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V. Vanzani : NUCLEAR REARRANGEMENT PROCESSES IN AN EXACT N-BODY THEORY. -

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SUMMARY. -

Nuclear rearrangement processes are investigated within the framework of the Yakubovskiĭ approach to the N-body problem. Transition operator components having prescribed connectedness properties are proposed in order to extract from exact equations the relevant reaction mechanisms. Specific applications to single-particle and two-particle transfer reactions are treated.

1. - INTRODUCTION. -

As is well known, conventional approaches to nuclear rearrangement reactions are usually not based on a rigorous mathematical ground. For calculational purposes, the N-body rearrangement scattering problem is replaced by effective two-particle models and approximate expressions for the transition amplitude are proposed on the basis of intuitive arguments or of phenomenological suggestions.

In such methods the incorrect mathematical description of the problem is usually associated with a not well clear physical interpretation of the approximations one made. For instance, the distorted-wave Born approximation (DWBA) amplitude has a form rather unsymmetrical with respect to the initial and final channel, because the reaction mechanism is not described correctly. The well-known post and prior representation discrepancy would not exist if the proposed amplitude had a well-defined physical picture(1, 2, 3).

Some new insight into the general questions involved in the rearrangement processes has been obtained within the framework of the non-relativistic Feynman-diagram approach to direct nuclear reactions(4-7). This approach gives useful suggestions in order to formulate a rigorous mathematical theory, in which all the terms, appearing in a meaningful expansion of the exact amplitude, have a well-defined physical interpretation (in a diagrammatic form). Such a program has ben carried out, in an exact three body context, for single-particle transfer reactions(2, 3). For more complex reactions, one needs to formulate the rearrangement problem in a N-body context. In this paper we try to get such a formulation.

The N-body rearrangement scattering problem can be treated on a rigorous theoretical ground by generalizing the three-particle Faddeev equations to any number of particles. However, as Blankenbecler and Sugar showed[8], there are infinite ways of constructing integral equations with connected kernels. This explains the large number of generalizations proposed in the literature[9-18]. Among them, the Yakubovskiĭ formulation represents, as recently pointed out by Faddeev[19], the most natural method for attacking the N-body problem.

In order to make the Yakubovskiĭ approach suitable for describing nuclear rearrangement mechanisms, we shall introduce scattering operator components having prescribed connectedness properties from the left as well as from the right-hand side (two-sided components). An earlier different definition of two-sided components has been proposed in the study of the analytic structure of the N-body scattering amplitude[20]. However, both with respect to these latter components and with respect to the original one-sided Yakubovskiĭ ones, the components we define have the advantage of leading directly to a physically meaningful expansion of the transition operators.

It will be proved that the two-sided components satisfy Yakubovskiĭ-like equations. Therefore the attractive features of the Yakubovskiĭ approach are preserved in the proposed reformulation. Furthermore, we shall see that the contributions to the transition amplitude, arising from the new inhomogeneous terms, have an immediate interpretation in terms of reaction mechanisms.

In Sect. 2 we introduce generalized expressions for the transition operators and we define their two-sided components. In Sect. 3 integral equations for these components are derived and transition operators are splitted into parts having a well-defined connectedness. Sect. 4 is devoted to specific applications : single-particle and two-particle rearrangement processes are discussed in a three-body and in a four-body context, respectively. The different role played by competitive mechanisms will be emphasized.

Mathematical details concerning integral equations and a technical discussion on the operator connectedness properties are given in the Appendices.

2. - GENERAL FORMALISM. -

2. 1. - Transition operators.

We assume our N-body system to be described by the Hamiltonian :

$$H = H_0 + \sum_i{}' V_i \, , \qquad (2.1)$$

where H_0 is the kinetic energy and V_i are two-body potentials. The sum in the eq. (2.1) extends over all the pairs of particles.

In order to describe scattering processes, one has to consider the possible initial and final configurations. The Yakubovskiĭ notation will be followed[18].

Let a_k be a partition of the N-particle system into k clusters ($1 \le k \le N$). Different partitions into k clusters will be denoted by different latin letters a_k, b_k, ... Partitions into k+1 clusters, obtained from a_k, b_k, ..., by breaking up one of their clusters, will be denoted by $a_{k+1}, b_{k+1}, \dots,$ respectively ($a_{k+1} \subset a_k$).

Each partition a_k corresponds to an asymptotic configuration in which the clusters of a_k are not interacting among them; only the interactions acting within the clusters do not vanish. If V_{a_k} is the sum of all these interactions, the channel Hamiltonian H_{a_k} is :

$$H_{a_k} = H_0 + V_{a_k}. \tag{2.2}$$

The resolvents of the channel Hamiltonians are :

$$G_{a_k}(z) = (z - H_{a_k})^{-1}. \tag{2.3}$$

For k=N one has the resolvent of the free Hamiltonian $G_0 = G_{a_N}$ ($V_{a_N}=0$); for k=1 one has the resolvent of the total Hamiltonian $G=G_{a_1}$ ($V_{a_1} = \sum_i V_i$).

Let us introduce the transition operators $U_{a_j b_k}(z)$:

$$U_{a_j b_k}(z) = (1-\delta_{a_j b_k})G_{a_j}^{-1}(z) + \bar{V}_{b_k} + \bar{V}_{a_j} G(z) \bar{V}_{b_k}, \tag{2.4a}$$

$$U_{a_j b_k}(z) = (1-\delta_{a_j b_k})G_{b_k}^{-1}(z) + \bar{V}_{a_j} + \bar{V}_{a_j} G(z) \bar{V}_{b_k}, \tag{2.4b}$$

from the initial configuration characterized by the partition b_k to the final one characterized by the partition a_j. $\bar{V}_{a_j}$ is the sum of all the interactions not contained in the clusters of the partition a_j.

The on-the-energy-shell matrix elements of the operators (2.4) lead directly to the S-matrix elements. This has been shown by Alt, Grassberger and Sandhas[14, 21], which proposed the nonstandard definition (2.4) and which pointed out its advantages with respect to the transition operators generally used in the literature. In the following, we shall see that eqs. (2.4) represent, in extracting the relevant reaction mechanisms, the most natural choice for the transition operators among all the possible off energy-shell continuations which coincide on-the-energy-shell.

For j=k=N, the eqs. (2.4) give the N-particle scattering operator

$$T(z) = U_{a_N a_N}(z) = V + VG(z)V, \tag{2.5}$$

where $V = \bar{V}_{a_N} = V_{a_1}$. If one takes into account only the interactions contained in the partition a_i, one has the scattering operator :

$$T_{a_i}(z) = V_{a_i} + V_{a_i} G_{a_i}(z) V_{a_i}. \tag{2.6}$$

For i=N-1, the eq. (2.6) gives the two-body scattering operators $T_{a_{N-1}}(z)$, acting in the N-particle space. Note that each partition a_{N-1} is in an one-to-one correspondence with a pair of particles. For i=1, the eq. (2.6) coincides with the eq. (2.5).

The formulas (2.4)-(2.6) can be considered as particular cases of the following generalized definition of transition operators between configurations within the partition a_i :

$$U_{a_j b_k; a_i}(z) = (1-\delta_{a_j b_k})G_0^{-1}(z) + V_{a_i} - V_{b_k} - (1-\delta_{a_j b_k})V_{a_j} + (V_{a_i} - V_{a_j})G_{a_i}(z)(V_{a_i} - V_{b_k}). \tag{2.7}$$

2.2. - One-sided components of the transition operators.

As is well-known, after Faddeev comments on the possible formulations of the N-body scattering problem(19), only in the three-body case one can obtain in a natural way integral equations directly for the transition operators. These are the well-known equations derived by Lovelace in an approach which is essentially equivalent to the Faddeev original one(22-24).

Although integral equations directly for the transition operators have been proposed in the N-body case, difficulties appear in proving both the compactness of the kernel and the absence of spurious bound-state solutions. These two important features are preserved if one splits the transition operators into components having well-defined connectedness properties and one derives integral equations for such components by means of the operator inversion technique, firstly suggested by Faddeev.

