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L. Taffara and V. Vanzani : BINDING ENERGY SHIFT AND RECOIL CORRECTIONS IN TUNNELING REACTIONS.-

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L. Taffara[(x)] and V. Vanzani[(+)]: BINDING ENERGY SHIFT AND RECOIL CORRECTIONS IN TUNNELING REACTIONS[(o)].

1. - Recently[(1)] we have pointed out the possibility of applying the Feynman-diagram method in the heavy-ion direct interactions. The low energy transfer process, usually called "tunneling"[(2)], has been represented by a triangle-graph, in which a particle c is transferred from the initial bound state A=b+c to the final bound state B=a+c, while the "cores" a and b scatter via the Coulomb potential.

The amplitude of this graph, obtained by assuming for the off-energy-shell core-core Coulomb amplitude the same form as the one in the on-energy-shell case, corresponds to the reaction amplitude obtained by Greider in the T-matrix approximation (TMA)[(2)]. In the latter approach, in order to perform numerical calculations the initial and final state binding energies are assumed equal or averaged to a same value. Moreover terms of order $m_c/m_A$ or $m_c/m_B$ ($m_j$ is the mass of j-particle) are neglected. Recoil effects have been taken into account in the distorted-wave-Born approximation[(3)] (DWBA) and in the diffraction model at high energies[(4)], but not in the TMA or Feynman-diagram theories.

The aim of the present letter is to evaluate in the Feynman-diagram approach the corrections which arise both from the binding energy difference between the initial and final bound states and from the recoil terms.

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The recoil corrections allow us to extend the proposed method to the transfer processes between two light nuclei or from a very light incident particle and a heavy target (as in the case of (d, p), (t, d) reactions on heavy nuclei, below the Coulomb barrier).

2. - Let us briefly review the essential features of the Feynman diagram method applied to the neutron tunnelling processes. In Fig. 1 the transfer reaction A(a, B)b is described by means of a virtual decay of A in b'+c, a virtual synthesis of a'+c in B, and a core-core scattering a+b' ⟶ a'+b. The three-ray vertices are represented by the neutron-core nuclear form factors $f_N$ and $f'_N$ for the initial and final bound states. The four-ray vertex is represented by the off-energy-shell core-core Coulomb amplitude $f_C$.

