quantum number and we shall omit the related label. The channels will be labelled c, c', ...; with L we indicate a diagonal matrix, the elements of which are the orbital angular momenta pertaining to the various channels: $L_{cc'} = \ell_c \delta_{cc'}$; in any case, quantities where L appears are to be intended as diagonal matrices, the c-th element of which should be labelled ℓ_c.

When the collision process is primed by waves incoming in the c-channel, we write the Schroedinger equation in the form:

$$\left[-\frac{d^2}{dr^2} + \frac{L(L+1)}{r^2} + V(r) \right] \psi^c(r) = K^2 \psi^c(r) ; \tag{8}$$

$\psi^c(r)$ is a unicolumnar matrix, whose asymptotic behaviour is characterized by the presence of an ingoing wave in the only channel c, whereas there are outgoing waves in all open channel. In eq. (8) K^2 is a diagonal matrix with elements $K^2_{c'} = E - \mathcal{E}_{c'}$, where E is the total energy of the system and $\mathcal{E}_{c'}$ is the energy at which the generic channel c' opens. (We pose $\mathcal{E}_o = 0$ for the lowest energy channel). The quantity V(r) is a symmetric matrix responsible for the channel coupling; its elements take into account the target structure and the angular momenta coupling scheme.

We can write a matrix solution, independent of the incoming channel, which satisfies the following condition

$$\psi^{(+)}(K, r) = \frac{i}{2} \left[I_L(K, r) - 0_L(K, r) \hat{S}(K) \right] K^{-L-1} \qquad r \geq a ; \tag{9}$$

together with the condition that $\psi^{(+)}(K, 0) = 0$, eq. (9) defines the physical solution.

Through the modified $\hat{S}$ matrix we have introduced in (9), we can express the S matrix:

$$S = K^{1/2} \hat{S} K^{-1/2} \tag{10}$$

The S matrix is unitary in the open channel subspace.

In close analogy with the potential scattering problem, let us introduce the regular-solution matrix and the outgoing Jost solution matrix. As well known[5, 12], contrary to the case of potential scattering, the boundary conditions on the regular solution matrix are not easily written explicitly; for this, it is preferable to define such solution through an integral equation. The regular solution is defined by

$$\phi(K, r) = \phi_o(K, r) + \int_o^r dr' G(K; r, r') V(r') \phi(K, r'),$$

where $\phi_o(K, r)$ is the following diagonal matrix:

$$\phi_o(K, r) = K^{-L-1} u_L(K, r).$$

Also the outgoing solution can be expressed in integral form

$$F(K, r) = F_o(K, r) - \int_r^\infty dr' G(K; r, r') V(r') F(K, r'), \tag{11}$$

where

$$F_o(K, r) = K^L 0_L(K, r).$$

In both cases the unperturbed Green function matrix is defined by the expression

$$G(K; r, r') = K^{-1} \left[v_L(K, r) u_L(K, r') - u_L(K, r) v_L(K, r') \right],$$

where v_L is the Bessel function matrix which at infinity behaves like $v_L \to \cos(Kr - L\pi/2)$. From eq. (11) one sees that at $r = a$ the solution $F(K, r)$ is a diagonal matrix (coincident with $F_o(K, r)$) and satisfies the conditions

$$F'(K, a) - \mathcal{L}_L(K, a) F(K, a) = 0 \; ; \qquad F(K, a) = K^L 0_L(K, a)$$

where $\mathcal{L}_L(K, a)$ is the diagonal matrix $0'_L(K, a)/0_L(K, a)$.

The physical solution can be put in the form

$$\psi^{(+)}(K, r) = \phi_o(K, r) + \int_o^a \mathcal{G}^{(+)}(K; r, r') V(r') \phi_o(K, r') dr' ;$$

the Green matrix function $\mathcal{G}^{(+)}(K;r,r')$ is defined by

$$\mathcal{G}^{(+)}(K;r,r') = \begin{cases} -\phi(K,r)\dfrac{1}{\mathcal{F}(K)}\tilde{F}(K,r') & r \leq r' \leq a \\ -F(K,r)\dfrac{1}{\tilde{\mathcal{F}}(K)}\tilde{\phi}(K,r') & r' \leq r \leq a \end{cases}$$

where

$$\mathcal{F}(K) = \tilde{F}(K,r)\,\phi'(K,r) - \tilde{F}'(K,r)\,\phi(K,r) \tag{12}$$

Eq. (12) is a Wronskian-type relation and $\mathcal{F}(K)$ is independent of r; most of the times we shall explicitate it at r = a.