The one-sided components of the scattering operators defined by Yakubovskiĭ are:

(2.8a) $$M_{a_i}^{a_{N-1} b_{N-1}} = V_{a_{N-1}} \delta_{a_{N-1} b_{N-1}} + V_{a_{N-1}} G_{a_i} V_{b_{N-1}},$$

(2.8b) $$M_{a_i}^{\alpha_{k-1} \beta_{k-1}} = M_{a_i(1)}^{\alpha_{k-1} \beta_{k-1}} + \sum_{\substack{\gamma_k \delta_k \\ d_k \subset a_{k-1}}} M_{a_i(0)}^{\alpha_k \gamma_k} X_{a_i}^{\gamma_k \delta_k} M_{a_i}^{\delta_k \beta_k} \quad (3 \leq k \leq N-1);$$

(2.9a) $$\overline{M}_{a_i}^{a_{N-1} b_{N-1}} = M_{a_i}^{a_{N-1} b_{N-1}},$$

(2.9b) $$\overline{M}_{a_i}^{\alpha_{k-1} \beta_{k-1}} = \overline{M}_{a_i(1)}^{\alpha_{k-1} \beta_{k-1}} + \sum_{\substack{\epsilon_k \zeta_k \\ e_k \subset b_{k-1}}} \overline{M}_{a_i}^{\alpha_k \epsilon_k} \overline{X}_{a_i}^{\epsilon_k \zeta_k} \overline{M}_{a_i(0)}^{\zeta_k \beta_k} \quad (3 \leq k \leq N-1);$$

where

(2.10) $$X_{a_i}^{\alpha_k \beta_k} = \prod_{j=k}^{N-1} (1 - \delta_{a_j b_j}) \prod_{j=k}^{N-2} \delta_{b_{j+1} \subset a_j} G_0, \quad \overline{X}_{a_i}^{\alpha_k \beta_k} = X_{a_i}^{\beta_k \alpha_k}.$$

In the definitions (2.8)-(2.10) the indices α_k, β_k, ... denote sequences of partitions beginning with a_k, b_k, ..., respectively, e.g.:

$$\alpha_k = (a_k, a_{k+1}, \ldots, a_{N-1}) = (a_k, \alpha_{k+1});$$

the partitions a_k, b_k in the matrix elements $A_{a_i}^{\alpha_k \beta_k}$ fulfil the condition $a_k, b_k \subset a_i$ (the index a_i will be omitted when i=1); the quasidiagonal matrix elements $A_{a_i(0)}^{\alpha_k \beta_k}$, $A_{a_i(1)}^{\alpha_k \beta_k}$ are defined as:

(2.11) $$A_{a_i(0)}^{\alpha_k \beta_k} = A_{a_k}^{\alpha_{k+1} \beta_{k+1}} \delta_{a_k b_k}, \quad A_{a_i(1)}^{\alpha_k \beta_k} = A_{a_i(0)}^{\alpha_{k+1} \beta_{k+1}} \delta_{a_k b_k}.$$

The recurrence relations (2.8b) and (2.9b) contain the graphic connectedness properties of $M_{a_i}^{\alpha_k \beta_k}$ (from the left hand side) and of $\overline{M}_{a_i}^{\alpha_k \beta_k}$ (from the right hand side)(20,25).

The transition operator components $M^{\alpha_2\beta_2}$, $\overline{M}^{\alpha_2\beta_2}$ for the fully interacting N-body system satisfy the Faddeev-Yakubovskiǐ equations[18]

(2.12a) $$M^{\alpha_2\beta_2}=M^{\alpha_2\beta_2}_{(0)}+\sum_{\gamma_2\delta_2} M^{\alpha_2\gamma_2}_{(0)}X^{\gamma_2\delta_2}M^{\delta_2\beta_2},$$

(2.12b) $$\overline{M}^{\alpha_2\beta_2}=\overline{M}^{\alpha_2\beta_2}_{(0)}+\sum_{\varepsilon_2\zeta_2} \overline{M}^{\alpha_2\varepsilon_2}\overline{X}^{\varepsilon_2\zeta_2}\overline{M}^{\zeta_2\beta_2}_{(0)}.$$

Note that the eqs. (2.12) in matrix notation have a formal Lippmann-Schwinger-like structure ($M_{(0)}$ and X play the role of the potential and of the free propagator, respectively).

From the relation connecting $M^{\alpha_k\beta_k}_{a_i}$ with $M^{\alpha_{k+1}\beta_{k+1}}_{a_i}$ (eq. (3.4) of ref. (18)) and from the definition (2.4) of the transition operators, it follows that:

(2.13) $$U_{a_jb_k}=(1-\delta_{a_jb_k})\Big(G_0^{-1}-\sum_{c_{N-1}\subseteq a_j,b_k} V_{c_{N-1}}\Big)+\sum_{\substack{\gamma_2;\, d_{N-1}\not\subseteq b_k\\ c_{N-1}\not\subseteq a_j}} M^{\gamma_2\delta_2}$$

for arbitrary $d_2,\dots,d_{N-2}$. Introducing the eq. (2.12a) in the eq. (2.13), one sees that there is a large arbitrariness in constructing convergent expansions of the transition operators (one can start from different first-order terms). Furthermore the expression (2.12) shows a rather unsymmetrical structure with respect to the different possible intermediate state contributions. In order to obtain unique convergent expansions and symmetric expressions for the transition operators, we shall introduce two-sided components having prescribed connectedness properties from the left as well as from the right hand side.

2.3. - Two-sided components of the transition operators.

Let us now construct two-sided components $N^{\alpha_k\beta_k}_{a_i}$ of the transition operators by means of the one-sided Yakubovskiǐ ones (2.8) and (2.9). We start from

(2.14) $$N^{a_{N-1}b_{N-1}}_{a_i}=M^{a_{N-1}b_{N-1}}_{a_i}$$

and introduce the recurrence relations

(2.15) $$N^{\alpha_{k-1}\beta_{k-1}}_{a_i}=N^{\alpha_{k-1}\beta_{k-1}}_{\alpha_i(0)}-$$
$$-\sum_{(\gamma\delta\varepsilon\zeta)_{k-1}} M^{\alpha_{k-1}\gamma_{k-1}}_{a_i(1)}X^{\gamma_{k-1}\delta_{k-1}}_{a_i(0)}N^{\delta_{k-1}\varepsilon_{k-1}}_{a_i(0)}\overline{X}^{\varepsilon_{k-1}\zeta_{k-1}}_{a_i(0)}\overline{M}^{\zeta_{k-1}\beta_{k-1}}_{a_i(1)}+$$
$$+\sum_{\substack{(\gamma\delta\varepsilon\zeta)_k\\ d_k\subset a_{k-1}\\ e_k\subset b_{k-1}}} M^{\alpha_k\gamma_k}_{a_i(0)}X^{\gamma_k\delta_k}_{a_i}N^{\delta_k\varepsilon_k}_{a_i}\overline{X}^{\varepsilon_k\zeta_k}_{a_i}\overline{M}^{\zeta_k\beta_k}_{a_i(0)} \quad (3\leq k\leq N-1),$$

where $(\gamma\delta\varepsilon\zeta)_i = \gamma_i\delta_i\varepsilon_i\zeta_i$ and

(2.16) $$N^{a_{N-1}b_{N-1}}_{\alpha_i(0)} = T_{a_{N-1}}\delta_{a_{N-1}b_{N-1}},$$

(2.17) $$N^{\alpha_{k-1}\beta_{k-1}}_{\alpha_i(0)} = N^{\alpha_{k-1}\beta_{k-1}}_{a_i(0)} - N^{\alpha_{k-1}\beta_{k-1}}_{\alpha_i(1)}, \qquad (3\leq k\leq N-1)$$

(2.18) $$N^{\alpha_{k-1}\beta_{k-1}}_{\alpha_i(1)} = N^{\alpha_k\beta_k}_{\alpha_i(0)}\delta_{a_{k-1}b_{k-1}}, \qquad (3\leq k\leq N-1).$$

The eqs. (2.14) and (2.15) define $N^{\alpha_k\beta_k}_{a_i}$ as a sum of a quasidiagonal part and of a part which contains the resolvent G_{a_i}. This latter one arises from a direct coupling of the resolvent part of the left-hand-sided operators (2.8b) with the resolvent part of the right-hand-sided operators (2.9b). The proposed definition of the quasidiagonal part will assure a k-independent structure of the inhomogeneous part of the Yakubovskiĭ-like integral equation system for $N^{\alpha_k\beta_k}_{a_i}$. This essential requisite is not preserved if quasidiagonal terms do not appear in the eq. (2.15)[20].