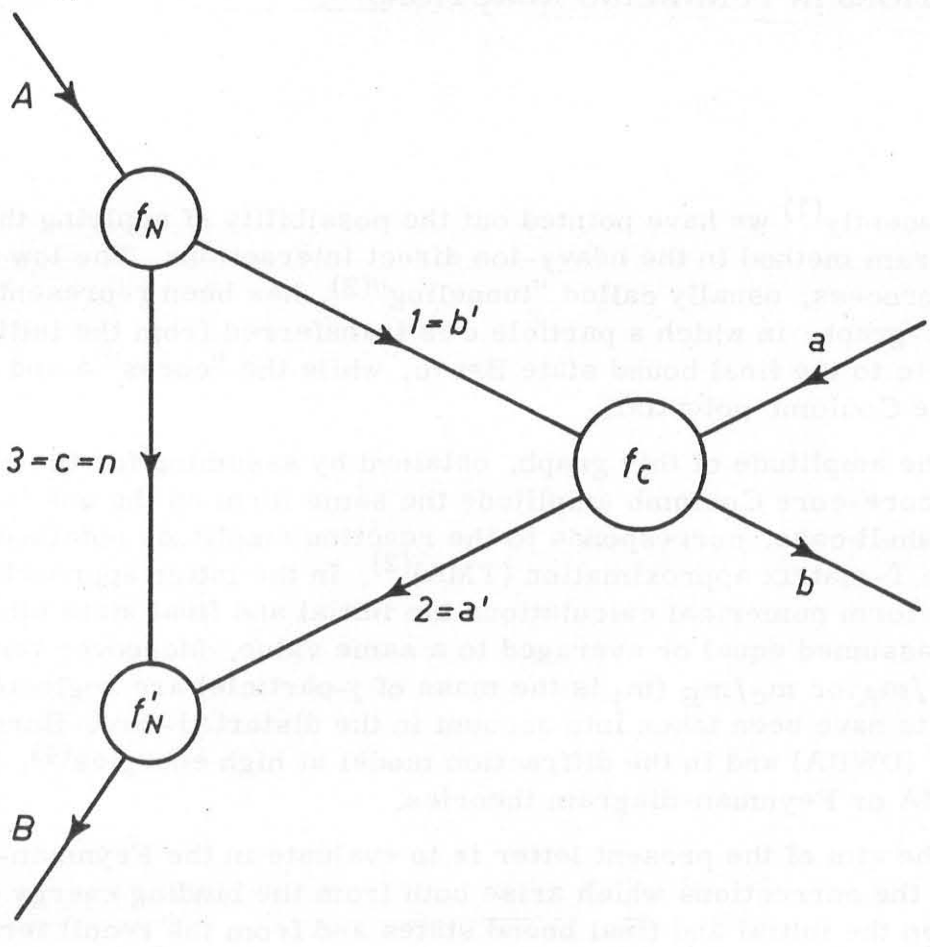


FIG. 1 - Triangle graph for a neutron tunneling process.

Owing to the strong repulsive core-core Coulomb interaction, only the asymptotic behaviour of the neutron-core bound state wave functions contributes to the transfer amplitude. It is known(5) that, at energies below the Coulomb barrier, the form of the angular distribution and excitation function (but not the over-all normalization) are fairly insensitive to the orbital momentum of the bound states. For these reasons it is assumed that the dependence of the nuclear form factors from the modulus of the relative linear momentum of the particles (1, 3) and (2, 3) is the same as the one in the s-wave case. This approximation appears to be more legitimate, in this problem, than the zero-range nuclear interaction assumption.

The cross section evaluated according to the procedure outlined in Ref. (1) reads

$$(1) \qquad \frac{d\sigma_{fi}}{d\Omega} = \frac{\mu_i \mu_f}{(2\pi)^2} \frac{p_f}{p_i} \frac{2s_B+1}{2s_a+1} \left(\frac{Z_a Z_A e^2}{\sqrt{2}\,\pi}\right)^2 N_A^2 N_B^2 \left|\beta_{j_A \ell_A}\right|^2 \left|\beta_{j_B \ell_B}\right|^2 |J|^2$$

where the magnetic sums have been already carried out and $\mu_i$ and $\mu_f$, $p_i$ and $p_f$, $j_A \ell_A$ and $j_B \ell_B$ are the initial and final reduced masses, channel energies, bound state angular momentum quantum numbers; $s_j$, $Z_j e$ are spin and charge of the j-particle.

The normalization constants $N_A$ and $N_B$ for the initial and final bound states are related to the single-particle reduced widths $\theta_o^2(A)$ and $\theta_o^2(B)$, defined in terms of nuclear surface radii $R_A$ and $R_B$(6), by the relations

$$(2) \qquad \theta_o^2(A) = \frac{1}{3} R_A^3 \left|\alpha N_A h_{\ell_A}^{(1)}(i\alpha R_A)\right|^2, \quad \theta_o^2(B) = \frac{1}{3} R_B^2 \left|\beta N_B h_{\ell_B}^{(1)}(i\beta R_B)\right|^2.$$

The constant factors $\alpha$ and $\beta$ in Eq. (2) are given in terms of the binding energies of the neutron in the initial and final nucleus $\alpha^2 = 2\mu_{13} \varepsilon_A^{13}$, $\beta^2 = 2\mu_{23} \varepsilon_B^{23}$, $\mu_{13}$ and $\mu_{23}$ being the reduced masses of the particles (1, 3) and (2, 3).

The quantities $\beta_{j_A \ell_A}$ and $\beta_{j_B \ell_B}$ are connected with the spectroscopic factors $S_{j_A \ell_A}$ and $S_{j_B \ell_B}$ by the relations

$$(3) \qquad S_{j_A \ell_A} = n_A \left|\beta_{j_A \ell_A}\right|^2, \quad S_{j_B \ell_B} = n_B \left|\beta_{j_B \ell_B}\right|^2$$

where $n_A$ and $n_B$ represent the number of distinct ways the nuclei A and B may form the configurations b' + c and a' + c, respectively.