The S matrix defined by eqs. (9) and (10) can be put into the form

$$S(K) = 1 - 2iK^{L+1/2}\left[\int_0^a \phi_0(K,r)\,V(r)\,\Psi^{(+)}(K,r)\,dr\right]K^{L+1/2}.$$

b)- Residues calculation. -

In analogy with eq. (4) we write

$$S(E) = 1 - B(E) + K^{L+1/2}\,R(E)\,K^{L+1/2}$$

where B(E) is the Born integral

$$B(E) = 2iK^{L+1/2}\int_0^a dr'\,\phi_0(K,r)\,V(r)\,\phi_0(K,r)\,K^{L+1/2}$$

while **R(E)** is the resonant integral,

$$R(E) = 2i\int_0^a \phi_0(K,r)\,V(r)\left\{\int_0^r \phi(K,r')\frac{1}{\mathcal{F}(K)}\tilde{F}(K,r) + \right.$$

$$\left. + \int_r^a F(K,r')\frac{1}{\tilde{\mathcal{F}}(K)}\tilde{\phi}(K,r)\right\}V(r')\,\phi_0(K,r')\,dr\,dr'. \tag{13}$$

We must now study the term R(E), and we shall do this in analogy to what has been done in the previous section.

The fact that we are dealing with a many-channel problem implies the existence of a many sheet energy surface. As in the Humblet and Rosenfeld theory, our analysis will be restricted to only one sheet of the Riemann surface, and this will be constructed according to the Humblet review paper, quoted in ref.(4).

Another troublesome feature of the many channel problem is to invert the matrix $\mathcal{F}(K)$, since we are interested in such an operation just at the points where $\mathcal{F}(K)$ is singular. In doing this we shall follow a technique that various authors have already used, and which has been set up by Glöckle[12] in the form we are directly interested in.

The procedure is the following. If K_n is a zero simple of the determinant of $\mathcal{F}(K)$, there exist two vectors (in the normalization of which we are not interested) which satisfy the equations

$$\tilde{\mathcal{F}}(K_n)\,A_n = 0\,; \qquad \mathcal{F}(K_n)\,B_n = 0\,. \tag{14}$$

One can then demonstrate that

$$\lim (E-E_n)\frac{1}{\mathcal{F}(K)} = \frac{B_n \tilde{A}_n}{\tilde{B}_n \left[\frac{d}{dE}\,\tilde{\mathcal{F}}(K)\right]_{E=E_n} A_n}\,. \tag{15}$$

From this we immediately obtain the matrix residue of $R(E)$ $(\mathrm{Res}R(E_n)=r_n)$

$$r_n = 2i\,\frac{J(E_n)\,\tilde{J}(E_n)}{\tilde{B}_n \left[\frac{d}{dE}\,\tilde{\mathcal{F}}(K)\right]_{E=E_n} A_n}\,; \tag{16}$$

in eq. (16) $J(E_n)$ is the vector defined by the following integral

$$J(E_n) = \int_0^a \phi_0(K_n, r)\,V(r)\,H(K_n, r)\,dr,$$

where $J(K_n, r)$ is the vector(x)

$$H(K_n, r) = \phi(K_n, r) B_n = F(K_n, r) A_n . \tag{17}$$

It is clear from eq. (16) that the residue value is independent of the normalization of A_n and B_n. The denominator of eq. (16) can be explicited in the following way (see Appendix A)

$$\tilde{B}_n \left[\frac{d}{dE} \tilde{\mathcal{F}}(K)\right]_{E=E_n} A_n = -\left[\int_0^a \tilde{H}(K_n, r) H(K_n, r) dr + \tilde{H}(K_n, a) \dot{\mathcal{L}}_L(K_n, a) H(K_n, a)\right] \tag{18}$$