The operator $N^{\alpha_k\beta_k}_{\alpha_i(0)}$ have connectedness a_k, that is they can be represented as a sum of a_k-connected graphs[9, 13](x). In virtue of this interesting property one can expand the transition operators into parts having a well-defined graphic connectedness. As shown by the recurrence definitions (2.16)-(2.18), the a_{k-1}-connectedness property of $N^{\alpha_{k-1}\beta_{k-1}}_{\alpha_i(0)}$ for $\alpha_k=\beta_k$ is obtained by subtracting, with a sequential procedure, all terms with lower connectedness. The sequence terminates with $T_{a_{N-1}}\delta_{\alpha_{k-1}\beta_{k-1}}$, which is a_{N-1}-connected. The proof of the connectedness property of $N^{\alpha_{k-1}\beta_{k-1}}_{\alpha_i(0)}$ for arbitrary sequences α_k and β_k requires a detailed investigation. It will be given in Appendix A.

3. - THE N-BODY PROBLEM IN TERMS OF TWO-SIDED SCATTERING OPERATORS.

3.1. - Integral equations for two-sided scattering operators.

In order to derive, by means of an inductive procedure, integral equations for the two-sided operators $N^{\alpha_k\beta_k}_{a_i}$ $(2\leq k\leq N-2,\ k>i)$, we start from the Faddeev-like equations for $N^{ab}_{a_i}$ $(a=a_{N-1},\ b=b_{N-1})$

(3.1a) $$N^{ab}_{a_i} = T_a\delta_{ab} + T_a G_0 \sum_{d\neq a} N^{db}_{a_i},$$

(3.1b) $$N^{ab}_{a_i} = T_a\delta_{ab} + \sum_{e\neq b} N^{ae}_{a_i} G_0 T_b.$$

For partitions $a_i=a_{N-2}$ which contain a three-particle cluster the eqs. (3.1) coincide with the original Faddeev equations. For partitions $a_i=a_{N-2}$ which contain two two-par

(x) - A graph consisting of k connected parts is a_k-connected if exists an one-to-one correspondence between its connected parts and the clusters of the partition a_k (graphic lines belonging to the same connected part correspond to particles belonging to the same cluster).

ticle clusters the eqs. (3. 1) are rearranged expressions of the well-known (second) resolvent equation for $G_{a_{N-2}}$.

Using the eqs. (3. 1), one can show that the operators $N^{\alpha'\beta'}_{a_i}$ $(\alpha'=(a',a)\equiv(a_{N-2},a_{N-1})=\alpha_{N-2}$ and similarly for β') satisfy the integral equations (see Appendix B)

(3. 2a) $$N^{\alpha'\beta'}_{a_i}=N^{\alpha'\beta'}_{\alpha_i(0)}+\sum_{\substack{c;d'\neq a'\\ d\neq c,\subset a'}} N^{ac}_{a'}G_0N^{\delta'\beta'}_{a_i},$$

(3. 2b) $$N^{\alpha'\beta'}_{a_i}=N^{\alpha'\beta'}_{\alpha_i(0)}+\sum_{\substack{z;e'\neq b'\\ e\neq z,\subset b'}} N^{\alpha'\varepsilon'}_{a_i}G_0N^{zb}_{b'}.$$

For N=4 $(a_i=a_1)$ the eqs. (3. 2) represent the correct solution of the four-body problem in terms of two-sided operators. The kernel of the eqs. (3. 2) coincides with the Yakubovskiĭ kernel in the four-body case.

Let us now suppose that the operators $N^{\alpha_k\beta_k}_{a_i}$ satisfy the following linear integral equations

(3. 3a) $$N^{\alpha_k\beta_k}_{a_i}=N^{\alpha_k\beta_k}_{\alpha_i(0)}+\sum_{\gamma_k\delta_k} M^{\alpha_k\gamma_k}_{a_i(0)}X^{\gamma_k\delta_k}_{a_i}N^{\delta_k\beta_k}_{a_i},$$

(3. 3b) $$N^{\alpha_k\beta_k}_{a_i}=N^{\alpha_k\beta_k}_{\alpha_i(0)}+\sum_{\varepsilon_k\zeta_k} N^{\alpha_k\varepsilon_k}_{a_i}\overline{X}^{\varepsilon_k\zeta_k}_{a_i}\overline{M}^{\zeta_k\beta_k}_{a_i(0)}.$$

We shall prove that formally identical equations hold for $N^{\alpha_{k-1}\beta_{k-1}}_{a_i}$. In the eqs. (3. 3) the inhomogeneous term is the a_k-connected operator $N^{\alpha_k\beta_k}_{\alpha_i(0)}$. Note that for k=N-1 the eqs. (3. 3) coincide with the eqs. (3. 1), for k=N-2 they coincide with the eqs. (3. 2).

Introducing the eqs. (3. 3) in the definition (2. 15), one obtains the following alternative formulas for $N^{\alpha_{k-1}\beta_{k-1}}_{a_i}$:

(3.4 a) $$N^{\alpha_{k-1}\beta_{k-1}}_{a_i}=\sum_{\gamma_{k-1}\delta_{k-1}} M^{\alpha_{k-1}\gamma_{k-1}}_{a_i(1)}X^{\gamma_{k-1}\delta_{k-1}}_{a_i(0)}N^{\delta_{k-1}\beta_{k-1}}_{\alpha_i(1)}+\sum_{\substack{(\gamma\delta\varepsilon\zeta)_k\\ d_k\subset a_{k-1}\\ e_k\subset b_{k-1}}} M^{\alpha_k\gamma_k}_{a_i(0)}\times$$
$$\times\ X^{\gamma_k\delta_k}_{a_i}N^{\delta_k\varepsilon_k}_{a_i}\overline{X}^{\varepsilon_k\zeta_k}_{a_i}\overline{M}^{\zeta_k\beta_k}_{a_i(0)},$$

(3. 4b) $$N^{\alpha_{k-1}\beta_{k-1}}_{a_i}=\sum_{\varepsilon_{k-1}\zeta_{k-1}} N^{\alpha_{k-1}\varepsilon_{k-1}}_{\alpha_i(1)}\overline{X}^{\varepsilon_{k-1}\zeta_{k-1}}_{a_i(0)}\overline{M}^{\zeta_{k-1}\beta_{k-1}}_{a_i(1)}+\sum_{\substack{(\gamma\delta\varepsilon\zeta)_k\\ d_k\subset a_{k-1}\\ e_k\subset b_{k-1}}} M^{\alpha_k\gamma_k}_{a_i(0)}\times$$
$$\times\ X^{\gamma_k\delta_k}_{a_i}N^{\delta_k\varepsilon_k}_{a_i}\overline{X}^{\varepsilon_k\zeta_k}_{a_i}\overline{M}^{\zeta_k\beta_k}_{a_i(0)}.$$

Summing over a_{k-1} or b_{k-1} and taking into account the eqs. (3. 3) and the well-known properties of the partitions[18], one gets

(3. 5a) $$\sum_{a_{k-1}} N_{a_i}^{\alpha_{k-1}\beta_{k-1}} = \sum_{\substack{\epsilon_k\zeta_k \\ e_k\subset b_{k-1}}} N_{a_i}^{\alpha_k\epsilon_k} \overline{X}_{a_i}^{\epsilon_k\zeta_k} \overline{M}_{a_i(0)}^{\zeta_k\beta_k}$$

