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L. Lovitch and S. Rosati : BINDING ENERGIES AND WAVE FUNCTIONS OF THREE BOSONS INTERACTING THROUGH LOCAL POTENTIALS. -

330

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ABSTRACT. -

Results obtained using a variational calculation are presented for the three lowest bound states of a three identical boson system interacting through local two-body potentials. The ground state energy is also calculated for the best trial function of a product form. These results may be used for comparison in attempts to exactly integrate the Schrödinger equation of the system.

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Interest in the problem of obtaining the binding energy of a non-relativistic three-body system, whose component particles interact through attractive central two-body potentials, has gained added impetus due to the efforts of several authors to integrate the Fadeev equations even in the case of non-separable potentials[1,2]. Particularly worthy of attention is the unpublished work of Osborn[2] where the Fadeev equations are solved numerically under the assumption that the $l \neq 0$ states of the two-body subsystems do not contribute appreciably to the formation of the three-body system. The local two-body potentials considered in ref. (2) are of the exponential form

(1a) $$V(r) = -\lambda \frac{\hbar^2}{M} e^{-\sigma r} \qquad \text{with} \quad \sigma = 1.49 \text{ fm}^{-1},$$

and of the Yukawa form

$$(1b) \qquad V(r) = -\lambda \frac{\hbar^2}{M} \frac{e^{-\sigma r}}{\sigma r} \qquad \text{with} \quad \sigma = 0.633 \text{ fm}^{-1}.$$

The values of the range parameters σ were taken so as to fit the low energy neutron-proton triplet scattering. In the case of the exponential potential Osborn finds that as λ increases one first of all reaches the threshold value for a bound state of the three boson system. Subsequently, one finds both ground and first excited bound states of the system, while increasing λ still more the next level encountered is the two-body bound state. However, going to still higher values of λ a second excited bound state appears which is stable with respect to the emission of a particle. Furthermore as λ varies all these energies have quite a smooth behaviour.

For the Yukawa potential, on the other hand, he finds a striking behaviour for $\lambda \approx 1.6$ fm^{-2}. As λ increases through this value the rate of binding of the ground state jumps to seven times the preceeding value. At the same time the first excited state slides down and then assumes a path which is just a linear continuation of the first portion of the ground state trajectory, while the second excited state continues along a linear extension of the original first excited state trajectory.

In the belief that this anomalous behaviour for the Yukawa potential is due to a spurious behaviour of the solution in that region, and that the results of the binding energy for $\lambda \gtrsim 1.6$ fm^{-2}, which Osborn claims to correspond to the first and second excited states of the three-body system, are really the results for the ground and first excited states of the system, we decided to perform a variational analysis of the first three bound states. The purpose of this work is to report a set of numerical results which may be used for comparison in future attempts to integrate the Schrödinger equation of the three-body system exactly.

Since initiating this work a variational analysis of the system was published[3] where the trial wave function was chosen of the form

$$(2) \qquad \psi = \exp\left\{-\alpha(r_1+r_2+r_3)\right\} \sum'_{\substack{l+m+n<N \\ l \geq m \geq n}} C_{l,m,n}\, S(r_1^l r_2^m r_3^n),$$

where r_1, r_2, r_3 are the three interparticle distances and $S(r_1^l r_2^m r_3^n)$ is the symmetrical form of $r_1^l r_2^m r_3^n$. The requirement that the mean value of the total energy is stationary with respect to variations of the coefficients $C_{l,m,n}$ leads to a matrix equation whose eigenvalues correspond to the energy values of the system. The parameter α was then varied until the minimum of the ground state energy was reached, and by examining the eigenvalues as a function of N a feeling for the convergence to the best

trial function of the form of eq. (2) was obtained.