In analogy with the potential scattering problem, the residue can be given two alternative forms. In fact, by use of the relation

$$J(E_n) \equiv \int_0^a \phi_o(K_n, r) V(r) H(K_n, r) dr = -K^{-L} \frac{1}{0_L(K_n, a)} H(K_n, a), \tag{19}$$

which will be demonstrated in Appendix B, we have the two forms(o)

$$r_{ncc'} = -2i \frac{J_{nc} J_{nc'}}{\int_0^a \tilde{H}(K_n, r) H(K_n, r) dr + \tilde{J}(E_n) 0_L^2(K_n, a) J(E_n)} \tag{20}$$

$$r_{ncc'} = -2i \frac{1}{\nu_n} \frac{1}{k_{c'n}^{\ell c'}} \frac{1}{k_{cn}^{\ell c}} \frac{H_{nc'}}{0_{nc'}} \frac{H_{nc}}{0_{nc}} ; \tag{21}$$

in eq. (21) we have posed

(x) - The equality of the last two members of eq. (17) is demonstrated in ref. (12).

(o) - The element cc' of the generic matrix $Z(k_n)$ will be written $Z_{ncc'}$.

$$\nu_n = \int_0^a \tilde{H}(K_n, r)\, H(K_n, r)\, dr + \sum_c \dot{\mathcal{L}}_{nc} H_{nc}^2 .$$

We finally obtain for the S matrix the following expression

$$S_{cc'} = Q_{cc'} + k_c^{\ell c'+1/2} k_c^{\ell c+1/2} \sum \frac{r_{ncc'}}{E - E_n} . \tag{22}$$

Expressions (21) and (22) have exactly the same <u>form</u> as that given by H-R. To verify that also the <u>content</u> is the same we must demonstrate that the pole location is the same too, and that our H_{nc} coincides, for $r = a$, with the "radial factors" ϕ_{cn} of the Humblet and Rosenfeld paper[3]. The proof, although simple, is not immediate because of the complete dif<u></u>ference of the two approaches; and will be given in Appendix C.

3. - CONCLUDING REMARKS; THE PROBLEM OF REARRANGEMENT COLLISIONS. -

We have shown that as far as the parametrization of the S matrix is concerned, the approach based on the Green function method leads to results which are exactly equivalent to those obtained in the framework of the wave function matching method. It is of interest to observe that in our approach the channel-wise residue factorization arises from the proportionality between the regular solution and the Jost outgoing solution (in the coupled channel problem the "proportionality constants" are matrices: see eq. (17)).

The generalization of this formalism to nuclear reactions proper is not immediate, and this problem will be faced in a forthcoming paper. Nevertheless, owing to the close correspondence we established between the H-R theory and our formalism, we can follow this way: we start from the S matrix expression given by H-R, and apply to the resedues a transformation similar to that given in eq. (19). Indeed, we can demonstrate (see Appendix B) the following identity(x)

(x) - In order to avoid to modify the symbols so far used, which are those commonly used in coupled channel problems, we modify the notation of the H-R paper, with apologies to these authors. We shall write Ψ(H-R) $\rightarrow \chi$, while the expansion (6.1a) of the H-R paper will be written in the form $(r_\alpha \chi)_{r_\alpha = a_\alpha} = \sum_{c(\alpha)} \hat{\phi}_c \varphi_c$; φ_c is the surface function and $\hat{\phi}_c$ is the surface factor. We shall call $\chi^{c\alpha}$ the unperturbed state in the channel c_α, and we write $\chi^{c\alpha} = \phi_o^{c\alpha}(r_\alpha)\, \varphi_{c\alpha}$, where $\phi_o^{c\alpha}$ is the unperturbed radial function, defined as $\phi_o^c(r_\alpha) = k_{c_\alpha}^{-\ell_{c\alpha}-1} u_{\ell_\alpha}(r_\alpha)$.

$$(23) \qquad \int \chi^{*}_{-n} V_{\alpha} \chi^{c_\alpha}_{n} = -\frac{\hbar^2}{2M_\alpha} k^{-}_{n c_\alpha} \frac{1}{0_{n \ell_{c_\alpha}}} \hat{\Phi}_{n c_\alpha};$$

this way we obtain two alternative expressions for the residues, similar to eqs. (20) and (21).