(3. 5b) $$\sum_{b_{k-1}} N_{a_i}^{\alpha_{k-1}\beta_{k-1}} = \sum_{\substack{\gamma_k\delta_k \\ d_k\subset a_{k-1}}} M_{a_i(0)}^{\alpha_k\gamma_k} X_{a_i}^{\gamma_k\delta_k} N_{a_i}^{\delta_k\beta_k}.$$

From the eqs. (3. 4), (3. 5), (2. 17), (2. 18) and (3. 3) it follows that

(3. 6a) $$N_{a_i}^{\alpha_{k-1}\beta_{k-1}} = N_{\alpha_i(0)}^{\alpha_{k-1}\beta_{k-1}} - \sum_{\gamma_{k-1}\delta_{k-1}} M_{a_i(1)}^{\alpha_{k-1}\gamma_{k-1}} X_{a_i(0)}^{\gamma_{k-1}\delta_{k-1}} N_{\alpha_i(0)}^{\delta_{k-1}\beta_{k-1}} + \sum_{\substack{\gamma_k\delta_{k-1} \\ d_k\subset a_{k-1}}} M_{a_i(0)}^{\alpha_k\gamma_k} X_{a_i}^{\gamma_k\delta_k} N_{a_i}^{\delta_{k-1}\beta_{k-1}}$$

(3. 6b) $$N_{a_i}^{\alpha_{k-1}\beta_{k-1}} = N_{\alpha_i(0)}^{\alpha_{k-1}\beta_{k-1}} - \sum_{\epsilon_{k-1}\zeta_{k-1}} N_{\alpha_i(0)}^{\alpha_{k-1}\epsilon_{k-1}} \overline{X}_{a_i(0)}^{\epsilon_{k-1}\zeta_{k-1}} \overline{M}_{a_i(1)}^{\zeta_{k-1}\beta_{k-1}} + \sum_{\substack{\epsilon_{k-1}\zeta_k \\ e_k\subset b_{k-1}}} N_{a_i}^{\alpha_{k-1}\epsilon_{k-1}} \overline{X}_{a_i}^{\epsilon_k\zeta_k} \overline{M}_{a_i(0)}^{\zeta_k\beta_k}.$$

By analogy with the Faddeev-Yakubovskiǐ arguments for the one-sided operator case, we extract from the latter sum at the right hand side of the eqs. (3. 6) the part corresponding to $d_{k-1}=a_{k-1}$ or $e_{k-1}=b_{k-1}$ and we transfer it to the left hand side. Thus one obtains, in matrix notation,

(3. 7a) $$\left(I - M_{a_i(1)}^{k-1} X_{a_i(0)}^{k-1}\right) N_{a_i}^{k-1} = \left(I - M_{a_i(1)}^{k-1} X_{a_i(0)}^{k-1}\right) N_{\alpha_i(0)}^{k-1} + M_{a_i(1)}^{k-1} X_{a_i}^{k-1} N_{a_i}^{k-1},$$

(3. 7b) $$N_{a_i}^{k-1}\left(I - \overline{X}_{a_i(0)}^{k-1} \overline{M}_{a_i(1)}^{k-1}\right) = N_{\alpha_i(0)}^{k-1}\left(I - \overline{X}_{a_i(0)}^{k-1} \overline{M}_{a_i(1)}^{k-1}\right) + N_{a_i}^{k-1} \overline{X}_{a_i}^{k-1} \overline{M}_{a_i(1)}^{k-1}.$$

If one takes into account the relations (4. 5), (4. 6) of ref. (18) and similar relations for the right hand-sided operators, one gets from the eqs. (3. 7) the integral equations for $N_{a_i}^{\alpha_{k-1}\beta_{k-1}}$ in the form (3. 3).

Finally, the above inductive procedure leads to the following result for the fully interacting N-body system

(3. 8a) $$N^{\alpha_2\beta_2} = N_{\alpha(0)}^{\alpha_2\beta_2} + \sum_{\gamma_2\delta_2} M_{(0)}^{\alpha_2\gamma_2} X^{\gamma_2\delta_2} N^{\delta_2\beta_2},$$

(3.8b) $$N^{\alpha_2\beta_2}=N^{\alpha_2\beta_2}_{\alpha(0)}+\sum_{\epsilon_2\zeta_2} N^{\alpha_2\epsilon_2}\overline{X}^{\epsilon_2\zeta_2}\overline{M}^{\zeta_2\beta_2}_{(0)},$$

where $\alpha(0)=\alpha_1(0)$. According to the connectedness properties discussed in Sect. 2. 3, one sees that the homogeneous part of the integral eqs. (3. 8) is fully connected (that is a_1-connected). The kernel of the $N^{\alpha_2\beta_2}$ equations coincides with the Yakubovskiĭ kernel, so that all the results concerning compactness apply equally to the kernel of the eqs. (3. 8). Furthermore the equivalence of the homogeneous equation system (which can be extracted from the system (3. 8a) or (3. 8b)) to the Schrödinger equation can be easily proved.

3. 2. - Expansion of the transition operators into connected components.

The generalized transition operators (2. 7) can be expanded in terms of two-sided components (2. 14)

(3. 9) $$U_{a_j b_k;a_i}=(1-\delta_{a_j b_k})(G_0^{-1}-\sum_{c_{N-1}\subseteq a_j;b_k} V_{c_{N-1}})+\sum_{\substack{c_{N-1}\not\subseteq a_j\\ d_{N-1}\not\subseteq b_k}} N_{a_i}^{c_{N-1}d_{N-1}}.$$

In order to separate in the eq. (3. 9) the terms having different graphic connectedness properties, we will derive some auxiliary formulas. Let us sum the eq. (3.5a) over b_{k-1} or the eq. (3. 5b) over a_{k-1}. Using the eqs. (3. 3), one obtains

(3. 10) $$\sum_{a_{k-1}b_{k-1}} N_{a_i}^{\alpha_{k-1}\beta_{k-1}}=N_{a_i}^{\alpha_k\beta_k}-N_{\alpha_i(0)}^{\alpha_k\beta_k}.$$

By applying this relation to less and less connected operators, up to $N_{\alpha_i(0)}^{a_{N-1}b_{N-1}}$, one gets

(3. 11) $$\sum_{\bar\alpha_{k-1}\bar\beta_{k-1}} N_{a_i}^{\alpha_{k-1}\beta_{k-1}}=N_{a_i}^{a_{N-1}b_{N-1}}-N_{\alpha_i(0)}^{a_{N-1}b_{N-1}}-\sum_{\bar\alpha_{N-2}\bar\beta_{N-2}} N_{\alpha_i(0)}^{\alpha_{N-2}\beta_{N-2}}-\dots-\sum_{\bar\alpha_k\bar\beta_k} N_{\alpha_i(0)}^{\alpha_k\beta_k};$$

where $\bar\alpha_j=(a_j, a_{j+1},\dots,a_{N-2})$. For $a_i=a_1$ one has :

(3. 12) $$N^{a_{N-1}b_{N-1}}=\sum_{n=3}^{N-1}\sum_{\bar\alpha_n\bar\beta_n} N_{\alpha(0)}^{\alpha_n\beta_n}+\sum_{\bar\alpha_2\bar\beta_2} N^{\alpha_2\beta_2}.$$

Let us consider the eq. (3. 9) for $a_i=a_1$. In virtue of the relation (3. 12), the N-particle scattering operator (2. 5) takes the form

(3. 13) $$T=\sum_{n=3}^{N-1}\sum_{\gamma_n\delta_n} N_{\alpha(0)}^{\gamma_n\delta_n}+\sum_{\gamma_2\delta_2} N^{\gamma_2\delta_2}.$$

Similarly the transition operators $U_{a_j b_k}$ become :

$$U_{a_j b_k}=(1-\delta_{a_j b_k})\Big(G_0^{-1}-\sum_{e_{N-1}\subseteq a_j,b_k} V_{e_{N-1}}\Big)+\sum_{n=3}^{N-1}\sum_{\substack{\gamma_n\delta_n\\ c_{N-1}\nsubseteq a_j\\ d_{N-1}\nsubseteq b_k}} N^{\gamma_n\delta_n}_{\alpha(0)}+\sum_{\substack{\gamma_2\delta_2\\ c_{N-1}\nsubseteq a_j\\ d_{N-1}\nsubseteq b_k}} N^{\gamma_2\delta_2}. \tag{3.14}$$

The eqs. (3. 13) and (3. 14) give a cluster decomposition of the transition operators into terms having a well-defined connectedness. Taking into account the integral equations (3. 8), one sees that the terms fully connected appear separated from the terms $N^{\gamma_n\delta_n}_{\alpha(0)}$ ($n=2,\ldots,N-1$) having lower connectedness.