Our parametrization of the wave function is quite different. For describing the ground state we adopt the function

$$\psi = f(r_1)\, f(r_2)\, f(r_3) \tag{3}$$

with

$$f(r) = e^{-\alpha_o r} + x_o\, e^{-\beta_o r}, \tag{4}$$

and vary the parameters until a minimum in the energy ($E = \varepsilon\hbar^2/M$) is obtained for each value of interest; a similar calculation has been performed by Kok[4] - our values for α_o, β_o and x_o differ from his and we obtain slighly improved values for ε_o. We also calculate numerically the energy ε_f corresponding to the best variational function of the form (3) following the method presented in ref. (5) and, furthermore, a lower bound ε_L to the ground state energy using the prescription of Hall and Post[6]. The values of the parameters α_o, β_o, x_o together with ε_o, ε_f and ε_L are presented in Table Ia and Ib for the exponential and Yukawa potentials respectively (see also Figs. 1 and 2).

The results obtained for the Yukawa potential are in agreement with those of refs. (3,4) and do not manifest the collapse of the ground state found by Osborn[2] when integrating the Fadeev equations in the region of $\lambda \approx 1.6$ fm^{-2}. In fact, the curve of the wave number $k = \sqrt{-\varepsilon}$ versus λ is almost precisely linear for the ground state, throughout the region of values examined. For the first and second excited states we adopted for our trial wave function the form

$$\psi = f(r_1)\, f(r_2)\, g(r_3) + f(r_1)\, g(r_2)\, f(r_3) + g(r_1)\, f(r_2)\, f(r_3), \tag{5}$$

where f(r) is defined as in (4) and

$$g(r) = e^{-\gamma r} + y\, e^{-\delta r} + z\, e^{-\varrho r}. \tag{6}$$

In the case of the first excited state α, β, x, γ, δ, ϱ and z were taken as the free trial parameters in minimizing the total energy, while y was calculated so as to make the function (5) orthogonal to the ground state wave function found previously. The computations were performed with double precision arithmetic on the IBM 7090 of Pisa University. The results are given in Tables IIa and IIb. From an examination of the parameters of the resulting wave function we see that f(r) does not contain any

node whereas g(r) contains one node.

In the case of the second excited state $\alpha, \beta, x, \gamma, \delta, \varepsilon$ were taken as the free trial parameters while y and z were determined so as to make the wave function orthogonal to those of both the ground and first excited states. The results are given in Tables IIIa and IIIb where the two-body binding energy $\mathcal{E}_o^{(2)}$ is also presented. Examining the form of the wave function for this state (see eqs. (5), (4), (6)) we find on examining tables IIIa and IIIb that the function f has no node while g has two nodes in this case. Thus the possibility of having one node in f and no node in g is ruled out for energetic reasons.