As far as the dynamic problem is concerned, we observe that the dynamical content of the Lippman-Schwinger equation is thrown on the form the residues are expressed (see eq. (20)). This expression should be suitable for practical calculations based on model descriptions or approximate expressions of the resonant state, in the limit in which the integral expressions are suited to conceal the fine details of the resonant states. However, the accurary of such approximations cannot be tested, and there is no mean of introducing corrections.

This approach can lead to an actual dynamic theory when the Mittag-Leffler expansion is directly applied to the Green function; this will enable us to introduce a procedure similar to that introduced by Romo[6], with the consequent possibility to gain any degree of accuracy. However, this way we obtain a parametrization which is slightly different from the H-R one; such developments are reserved to a forthcoming paper.

APPENDIX A. -

We must evaluate the following quantity

$$\tilde{B}_n \left[\frac{d}{dE} \tilde{\mathcal{F}}(K) \right]_{E=E_n} A_n = \tilde{B}_n \left[\dot{\tilde{\phi}}'(K_n, a)H(K_n, a) - \dot{\tilde{\phi}}(K_n, a)H'(K_n, a) \right] +$$

$$+ \left[\tilde{H}'(K_n, a)\dot{F}(K_n, a) - \tilde{H}(K_n, a)\dot{F}'(K_n, a) \right] A_n .$$

In order to evaluate the first member on the right hand side, we use the identity

$$(E_n - E) \int_0^a \tilde{\phi}(K_n, r)H(K_n, r)dr = \left[\tilde{\phi}'(K_n, r)H(K_n, r) - \tilde{\phi}(K_n, r)H'(K_n, r) \right]_0^a .$$

By derivating with respect to E at $E=E_n$, and remembering the conditions at the origin and eq. (17), one gets

$$- \int_0^a \tilde{\phi}(K_n, r)H(K_n, r)dr = \dot{\tilde{\phi}}'(K_n, a)H(K_n, a) - \dot{\tilde{\phi}}(K_n, a)H'(K_n, a) .$$

By left-multiplication for $\tilde{B}_n$ we get finally

$$(A.1) \quad \tilde{B}_n \left[\dot{\tilde{\phi}}'(K_n, a)H(K_n, a) - \dot{\tilde{\phi}}(K_n, a)H'(K_n, a) \right] = \int_0^a \tilde{H}(K_n, r)H(K_n, r)\, dr$$

The second term is easily evaluated by taking into account the following boundary conditions:

$$F'(K_n, a) = \mathcal{L}(K_n)F(K_n, a); \qquad H'(K_n, a) = \mathcal{L}(K_n)H(K_n, a);$$

$$\dot{F}'(K_n, a) = \dot{\mathcal{L}}(K_n)F(K_n, a) + \mathcal{L}(K_n)\dot{F}(K_n, a)$$

It is then immediate to obtain

(A.2) $$\tilde{H}'(K_n, a)\dot{\tilde{F}}(K_n, a) - \tilde{H}(K_n, a)\dot{\tilde{F}}'(K_n, a) = -\tilde{H}(K_n, a)\,\mathcal{L}(K_n)H(K_n, a)\,.$$

Eq. (18) follows immediately from eqs. (A.1) and (A.2).

APPENDIX B. -

We integrate by part the first member of the following identity

$$\int_0^a \phi_0(K_n, r)(E_n - H_0)\,\phi(K_n, r)\,dr \equiv \int_0^a \phi_0(K_n, r)\,V\,\phi(K_n, r)\,dr$$

and obtain

$$(B.1)\quad \phi_0(K_n, r)\,\phi'(K_n, r) - \phi_0'(K_n, r)\,\phi(K_n, r)\Big|_0^a \equiv \int_0^a \phi_0(K_n, r)V\phi(K_n, r)\,dr$$

(we have taken into account that $(H_0+V-E_n)\,\phi(K_n, r) = (H_0-E_n)\,\phi_0(K_n, r) = 0$.