4. - SINGLE-PARTICLE AND TWO-PARTICLE REARRANGEMENT PROCESSES. -

4. 1. - Single-particle rearrangement reactions in a three-body formulation.

Single-particle or single-cluster rearrangement processes can be represented schematically as :

$$a+(b+c)\rightarrow b+(a+c) \tag{4.1}$$

if one deals with a transfer or exchange reaction, or as :

$$a+(b+c)\rightarrow a+b+c \tag{4.2}$$

if one deals with a break-up reaction. Stripping processes (b+c, b), pick-up processes (a, a+c) and knock-out processes (a, b) enter in the first scheme (4. 1). If the particles or nuclear clusters a, b, c are assumed to act as inert units, the reactions (4. 1) and (4. 2) can be treated in a three-body context. In the usual notation for the three-body problem[23], one has from the general eq. (3. 14) :

$$U_{ba}=G_0^{-1}+\sum_{\substack{c\neq b\\ d\neq a}} N^{cd} \tag{4.3}$$

for processes (4. 1) and

$$U_{0a}=G_0^{-1}+\sum_{\substack{c\\ d\neq a}} N^{cd} \tag{4.4}$$

for processes (4. 2). The components N^{cd} satisfy the Faddeev equations (see eqs. (3.1) for N=3 and $a_i=a_1$). It is immediately seen that the transition operators U_{ba} and U_{0a} are constructed by means of quantities having a well-defined physical meaning in terms of reaction mechanisms. For instance, in correspondence to the inhomogeneous terms $T_c\,\delta_{cd}$ in the eqs. (3. 1) (N=3, $a_i=a_1$), we have :

$$U^{(0)}_{ba}=G_0^{-1}+T_c, \tag{4.5}$$

$$U^{(0)}_{0a}=G_0^{-1}+T_b+T_c. \tag{4.6}$$

Then, in the channel state representation $|\varphi_\alpha>$ $(\alpha=0,a,b,c;\ |\varphi_0>=|\bar{p}\bar{q}>)$ one gets on-the-energy-shell[3]:

(4.7) $$<\varphi_b|U^{(0)}_{ba}|\varphi_a> = <\varphi_b|V_b G_0 V_a|\varphi_a> + <\varphi_b|V_b G_0 T_c G_0 V_a|\varphi_a>,$$

(4.8) $$<\bar{p}\bar{q}|U^{(0)}_{0a}|\varphi_a> = <\bar{p}\bar{q}|V_a|\varphi_a> + <\bar{p}\bar{q}|T_b G_0 V_a|\varphi_a> + <\bar{p}\bar{q}|T_c G_0 V_a|\varphi_a>.$$

The eq. (4.7) gives the transition amplitude for the pole diagram describing the transfer of the particle c and for the triangle diagram describing in addition to the c-transfer, the a-b off-shell interaction (Fig. 1). The eq. (4.8) gives the transition amplitudes for the diagrams represented in Fig. 2. The shaded circles represent form factors for the channel bound states. The shaded squares represent two-body off-the-energy-shell (in Fig. 1) or half-off-the-energy-shell (in Fig. 2) scattering amplitudes.

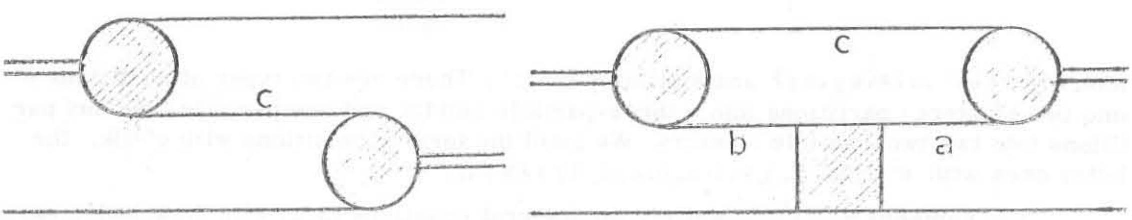

FIG. 1 - Pole and triangle diagrams for the transfer of the particle c.

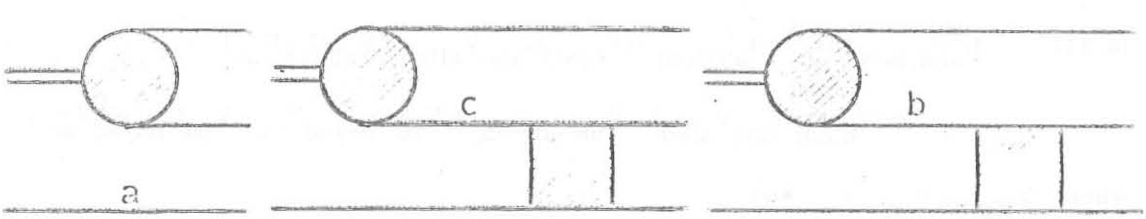

FIG. 2 - The simplest diagrams for break-up reactions.

Starting directly from integral equations for the three-body transition operators $U_{\beta\alpha}$ (Lovelace equations for the Alt-Grassberger-Sandhas operators) and once iterating them, one sees that the inhomogeneous term of such iterated equations (whose kernel is a Hilbert-Schmidt operator) coincides with the quantities (4.5) or (4.6). Within the framework of the Alt-Grassberger-Sandhas formulation the role of the pole and triangle diagrams in transfer and exchange nuclear reactions has been recently investigated[2, 3]. If multiple-rearrangement mechanisms are neglected (as it is reasonable in many cases), one can obtain from the exact theory a generalized distorted wave model which is constructed on the basis of the pole and triangle diagram mechanisms (not only on the simple pole mechanism).

As is well known, for a meaningful comparison between the contributions of the reaction mechanisms, one must take into account both the singularity positions and the vertex function magnitudes of the corresponding diagrams. It is worthwhile noting that for break-up reactions the kinematic features of the final state can affect considerably the position of the physical-region boundaries (at fixed energy) in the kinematic invariants.

4.2. - Mechanisms of two-particle transfer reactions in a four-body formulation.

A two-particle transfer reaction a(A, B)b can be represented schematically as

(4. 9) $$a+(b+c+d) \rightarrow b+(a+c+d)$$

where c and d are the particles transferred from the initial three-body bound state A = (b+c+d) to the final one B = (a+c+d). If the particles or nuclear clusters a, b, c, d are assumed to act as inert entities, the reaction (4. 9) can be treated in a four-body context. The process (4. 9) is described by the on-the-energy-shell matrix elements of the transition operator $U_{acd,bcd}$. From the eq. (3. 14) one obtains :

(4. 10) $$U_{acd,bcd}=G_0^{-1}-V_{cd}+T_{ab}+\sum_{\substack{\varepsilon'\zeta' \\ e\nsubseteq acd \\ z\nsubseteq bcd}} N^{\varepsilon'\zeta'}$$

where $\varepsilon'=(e',e)=(e_2,e_3)$ and similarly for ζ'. There are two types of partitions e' into two clusters : partitions into a three-particle cluster and one particle free and partitions into two two-particle clusters. We label the former partitions with e'=ijk, the latter ones with e'=ij/kl (i, j, k, l=a, b, c, d; $i\neq j\neq k\neq l$).