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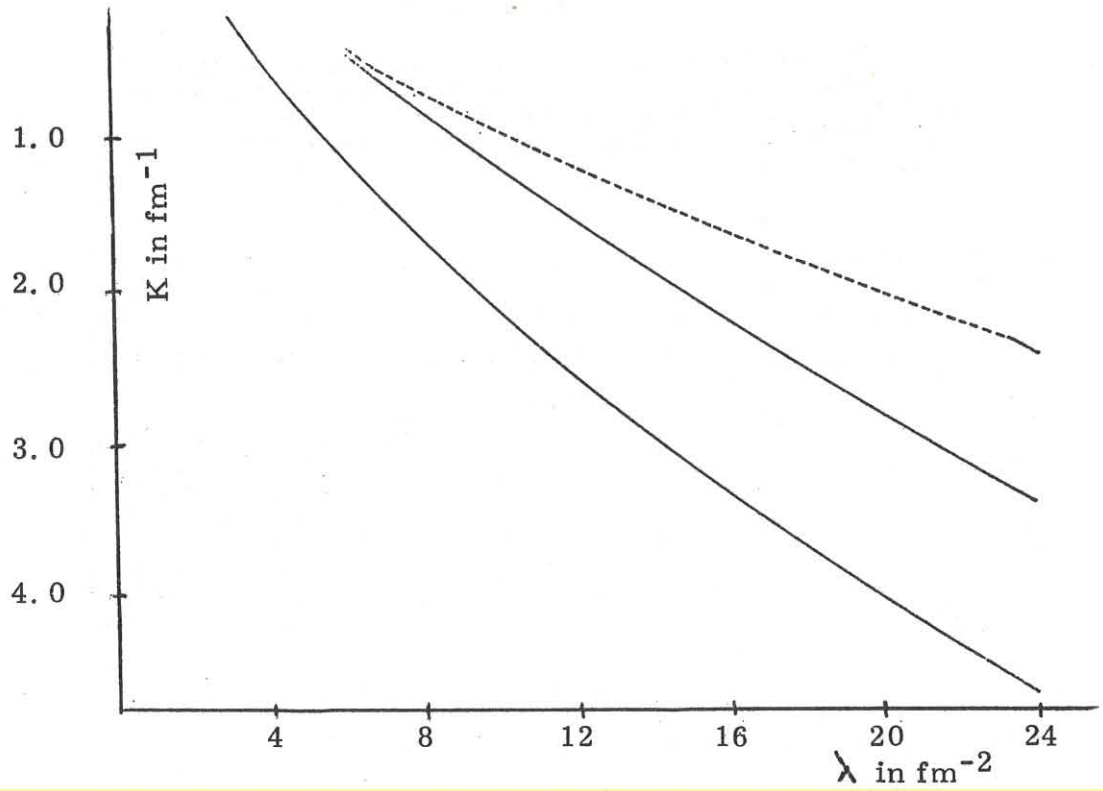

FIG. 1 - Wave number of the three boson system versus the coupling constant λ for the exponential potential. The dashed part of the curve for the second excited level corresponds the two-body bound state.