The angular and energetic dependence is contained in the J-factor; it reads

$$(4) \qquad J = \int \frac{d\vec{k}}{(\vec{k}+\vec{\Delta})^{2(i\eta+1)}(k_{13}^2+\alpha^2)(k_{23}^2+\beta^2)},$$

where the constant $\eta$ is the Sommerfeld parameter for the core-core Coulomb scattering and $\vec{\Delta} = \vec{p}_i - \vec{p}_f$, $\vec{k}_{13}$ and $\vec{k}_{23}$ are the relative momenta of the particles (1, 3) and (2, 3): they are expressed in terms of $\vec{k}$, $\vec{p}_i$ and $\vec{p}_f$ by the relations

$$\vec{k}_{13} = \vec{k} + \frac{m_3}{m_A}\vec{p}_i, \quad \vec{k}_{23} = \vec{k} - \frac{m_3}{m_B}\vec{p}_f.$$

In the heavy-ion neutron transfer reaction terms of order $m_3/m_A$, $m_3/m_B$ are neglected, and an average value for the binding energies(2) $\bar{\alpha}^2 = (\alpha^2 + \beta^2)/2$ is assumed in the calculation of the integral (4).

3. - Let us evaluate the binding energy difference corrections first by neglecting the recoil terms and then by taking them into account. If one puts $\vec{p} = \vec{k} + \vec{\Delta}$, under assumption of no recoil, the integral (4) becomes

$$J = \int \frac{d\vec{p}}{p^{2(i\eta+1)}\left[(\vec{p}-\vec{\Delta})^2+\alpha^2\right]\left[(\vec{p}-\vec{\Delta})^2+\beta^2\right]}. \tag{5}$$

By splitting in two terms the above integral and integrating over the angles, we obtain

$$J = \frac{\pi}{(\alpha^2-\beta^2)\Delta}\left\{T(\alpha)-T(\beta)\right\} \tag{6}$$

where

$$T(\alpha) = \int_0^{\infty} dp\, p^{-2i\eta-1} \log\frac{(p-\alpha_1)(p-\alpha_2)}{(p+\alpha_1)(p+\alpha_2)} \tag{7}$$

$\alpha_{1,2}$ being equal to $\Delta \pm i\alpha$. A similar expression holds also for $T(\beta)$, provided one puts $\beta_{1,2} = \Delta \pm i\beta$. The integral (7) is easily evaluated in the complex p-plane: first the integration path is extended from $-\infty$ to $+\infty$,

$$T(\alpha) = \frac{1}{1+e^{2\pi\eta}}\int_{-\infty}^{+\infty} dp\, p^{-2i\eta-1} \log\frac{(p-\alpha_1)(p-\alpha_2)}{(p+\alpha_1)(p+\alpha_2)}, \tag{8}$$

and then a contour in the upper half-plane is trivially chosen as consisting (a) of the segments of the real axis from -R to -r and from r to R, (b) of the semicircle $C_R$ of radius R enclosing the branch points $\alpha_1$ and $-\alpha_2$, and (c) of the semicircle $C_r$ of radius r enclosing the branch point at the origin. The contributions from $C_R$ and $C_r$ vanish for $R \to \infty$ and $r \to 0$, respectively, if we assume, for the latter semicircle, that $\eta$ has a small positive imaginary part[2, 7]. It follows that the integration contour collapses all around the branch cut from $\alpha_1$ to $-\alpha_2$. Therefore, it remains to integrate over the discontinuity of the function under the integral on the cut; it is found

$$T(\alpha) = \frac{2\pi i}{1+e^{2\pi\eta}} \int_{\alpha_1}^{-\alpha_2} p^{-2i\eta-1}\,dp = \frac{\pi}{\eta}\frac{\rho^{-2i\eta}}{\cosh\pi\eta} \sinh\left[\eta\,(2\theta-\pi)\right], \tag{9}$$