The components $N^{\varepsilon'\zeta'}$ satisfy the integral equations (3. 2) with N=4 and $a_i=a_1$. The simplest relevant reaction mechanisms can be extracted from the eq. (4. 10) by replacing $N^{\varepsilon'\zeta'}$ with its e'-connected inhomogeneous term $N^{ez}_{e'}-T_e\delta_{ez}$. One gets

(4. 11) $$U^{(0)}_{acd,bcd}=G_0^{-1}+T_{ab/(cd)}+(V_{b(ac)}G_{abc}V_{a(bc)}-V_{ab}G_{ab}V_{ab})+$$
$$+(V_{b(ad)}G_{abd}V_{a(bd)}-V_{ab}G_{ab}V_{ab})+V_{bc}G_{bc/ad}V_{ad}+V_{bd}G_{bd/ac}V_{ac},$$

where $V_{i(jk)}=V_{ijk}-V_{jk}$ and

(4. 12) $$T_{ij/(kl)}=V_{ij}-V_{ij}G_{ij/kl}V_{ij}.$$

$T_{ij/(kl)}$ represent the two-body transition operators for the ij-subsystem (disconnected from the subsystem of interacting particles k and l) in the four-body space. They are of the type $U_{ee;e'}$ with e' = ij/kl and e = kl. Note that the subtraction of the quantity $V_{ab}G_{ab}V_{ab}$ in the third and fourth term of the right hand side of the eq. (4. 11) assures the abc or abd-connectedness of such terms.

Using the on-the-energy-shell relations[26]:

(4. 13) $$|\varphi_{ijk}\rangle=G_0V_{ijk}|\varphi_{ijk}\rangle, \quad |\varphi_{ijk}\rangle=G_{jk}V_{i(jk)}|\varphi_{ijk}\rangle$$

for the channel states $|\varphi_{ijk}\rangle$, one obtains :

(4. 14) $$\langle\varphi_{acd}|(G^{-1}_{cd}+T_{ab/(cd)})|\varphi_{bcd}\rangle=$$
$$=\langle\varphi_{acd}|V_{a(cd)}(G_{cd}+G_{cd}T_{ab/(cd)}G_{cd})V_{b(cd)}|\varphi_{bcd}\rangle,$$

(4.15a) $<\varphi_{acd}|V_{b(ax)}G_{abx}V_{a(bx)}|\varphi_{bcd}>=$

$=<\varphi_{acd}|V_{y(ax)}G_{ax}V_{b(ax)}G_{abx}V_{a(bx)}G_{bx}V_{y(bx)}|\varphi_{bcd}>,$

(4.15b) $<\varphi_{acd}|V_{ab}G_{ab}V_{ab}|\varphi_{bcd}>=<\varphi_{acd}|V_{acd}G_0V_{ab}G_{ab}V_{ab}G_0V_{bcd}|\varphi_{bcd}>,$

(4.16) $<\varphi_{acd}|V_{by}G_{by/ax}V_{ax}|\varphi_{bcd}>=<\varphi_{acd}|V_{y(ax)}G_{ax}V_{by}G_{by/ax}V_{ax}G_{by}V_{x(by)}|\varphi_{bcd}>,$

with $x, y = c, d$ $(x \neq y)$ in the eqs. (4.15a) and (4.16).

From the eq. (4.14) one sees that the first two terms in (4.11) give the transition amplitude for the pole diagram describing the transfer of the subsystem of the interacting particles c and d and for the triangle diagram describing in addition to such a transfer the a-b off-shell interaction (Fig. 3). If the particles c and d are transferred in one of their bound states, one obtains again the basic mechanisms for single-cluster transfer processes. The well-known plane-wave theory for two-nucleon transfer reactions corresponds to the first term in the formula (4.14).

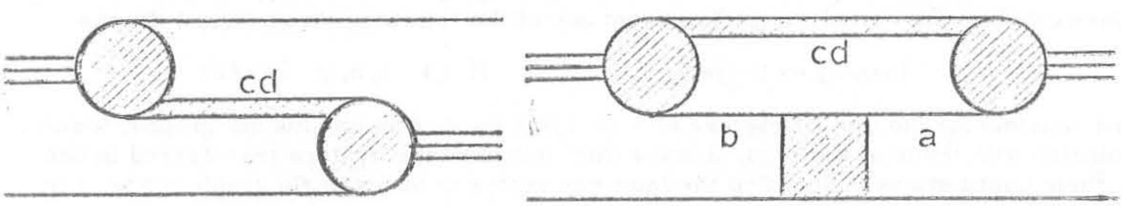

FIG. 3 - Pole and triangle diagrams for the transfer of the cd-subsystem.

The terms in bracket in the eq. (4.11) are represented by quadrangle diagrams in which the particle y is directly transferred from the initial to the final state and the particle x interacts successively with a, ab and b (Fig. 4a). Because of the abx-

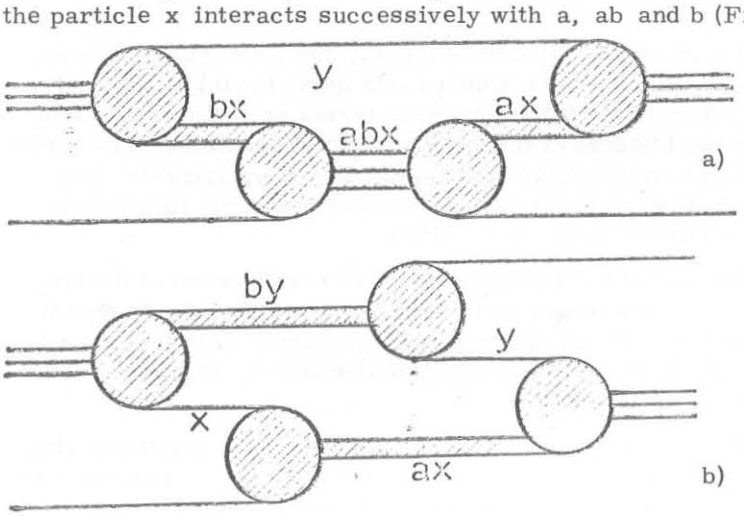

FIG. 4 - Quadrangle diagrams describing the transfer of the particles x and y (x, y=c, d; x≠y) and involving a) three-particle intermediate states ; b) two two-particle intermediate states.

connectedness of these terms, the particle x cannot freely propagate in the intermediate state. The abx-subsystem can propagate in one of its three-particle bound states.

The two last terms in the eq. (4. 11) give the amplitudes of quadrangle diagrams describing the successive and independent transfer of the particles x and y; the particle x is transferred while b and y interact and the particle y is transferred while a and x interact (Fig. 4b).

Details on the form factors, describing the vertices in the diagrams of the Figss. 3 and 4, can be found in refs. (26, 27).

If the resolvent operators G_α for the α -subsystems are approximated in the eqs. (4. 14)-(4. 16) by their dominating bound state separable parts, the graphs in Figs. 3 and 4 coincide with the simplest graphs for two-particle transfer processes which can be obtained in the non-relativistic Feynman-diagram approach to direct reactions proposed by Shapiro[(4)]. Obviously, in the above four-body context we cannot investigate diagrams with internal lines corresponding to virtual particles different from a, b, c, d. Within the framework of the Shapiro approach quadrangle diagrams describing the successive transfer of two nucleons (see Fig. 4b in correspondence to a bound state both of the ax-subsystem and of by-subsystem) have been proposed in the study of the (t, p) and (τ, p) reactions[(28-30)].

Finally, let us outline that the physical contents of the eq. (4. 11) is in a direct correspondence with an intuitive description of two-particle transfer processes. In fact, from an intuitive point of view, the simplest graphs describing two-particle transfers can be constructed by starting from one of the following three virtual decays

$$(a+b+c) \rightarrow i+(j+k) \qquad (i, j, k = a, b, c; \; i \neq j \neq k)$$

and considering the virtual capture of i or (j+k) by d. One obtains six graphs, which coincide with those of the Figs. 3 and 4 (for intermediate clusters transferred in one of their bound states). Note that the four-ray vertex of the triangle graph in Fig. 3 takes the form of a pole graph, in correspondence to a bound state separable term of $T_{ab/(cd)}$.

5. - CONCLUSIONS. -

Within the framework of a rigorous N-particle theory, the transition operators for nuclear rearrangement processes have been splitted into parts having a well-defined graphic connectedness. The N-particle fully connected terms have the same structure of the homogeneous part of the Faddeev-Yakubovskiǐ integral equations. The terms with a lower connectedness are well-interpreted in terms of Feynman diagram basic rearrangement mechanisms. It follows that our approach gives a support to the non-relativistic Feynman diagram technique proposed by Shapiro.