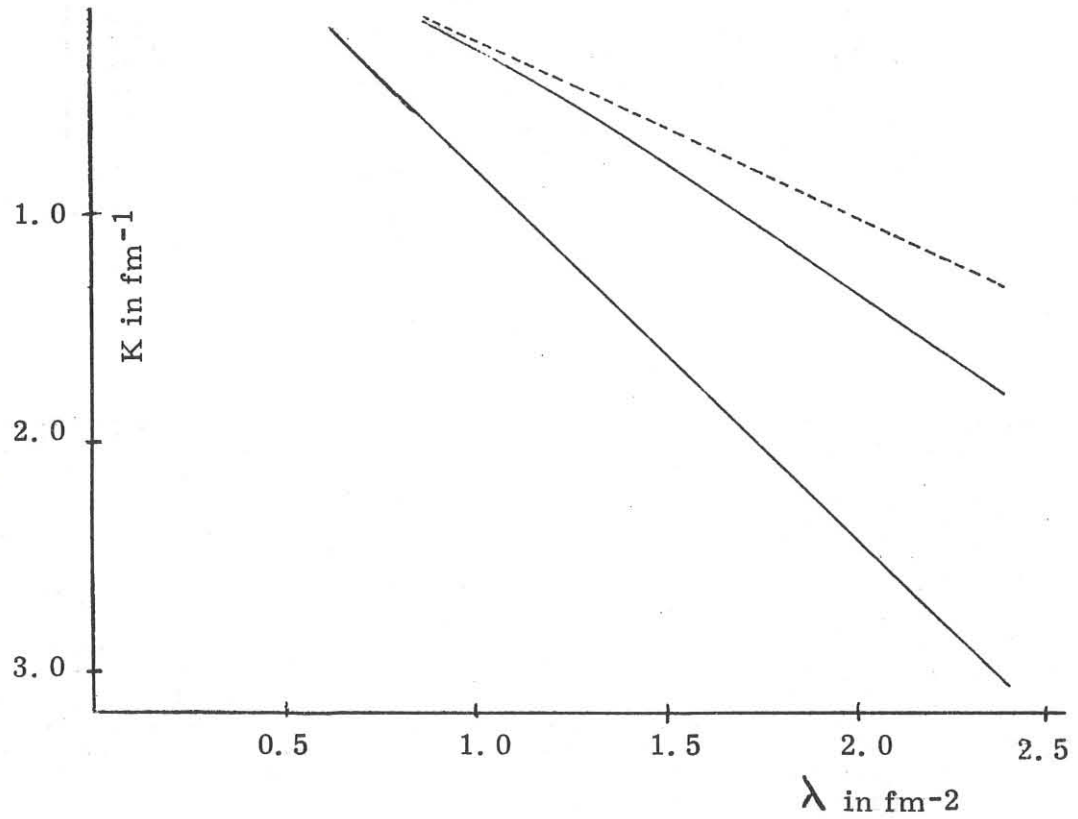

FIG. 2 - As in Fig. 1 for the Yukawa potential.

TABLE Ia

Parameters of the three particle ground state wave function for the exponential potential with the corresponding values of the coupling constant λ and energy ε_o. The energy ε_f corresponding to the best wave function of product form and the lower limit ε_L obtained from the Hall and Post prescription are also listed.

λ (fm^{-2})	α_o (fm^{-1})	β_o (fm^{-1})	x_o	ε_o (fm^{-2})	ε_f (fm^{-2})	ε_L (fm^{-2})
24	2.783	2.891	-0.9680	-21.28	-21.28	-21.40
20	2.533	2.637	-0.9651	-15.93	-15.94	-16.05
18	2.346	2.546	-0.9291	-13.40	-13.41	-13.51
14	1.851	2.528	-0.7494	-8.687	-8.687	-8.786
12	1.598	2.550	-0.6460	-6.542	-6.543	-6.635
10	1.344	2.607	-0.5382	-4.578	-4.578	-4.664
8	1.095	2.681	-0.4341	-2.837	-2.838	-2.915
6	0.8344	2.868	-0.3229	-1.388	-1.389	-1.457
4	0.5427	3.813	-0.1932	-0.3490	-0.3512	-0.4048
3	0.3635	6.120	-0.1103	-0.04917	-0.05695	-0.09698

TABLE Ib

As in table Ia for the Yukawa potential.

λ (fm^{-2})	α_o (fm^{-1})	β_o (fm^{-1})	x_o	ε_o (fm^{-2})	ε_f (fm^{-2})	ε_L (fm^{-2})
2.4054	1.666	2.078	0.9292	-9.320	-9.321	-10.09
1.899	1.304	1.881	0.4099	-4.984	-4.984	-5.463
1.6458	1.120	1.840	0.2851	-3.325	-3.325	-3.683
1.3926	0.8850	1.519	0.4418	-2.004	-2.004	-2.259
1.1394	0.6678	1.292	0.5245	-1.019	-1.019	-1.187
0.8862	0.4292	0.9736	0.8338	-0.3658	-0.3658	-0.4633
0.633	0.1910	0.6565	1.351	-0.03742	-0.03751	-0.07855

TABLE IIa

Parameters of the first excited state wave function together with the corresponding values of the coupling constant λ and energy ε_1 for the exponential potential.

λ (fm^{-2})	α_1 (fm^{-1})	β_1 (fm^{-1})	x_1	γ_1 (fm^{-1})	δ_1 (fm^{-1})	δ_1 (fm^{-1})	y_1	z_1	ε_1 (fm^{-2})
24	2.500	3.168	-0.8089	1.082	1.289	3.759	-1.174	0.1348	-11.26
20	2.070	3.167	-0.6707	0.9739	1.360	2.966	-1.441	0.3691	-7.825
18	1.828	3.439	-0.5599	0.9377	1.208	2.849	-1.304	0.2495	-6.262
14	1.439	3.996	-0.4159	0.6470	1.193	2.359	-1.891	0.7413	-3.521
12	1.306	4.047	-0.3873	0.5528	0.6109	3.291	-1.062	0.03655	-2.376
10	1.092	4.710	-0.3134	0.4522	0.6342	2.538	-1.243	0.1521	-1.427
8	0.9011	5.352	-0.2556	0.1884	0.5503	2.970	-1.617	0.3007	-0.6907
6	0.6758	6.663	-0.1821	-0.08508	0.6635	2.642	-3.487	1.129	-0.2078

TABLE IIb

As in table IIa for the Yukawa potential.