where $\rho = \sqrt{\Delta^2+\alpha^2}$, $\theta = \mathrm{arctg}\,\frac{\alpha}{\Delta}$. We have finally

$$J = \frac{\pi^2}{\Delta\eta(\alpha^2-\beta^2)} \frac{1}{\cosh\pi\eta} \left\{(\Delta^2+\alpha^2)^{-i\eta} \sinh\left[\eta\,(2\mathrm{arctg}\frac{\beta}{\Delta}-\pi)\right] - (\Delta^2+\beta^2)^{-i\eta} \sinh\left[\eta\,(2\mathrm{arctg}\frac{\beta}{\Delta}-\pi)\right]\right\}. \tag{10}$$

The correctness of the result can be checked by using the Mellin transform[8]. In the $\alpha \simeq \beta \simeq \bar{\alpha}$ case, Eq. (10) becomes

$$J(\bar{\alpha}) = \frac{\pi^2}{\Delta\bar{\alpha}} \frac{(\Delta^2+\bar{\alpha}^2)^{-i\eta-1}}{\cosh\pi\eta} \left\{\Delta\cosh\left[\eta\,(2\mathrm{arctg}\frac{\bar{\alpha}}{\Delta}-\pi)\right] - i\bar{\alpha}\sinh\left[\eta\,(2\mathrm{arctg}\frac{\bar{\alpha}}{\Delta}-\pi\right]\right\}, \tag{11}$$

and corresponds to the one obtained by Greider[2]: it has to be pointed out that the usual approximation of neglecting terms of the type $\exp(-2\pi\eta)$ is incorrect for $\Delta \to 0$ (forward scattering).

Obviously, the result (11) is consistent provided the Sommerfeld parameter $\eta$, which is the mean of $\eta_i = Z_a Z_b e^2 \mu_i/p_i$ and $\eta_f = Z_a Z_b e^2 \mu_f/p_f$, is nearly equal to $\eta_i$ and $\eta_f$. This condition corresponds to $\varepsilon_i \gg |\varepsilon_B - \varepsilon_A|$ (where $\varepsilon_i$ is the initial channel energy), and it is satisfied for incident energies sufficiently higher than the binding energies in the heavier-ion interactions, where the Coulomb barrier is relatively high. The cases in which this condition fails will be discussed in Sect. 5.

4. - To evaluate the recoil corrections, in the integral (4) we put

$$(12)\qquad \vec{p}=\vec{k}+\vec{\Delta}\,, \qquad \vec{p}_1 = \frac{m_b}{m_A}\vec{p}_i - \vec{p}_f, \qquad \vec{p}_2 = \vec{p}_i - \frac{m_a}{m_B}\vec{p}_f\,;$$

it is obtained

$$(13)\qquad J = \int \frac{d\vec{p}}{p^{2(i\eta+1)} r_1 r_2}\,,$$

where $r_1 = (\vec{p}-\vec{p}_1)^2+\alpha^2$, $r_2 = (\vec{p}-\vec{p}_2)^2+\beta^2$. By means of the Feynman parametrization

$$(14)\qquad \frac{1}{r_1 r_2} = \int_0^1 \frac{dx}{\left[x r_1+(1-x) r_2\right]^2} = \int_0^1 \frac{dx}{(p^2+2\vec{B}\cdot\vec{p}+C)^2}$$

with $\vec{B} = (\vec{p}_2-\vec{p}_1)x-\vec{p}_2$, $C = (p_1^2-p_2^2+\alpha^2-\beta^2)x+p_2^2+\beta^2$; we have, after integrating over the angles

$$(15)\qquad J = 4\pi \int_0^1 dx \int_0^{\infty} \frac{p^{-2i\eta}\,dp}{p^4+2(C-2B^2)p^2+C^2}\,.$$

This integral can be carried out by changing the variable, namely $q=p^2$, and by the aid of the Mellin transform(9), which can be applied because $C-B^2>0$, as it readily follows from the definition of $\vec{B}$ and C. After some tedious manipulations, one obtains the simple integral

$$(16)\qquad J = \frac{2\pi^2}{\cosh\pi\eta}\int_0^1 \frac{C^{-i\eta-1}}{2B(C-B^2)^{1/2}}\left\{B\cosh(\eta t)+i(C-B^2)^{1/2}\sinh(\eta t)\right\}dx,$$

where $t = \arccos\left[1-(2B^2)/C\right]$; the integral (16) can be evaluated numerically.