We have started from the nonstandard off-energy-shell continuation of the transition operators suggested by Alt, Grassberger and Sandhas and we have found that it leads to a physicslly transparent formulation of the N-body problem. This fact shows the advantage of the above special choice for the transition operators, which has not yet been sufficiently appreciated in the literature.

Practical applications of the proposed formulation depend on the progress which will be made in solving Yakubovskiǐ equations. A first attempt in the four-body case has been recently tried[(31)]. However, it is wothwhile noting that, in virtue of the well-defi ned physical picture of the general formulas, some meaningful approximations can be easily proposed.

Finally, we notice that, in the context of the suggested approach, one can get some insight into many-particle heavy-ion transfer processes, for which accurate experimental data can be now obtained using modern Tandem Van de Graaff accelerators.

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APPENDIX A. -

We shall prove here, by induction, that the operators $N_{\alpha_i(0)}^{\alpha_{k-1}\beta_{k-1}}$ have connectednes a_{k-1} $(3 \le k \le N)$.

By definition the operators $N_{\alpha_i(0)}^{\alpha_{N-1}\beta_{N-1}}$ are a_{N-1}-connected. It is immediately seen that the operators $N_{\alpha_i(0)}^{\alpha_{N-2}\beta_{N-2}}$ are a_{N-2}-connected both for $a_{N-1} \neq b_{N-1}$ and for $a_{N-1} = b_{N-1}$; in this latter case one has $(a_{N-2} = a'; \ a_{N-1} = a)$

$$N_{a'}^{aa} - T_a = V_a G_a (V_{a'} - V_a) G_{a'} V_a . \tag{A.1}$$

Let us consider $N_{\alpha_i(0)}^{\alpha_{k-1}\beta_{k-1}}$ for $3 \le k \le N-2$ and let us distinguish the case $a_k = b_k$ from the case $a_k \neq b_k$. For $a_k = b_k$ one has

$$N_{a_{k-1}}^{\alpha_k, a_k \beta_{k+1}} - N_{\alpha_i(0)}^{\alpha_k, a_k \beta_{k+1}} = \sum_{(\gamma\delta\varepsilon\zeta)_{k+1}} M_{a_i(0)}^{\alpha_{k+1}\gamma_{k+1}} X_{a_k}^{\gamma_{k+1}\delta_{k+1}} \left(N_{a_{k-1}}^{\delta_{k+1}\varepsilon_{k+1}} - N_{a_k}^{\delta_{k+1}\varepsilon_{k+1}}\right) \bar{X}_{a_k}^{\varepsilon_{k+1}\zeta_{k+1}} \bar{M}_{a_i(0)}^{\zeta_{k+1}\beta_{k+1}} . \tag{A.2}$$

From the eq. (2.15) one obtains the following decomposition for $N_{a_{k-1}}^{\delta_{k+1}\varepsilon_{k+1}}$

$$\begin{aligned} N_{a_{k-1}}^{\delta_{k+1}\varepsilon_{k+1}} = {} & N_{\alpha_i(0)}^{\delta_{k+1}\varepsilon_{k+1}} - \sum_{(\gamma\delta\varepsilon\zeta)^1_{k+2}} (MX)^{k+2} N_{d_{k+1}}^{\delta^1_{k+2}\varepsilon^1_{k+2}} (\bar{X}\bar{M})^{k+2} \delta_{d_{k+1} e_{k+1}} + \\ & + \sum_{\substack{(\gamma\delta\varepsilon\zeta)^1_{k+2} \\ d^1_{k+2} \subset d_{k+1} \\ e^1_{k+2} \subset e_{k+1}}} (MX)^{k+2} \left\{ N_{\alpha_i(0)}^{\delta^1_{k+2}\varepsilon^1_{k+2}} - \sum_{(\gamma\delta\varepsilon\zeta)^2_{k+3}} (MX)^{k+3} N_{d^1_{k+2}}^{\delta^2_{k+3}\varepsilon^2_{k+3}} (\bar{X}\bar{M})^{k+3} \delta_{d^1_{k+2} e^1_{k+2}} \right\} (\bar{X}\bar{M})^{k+2} + \\ & + \dots + \sum_{\substack{(\gamma\delta\varepsilon\zeta)^1_{k+2} \\ d^1_{k+2} \subset d_{k+1} \\ e^1_{k+2} \subset e_{k+1}}} (MX)^{k+2} \dots \sum_{\substack{(\gamma\delta\varepsilon\zeta)^{N-k-2}_{N-1} \\ p \subset d^{N-k-3}_{N-2} \\ q \subset e^{N-k-3}_{N-2}}} (MX)^{N-1} (V_p \delta_{pq} + V_p G_{a_{k-1}} V_q) (\bar{X}\bar{M})^{N-1} \dots (\bar{X}\bar{M})^{k+2} \end{aligned} \tag{A.3}$$

where $(\gamma\delta\varepsilon\zeta)^i_{k+1} = \gamma^i_{k+1} \delta^i_{k+1} \varepsilon^i_{k+1} \zeta^i_{k+1}$ and

$$(MX)^{k+j} = M_{a_i(0)}^{\delta^{j-2}_{k+j} \gamma^{j-1}_{k+j}} X_{d^{j-2}_{k+j-1}}^{\gamma^{j-1}_{k+j} \delta^{j-1}_{k+j}} ;$$

$$(\bar{X}\bar{M})^{k+j} = \bar{X}_{e^{j-2}_{k+j-1}}^{\varepsilon^{j-1}_{k+j} \zeta^{j-1}_{k+j}} \bar{M}_{a_i(0)}^{\zeta^{j-1}_{k+j} \varepsilon^{j-2}_{k+j}} ;$$

$$p = d^{N-k-2}_{N-1} , \quad q = e^{N-k-2}_{N-1}$$

with $2 \le j \le N\text{-}k\text{-}1$; $\delta^{0}_{k+2} = \delta_{k+2}$, $\varepsilon^{0}_{k+2} = \varepsilon_{k+2}$. By extracting G_{a_k} from $G_{a_{k-1}}$ one gets from the eq. (A. 3), because p, $q \subset a_k$,

(A. 4)
$$-N^{\delta_{k+1}\varepsilon_{k+1}}_{a_{k-1}} = N^{\delta_{k+1}\varepsilon_{k+1}}_{a_k} + \sum_{\substack{(\gamma\delta\varepsilon\zeta)^1_{k+2} \\ d^1_{k+2} \subset d_{k+1} \\ e^1_{k+2} \subset e_{k+1}}} (MX)^{k+2} \dots$$

$$\dots \sum_{\substack{(\gamma\delta\varepsilon\zeta)^{N-k-2}_{N-1} \\ p \subset d^{N-k-3}_{N-2} \\ q \subset e^{N-k-3}_{N-2}}} (MX)^{N-1} (V_p G_{a_k} V_{a_{k-1}/a_k} G_{a_{k-1}} V_q)(\bar{X}\bar{M})^{N-1} \dots (\bar{X}\bar{M})^{k+2}$$

where $V_{a_i/a_{i+1}} = V_{a_i} - V_{a_{i+1}}$. Introducing this result in the eq. (A. 2) and taking into account the identity[20]

(A. 5)
$$\sum_{\substack{(\gamma\delta)_{k+1} \\ d_{k+1} \subset a_k}} M^{\alpha_{k+1}\gamma_{k+1}}_{a_i(0)} X^{\gamma_{k+1}\delta_{k+1}}_{a_i} \dots \sum_{\substack{(\gamma\delta)^{N-k-2}_{N-1} \\ p \subset d^{N-k-3}_{N-2}}} M^{\delta^{N-k-3}_{N-1}\gamma^{N-k-2}_{N-1}}_{a_i(0)} X^{\gamma^{N-K-2}_{N-1}\delta^{N-k-2}_{N-1}}_{a_i} V_p =$$

$$= V_{a_{N-1}} G_{a_{N-1}} V_{a_{N-2}/a_{N-1}} G_{a_{N-2}} \dots V_{a_k/a_{k+1}}$$

and a similar identity for the right-hand-sided Yakubovskiǐ operators, one obtains

(A. 6)
$$N^{\alpha_k, a_k\beta_{k+1}}_{a_{k-1}} - N^{\alpha_k, a_k\beta_{k+1}}_{\alpha_i(0)} = V_{a_{N-1}} G_{a_{N-1}} V_{a_{N-2}/a_{N-1}} G_{a_{N-2}} \dots$$

$$\dots V_{a_k/a_{k+1}} G_{a_k} V_{a_{k-1}/a_k} G_{a_{k-1}} V_{b_k/b_{k+1}} G_{b_{N-2}} V_{b_{N-2}/b_{N-1}} G_{b_{N-1}} V_{b_{N-1}}.$$

From this equation one sees that the operator $N^{\alpha_{k-1}, a_{k-1}a_k\beta_{k+1}}_{\alpha_i(0)}$ is a_{k-1}-connected.