We now right-multiply eq. (B.1) by B_n, and take into account the boundary condition

$$H'(K_n, a) = \mathcal{L}(K_n)\,H(K_n, a)\,;$$

one obtains

$$(B.2)\quad \left[\mathcal{L}(K_n)\,\phi_0(K_n, a) - \phi_0'(K_n, a)\right] H(K_n, a) = \int_0^a \phi_0(K_n, r)VH(K_n, r)\,dr.$$

Formula (19) follows from (B.2) remembering the definition of $\mathcal{L}(K_n)$ and ϕ_0.

Let us consider now nuclear reactions proper. In the generic partition $\alpha, \beta, \ldots$ we write

$$H = H_\alpha + V_\alpha = H_\beta + V_\beta = \ldots$$

with obvious meaning of the symbols.

The wave function ψ of the Humblet and Rosenfeld paper will be

called χ and will be expanded in the following way (see footnote to the concluding remarks)

$$(r_\alpha \chi)_{r=a} = \sum_{c(\alpha)} \hat{\phi}_c \varphi_c .$$

The unperturbed solution of H_α in the channel c_α will be called χ^{c_α}, and has the expression

$$\chi^{c_\alpha} = \phi_o^{c_\alpha}(r_\alpha)\, \varphi_{c_\alpha}$$

where the function $\phi_o^{c_\alpha}(r_\alpha)$ describes the unperturbed relative motion of the fragments characterizing the channel c_α, and is normalized as the unperturbed functions at section 2. If χ_n is the wave function of the state at the pole n, we call χ_{-n} its time reversed.

Let us now consider the following identity

$$\int_\omega \left[(H-E_n)\chi_{-n}\right]^* \chi_n^{c_\alpha}\, d\omega - \int_\omega \chi_{-n}^* (H-E_n)\chi_n^{c_\alpha}\, d\omega \equiv \int \chi_{-n}^* V_\alpha \chi_n^{c_\alpha}\, d\omega$$

(ω is the interaction volume, $\chi_n^{c_\alpha}$ is the state χ^{c_α} evaluated at E_n), and handle the first member in analogy to what is done in the H-R paper. We obtain successively

$$-\int \chi_{-n}^* V_\alpha \chi_n^{c_\alpha}\, d\omega = \frac{\hbar^2}{2M_\alpha}\left[\hat{\phi}_{nc_\alpha}\, \phi'_{o\,c_\alpha} - \hat{\phi}'_{nc_\alpha}\, \phi_{o\,c_\alpha}\right] =$$

$$= \frac{\hbar^2}{2M_\alpha}\hat{\phi}_{nc_\alpha}\left[\phi_{onc_\alpha} - \mathscr{L}_{nc_\alpha}\phi_{onc_\alpha}\right] = \frac{\hbar^2}{2M_\alpha}\left[-k_{nc_\alpha}^{-\ell_{c_\alpha}}\, \frac{1}{0_{n\,\ell_{c_\alpha}}}\, \hat{\phi}_{nc_\ell}\right].$$

Formula (23) is then demonstrated.

APPENDIX C.-

If we specialize the H-R theory to the coupled channel problem, the Ψ of H-R paper is related to our regular solution Φ. Now we observe that at the poles our H_n is a vector which vanishes at $r = 0$ and has the property $|H_n(r)|^2 =$ finite in the internal region; furthermore its asymptotic behaviour is that of an outgoing wave (eq. 17). Therefore at $r = a$, H_{n_c} can be identified with the H-R radial factors $\Phi_{c\alpha}$. As concerns the pole location we make the following considerations. We multiply the expression

$$\mathcal{J}(K_n) = \tilde{F}(K_n, a)\,\Phi'(K_n, a) - \tilde{F}(K_n, a)\,\Phi(K_n, a)$$

at the right by B_n and obtain, because of eqs. (14) and (17) the identity

$$\tilde{F}(K_n, a)\,H'(K_n, a) - \tilde{F}'(K_n, a)\,H(K_n, a) \equiv 0 .$$

Since $F(K_n, a) = K_n^L 0_L(K_n, a)$, we have

$$0_{n\ell\alpha}(a)\,H'_{n\alpha}(a) - 0'_{n\ell\alpha}(a)\,H_{n\alpha}(a) = 0$$

which is the definition of resonant state given in ref. (3).

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