λ (fm^{-2})	α_1 (fm^{-1})	β_1 (fm^{-1})	x_1	γ_1 (fm^{-1})	δ_1 (fm^{-1})	ϱ_1 (fm^{-1})	y_1	z_1	ε_1 (fm^{-2})
2.4054	1.197	2.529	1.376	0.7907	0.9677	2.795	-1.150	0.07258	-3.157
1.899	0.8212	1.869	2.075	0.4479	0.9395	1.841	-1.748	0.4374	-1.452
1.6458	0.6445	1.559	2.578	0.3000	0.8759	1.410	-2.333	0.8680	-0.8580
1.3926	0.4755	1.276	3.168	0.1965	0.7056	1.135	-2.424	0.8464	-0.4310
1.1394	0.3256	0.9981	3.752	0.08484	0.4908	0.9374	-2.151	0.3299	-0.1613
0.8862	0.1844	0.7200	4.246	0.00045	0.3626	0.4145	-2.938	0.5030	-0.03020

TABLE IIIa

Parameters of the second excited state wave function with the corresponding values of the coupling constant λ and energy ε_2 for the exponential potential. The two-body binding energy $\varepsilon_o^{(2)}$ is also presented. Where ε_2 is given in parentheses indicates a resonance state of the system.

λ (fm^{-2})	α_2 (fm^{-1})	β_2 (fm^{-1})	x_2	γ_2 (fm^{-1})	δ_2 (fm^{-1})	ϱ_2 (fm^{-1})	y_2	z_2	ε_2 (fm^{-2})	$\varepsilon_o^{(2)}$ (fm^{-2})
24	2.010	7.531	-0.5779	0.2989	0.2062	0.1696	-3.217	2.221	-5.600	-5.543
20	1.536	7.309	-0.4580	0.6072	0.5267	0.1681	-1.122	0.1366	(-3.641)	-4.026
18	1.385	8.062	-0.4149	0.5861	0.4419	0.2500	-1.480	0.5025	(-2.803)	-3.318
14	1.105	9.123	-0.3175	0.5271	0.3033	0.1922	-2.202	1.249	(-1.412)	-2.027
12	0.9653	9.747	-0.2757	0.4204	0.2904	0.1722	-1.703	0.7304	(-0.8772)	-1.458
10	0.7905	10.96	-0.1885	0.320	0.2609	0.2018	-1.789	0.7976	(-0.4639)	-0.9526
8	0.5647	30.0	-0.09236	0.30	0.2500	0.100	-1.186	0.2036	(-0.1664)	-0.5265
6	0.2167	30.0	-0.010	0.2800	0.2500	0.010	-1.014	0.0426	(-0.0090)	-0.2024

TABLE IIIb

As in table IIIa for the Yukawa potential.

λ (fm^{-2})	α_2 (fm^{-1})	β_2 (fm^{-1})	x_2	γ_2 (fm^{-1})	δ_2 (fm^{-1})	ϱ_2 (fm^{-1})	y_2	z_2	ε_2 (fm^{-2})	$\varepsilon_o^{(2)}$ (fm^{-2})
2.4054	0.1768	1.822	219.4	0.3287	0.2447	0.03004	-1.259	0.2703	(-1.669)	-1.711
1.899	0.1127	1.383	216.8	0.300	0.2101	0.0330	-1.308	0.3271	(-0.8043	-0.8398
1.6458	0.3151	1.50	14.29	0.3456	0.2835	0.04988	-1.119	0.1406	(-0.4235)	-0.5213
1.3926	0.1986	1.013	17.98	0.3461	0.2835	0.02	-1.060	0.09604	(-0.2280)	-0.2800