5. - Now we deal with the problem arising from the change of the Sommerfeld parameter from the initial to the final state. To this end, after having integrated over the center-of-mass kinetic energy for the core-core scattering E (defined by Eq. (8c) of Ref. (1)), we split the integral over the momentum, related to the triangle-graph amplitude, in two parts, the

former containing the propagator $1/(k_{13}^2 + \alpha^2)$ and the latter the propagator $1/(k_{23}^2 + \beta^2)$. It is reasonable to introduce in the former the on-energy-shell form for the Coulomb amplitude $f_C(\eta_i)$ and in the latter the on-energy-shell form $f_C(\eta_f)$. The cross section derived in this way, with the aid of appropriate approximations, differs from the Eq. (1) by the replacement of the factor $|J|^2$ by the factor

$$|J'|^2 = \left(\frac{\mu_{ab}}{\mu_{13}\mu_{23}(p_i^2 - p_f^2)}\right)^2 \Big| \mu_{13}\exp(2i\sigma_i)(4p_i^2)^{i\eta_i} J_i - \tag{17}$$

$$- \mu_{23}\exp(2i\sigma_f)(4p_f^2)^{i\eta_f} J_f \Big|^2 ,$$

where $\sigma_i = \arg\Gamma(1+i\eta_i)$, $\sigma_f = \arg\Gamma(1+i\eta_f)$ and

$$J_i = \int \frac{d\vec{k}}{(\vec{k}+\vec{\Delta})^{2(1+i\eta_i)}(k_{13}^2+\alpha^2)} , \quad J_f = \int \frac{d\vec{k}}{(\vec{k}+\vec{\Delta})^{2(1+i\eta_f)}(k_{23}^2+\beta^2)} . \tag{18}$$

Introducing the transformation (12) and performing the angle integrations, one immediately sees that the form of the integrals $J_i$, $J_f$ is the same of $T(\alpha)$, $T(\beta)$ (7). Therefore, the integrals (18) read

$$J_i = - \frac{\pi^2}{\eta_i p_1} \frac{(p_1^2 + \alpha^2)^{-i\eta_i}}{\cosh\pi\eta_i} \sinh\left[\eta_i(2\,\mathrm{arctg}\frac{\alpha}{p_1} - \pi)\right] \tag{19}$$

$$J_f = - \frac{\pi^2}{\eta_f p_2} \frac{(p_2^2 + \beta^2)^{-i\eta_f}}{\cosh\pi\eta_f} \sinh\left[\eta_f(2\,\mathrm{arctg}\frac{\beta}{p_2} - \pi)\right] \tag{20}$$

Equations (19), (20) together with the formulas (1), (12b, c) and (17) give the desired expressions which take into account the binding energies change and the recoil corrections in the neutron transfer reactions. The recoil effects cannot be neglected in the (p, d), (d, t) and their inverse reactions on heavy targets at incident and outgoing energies below the Coulomb barrier.

The use of both amplitudes $f_C(\eta_i)$ and $f_C(\eta_f)$ implies a better approach to the tunneling problems than the use of only $f_C(\eta)$. Although the introduction of the on-energy-shell form for the Coulomb amplitude is supported by the satisfactory results of the TMA theory, it is worth-while to seek for a further justification of this approximation, by investigating all the off-energy-shell contributions. By starting from the first-order off-energy-shell Coulomb amplitude, the approximation implied by the introduction of the amplitudes $f_C(\eta_i)$ and $f_C(\eta_f)$ can be justified. This approach will be outlined in a forthcoming paper, and an attempt will be made to reformulate the TMA and DWBA theories on the basis of the Feynman-diagram techniques.

The increasing theoretical interest in the heavy-ion rearrangement scattering processes requires accurate experimental data of the type which can be obtained using preferentially modern Tandem Van de Graaff accelerators.

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