For $a_k \neq b_k$, one has

(A. 7)
$$N^{\alpha_k\beta_k}_{a_{k-1}} - N^{\alpha_k\beta_k}_{\alpha_i(0)} = \sum_{\substack{(\gamma\delta\varepsilon\zeta)_{k+1} \\ d_{k+1} \subset a_k \\ e_{k+1} \subset b_k}} M^{\alpha_{k+1}\gamma_{k+1}}_{a_i(0)} X^{\gamma_{k+1}\delta_{k+1}}_{a_{k+1}} N^{\delta_{k+1}\varepsilon_{k+1}}_{a_{k-1}} \bar{X}^{\varepsilon_{k+1}\zeta_{k+1}}_{a_{k-1}} \bar{M}^{\zeta_{k+1}\beta_{k+1}}_{a_i(0)}$$

By means of suitable splittings of the sums over e^{i-1}_{k+i} $(1 \le i \le N\text{-}k\text{-}2)$ in the eqs. (A. 7) and (A. 3), one obtains (with obvious meaning for $(MX)^{k+1}$ and $(\bar{X}\bar{M})^{k+1}$:

(A. 8)
$$N^{\alpha_k\beta_k}_{a_{k-1}} - N^{\alpha_k\beta_k}_{\alpha_i(0)} = \sum_{\substack{(\gamma\delta\varepsilon\zeta)_{k+1} \\ d_{k+1} \subset a_k, b_k}} (MX)^{k+1} N^{\delta_{k+1}\varepsilon_{k+1}}_{\alpha_i(0)} (\bar{X}\bar{M})^{k+1} -$$

$$- \sum_{\substack{(\gamma\delta\varepsilon\zeta)_{k+1} \\ d_{k+1} \subset a_k, b_k}} (MX)^{k+1} \sum_{\substack{(\gamma\delta\varepsilon\zeta)^1_{k+2} \\ d^1_{k+2}, e^1_{k+2} \subset d_{k+1}}} (MX)^{k+2} N^{d_{k+1}\delta^1_{k+2}, e_{k+1}\varepsilon^1_{k+2}}_{\alpha_i(0)} (\bar{X}\bar{M})^{k+2} (\bar{X}\bar{M})^{k+1} +$$

$$+\sum_{\substack{(\gamma\delta\varepsilon\zeta)_{k+1}\\ d_{k+1}\subset a_k\\ e_{k+1}\subset b_k\\ d_{k+1}\neq e_{k+1}}} (MX)^{k+1} \sum_{\substack{(\gamma\delta\varepsilon\zeta)^1_{k+2}\\ d^1_{k+2}\subset d_{k+1}\\ e^1_{k+2}\subset e_{k+1}}} (MX)^{k+2} N_{\alpha_i(0)}^{\delta^1_{k+2}\varepsilon^1_{k+2}} (\bar{X}\bar{M})^{k+2}(\bar{X}\bar{M})^{k+1}+\ldots+$$

$$+\sum_{\substack{(\gamma\delta\varepsilon\zeta)_{k+1}\\ d_{k+1}\subset a_k\\ e_{k+1}\subset b_k}} (MX)^{k+1}\ldots \sum_{\substack{(\gamma\delta\varepsilon\zeta)^{N-k-2}_{N-1}\\ p\subset d^{N-k-3}_{N-2}\\ q\subset e^{N-k-3}_{N-2}}} (MX)^{N-1} V_p G_p \sum_{r\neq p} N^{rq}_{a_{k-i}} (\bar{X}\bar{M})^{N-1}\ldots(\bar{X}\bar{M})^{k+1}.$$

The integral equations (3. 1) for $N^{pq}_{a_{k-1}}$ has been used in order to derive the eq. (A. 8). The a_{k-1}-connectedness of all the terms in the eq. (A. 8) follows immediately from the inductive assumption and from the fact that every term in the sum

$$\text{(A. 9)}\qquad A^{\alpha_k}_{a_i} = \sum_{\substack{\gamma_{k+1}\delta_{k+1}\\ d_{k+1}\subset a_k}} M^{\alpha_{k+1}\gamma_{k+1}}_{a_i(0)} X^{\gamma_{k+1}\delta_{k+1}}_{a_i} A^{\delta_{k+1}}_{a_i}$$

is a_k-connected if $A^{\delta_{k+1}}_{a_i}$ is d_{k+1}-connected. The eq. (A. 9) is the well-known recurrence relation for the left-hand sided components of the operators (2. 6)[19, 20, 25]. We notice that, owing to the severe restrictions one imposed on the indices of the sums in the eq. (4. 8), several terms vanish.

APPENDIX B. -

Let us derive, from the Faddeev-like equations (3. 1) for $N^{ab}_{a_i}$, the integral equations (3. 2) for $N^{\alpha'\beta'}_{a_i}$.

By using the eqs. (3. 1), the operator $N^{\alpha'\beta'}_{a_i}$, defined by eq. (2. 15) for k=N-1, can be written in the form

(B. 1) $$N^{\alpha'\beta'}_{a_i}=T_aG_0T_b(1-\delta_{ab})\delta_{a'b'}+T_aG_0\sum_{\substack{d\neq a,\subset a'\\ e\neq b,\subset b'}} N^{de}_{a_i}G_0T_b .$$

From the well-known properties of the partitions and from the eqs. (3. 1) it follows that :

(B. 2) $$\sum_{a'\supset a} N^{\alpha'\beta'}_{a_i}=\sum_{e\neq b,\subset b'} N^{ae}_{a_i}G_0T_b .$$

Using this relation in the eq. (B. 1) and extracting the terms with d'=a', one obtains :

(B. 3) $$N^{\alpha'\beta'}_{a_i}-T_aG_0\sum_{d\neq a} N^{a',d\beta'}_{a_i}=T_aG_0T_b(1-\delta_{ab})\delta_{a'b'}+T_aG_0\sum_{\substack{d\neq a,\subset a'\\ d'\neq a'}} N^{\delta'\beta'}_{a_i} .$$

If one takes into account the eqs. (2. 17), (2. 18) and (3. 1), one can easily show that :

(B. 4) $$T_aG_0T_b(1-\delta_{ab})\delta_{a'b'}=N^{\alpha'\beta'}_{\alpha_i(0)}-T_aG_0\sum_{d\neq a} N^{a',d\beta'}_{\alpha_i(0)},$$

(B. 5) $$T_aG_0\sum_{\substack{d\neq a,\subset a'\\ d'\neq a'}} N^{\delta'\beta'}_{a_i}=\sum_{\substack{c;e'\neq a'\\ e\neq c,\subset a'}} N^{ac}_{a'}G_0N^{\varepsilon'\beta'}_{a_i}-T_aG_0\sum_{d\neq a}\sum_{\substack{c;e'\neq a'\\ e\neq c,\subset a'}} N^{dc}_{a'}G_0N^{\varepsilon'\beta'}_{a_i} .$$

From the eqs. (B. 3)-(B. 5) one can immediately derive the integral equations (3. 2a). Similarly one obtains the eqs. (3. 2b).

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