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A. Pascolini, G. Pisent and F. Zardi: MULTILEVEL STRUCTURES IN A SURFACE COUPLING MODEL. -

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SUMMARY. -

A surface coupling model, able to generate intermediate structures in the nucleon-nucleus scattering is dealt with. Particular emphasis is given to the case of several interferring resonances, with the aim of reconsidering, on the ground of a physical model, the R and S matrix approach to the problem. In the case of two interferring levels, the pole collision has been investigated by variation of some meaningful parameters of the model. Some examples of actual cross section analyses are finally given.

1. - INTRODUCTION. -

In recent years an increasing interest has been devoted to the coupled channel problem, in the framework of the nuclear reaction theories. As is well known, in a large variety of cases a system of coupled differential equations is obtained[1, 2, 3], whose solution requires in general massive employment of electronic computers. The coupled system may be solved analytically only at the price of drastic schematizations, as is the case of the extreme surface[4] and of the pure volume[5] coupling model. Owing to the wide generality of the coupled channel problem, a complete investigation on the solutions disclosed by means of the solvable model although schematic it may be, is clearly of great euristic interest.

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We deal with an extreme surface model derived from the generalized optical model[1]. which describes the interaction between nucleons and "collective" nuclei by means of a deformed potential. This schematic model (referred to as Square well generalized optical model), has been discussed by two of us (G. P. and F. Z.) in previous paper[6].

In the present work the square well generalized optical model, considered as a generator of intermediate resonances[7], is employed as a tool of investigation of the problem of the interferring resonant structures. Much literature exists on the parametrization of two interferring resonances, and much light has been recently shed on the meaning of the R matrix and S matrix approach to the problem[8]. It is nevertheless believed that the results already obtained in a rather formal context could be usefully probed and interpreted in the light of a meaningful physical picture.

The problem of the interferring resonances will be developed in two steps: in Sections 2 and 3 a general survey on the cross section parametrization is outlined, while Sections 4 and 5 are more specifically devoted to the model application. Finally, in Section 6 the parametric formulas are applied to the analysis of some resonances in the neutron-^{12}C elastic scattering.

2. - MULTILEVEL LOCAL APPROXIMATION. -

The scattering matrix element for a pure elastic process in absence of coulomb interaction may be written as follows[x] (see eq. 2 of ref. (6)):

$$U = \exp(2i\delta) = \exp(-2i\phi)\, \frac{1-iR/2}{1+iR/2} \qquad (1)$$

where the R function

$$R = -2P/(x-S) \,, \qquad (2)$$

is defined as including the penetration factor P.

Our purpose is to write down a resonant formula of "local" validity for the analysis of a cross section exhibiting a number N of resonances (of the same quantum numbers), in a small energy interval.

(x) - The scattering matrix element refers to a given scattering channel. The channel index will be omitted (when no confusion is possible), or indicated by c throughout.

The R function may then be expanded as follows:

(3) $$R = R_o + \sum_{n=1}^{N} \frac{r_n}{E-E_n} .$$

The quantity R_o gives the "dynamical" background which can be present in addition to the hard sphere background, the latter being explicitly taken into account by the phase shift ϕ in eq. (1).

By introducing eq. (3) into eq. (1), one obtains

(4a) $$U = \exp(-2i\phi) \frac{(1-iR_o)\prod_n (E-E_n) - i \sum_n r_n \prod_{m\neq n}(E-E_m)}{(1+iR_o)\prod_n (E-E_n) + i \sum_n r_n \prod_{m\neq n}(E-E_m)} .$$

By proper manipulation, eq. (4a) can be written in the form

(4b) $$U = \exp(-2i\theta) \prod_n \frac{E-\mathcal{E}_n - i\Gamma_n/2}{E-\mathcal{E}_n + i\Gamma_n/2} ,$$

where the poles $\mathcal{E}_n - i\Gamma_n/2$ have been evidenced. The phase shift

(5) $$\theta = \phi + \mathrm{arctg}(R_o/2) ,$$

contains now the whole (hard sphere + dynamical) background.

Equations (4a) and (4b) will be referred to as "R matrix" and "S matrix" form of the scattering matrix element.

It is worthwhile to note that the unitarity of the scattering matrix (1) has been conserved in eqs. (4) at the price of the "local" validity of the approximation. By virtue of the unitarity the above expressions of U, in both R matrix and S matrix forms, are given in terms of all free parameters and may then be directly employed for the analysis of the multilevel structures. In connection with the "fitting" problem, we give for easy reference the phase shift expressions:

(6a) $$\mathrm{tg}(\delta + \phi) = -\left(R_o/2 + \sum_{n=1}^{N} \frac{r_n/2}{E-E_n}\right) ,$$

(6b) $$\mathrm{tg}(\delta + \theta) = -\sum_{n=1}^{N} \frac{\Gamma_n/2}{E-\mathcal{E}_n} .$$

In the next Section, four cases of prominent pratical interest will be discussed in detail, namely:

(a) $N=1;\ R_o=0\ (\theta=\phi)$. (b) $N=1;\ R_o\neq 0\ (\theta\neq\phi)$.

(c) $N=2;\ R_o=0\ (\theta=\phi)$. (d) $N=2;\ R_o\neq 0\ (\theta\neq\phi)$.

In all these cases (one or two resonances with or without dynamical background), the behaviour of the scattering matrix and of the cross section will be discussed in detail, in both R matrix and S matrix approaches. As well known, the S matrix parameters carry a higher degree of physical content, while the R matrix parameters are more directly related to the features of the resonant cross section. In the framework of the R matrix approach, where the hard sphere background has been extracted, a reference to the resonant scattering amplitude A^r is convenient. This is defined as in reference (6), namely

$$\left|1-U\right|^2=\left|A^p+A^r\right|^2, \tag{7a}$$

$$A^p=\exp(-2i\phi)-1, \tag{7b}$$

$$A^r=\frac{iR}{1+iR/2}, \tag{7c}$$

In this context, besides r and Γ an empirical width G, related to the half width of $|A^r|^2$ will be considered throughout. The usefulness of the empirical width in handling actual cross sections is easily recognized, mainly in cases when ϕ is small.

3. - ONE-LEVEL AND TWO-LEVEL FORMULAS. -

Let us now discuss the four cases listed above.

(a) - One resonance without dynamical background. -

This case leads to the usual Breit and Wigner expression, and it is mentioned here only for the sake of completeness. We have in this case $\mathcal{E}_1=E_1$ and $\Gamma_1=r_1=G_1$, G_1 being the half width of the peak of $|A^r|^2$.

(b) - One resonance with dynamical background. -

The connection between S matrix and R matrix parameters is given by the following equations[x]:

(x) - Here and in all cases with background, the relation between θ and R_o is given by eq. (5).

(8a) $$\mathcal{E}_1 = E_1 - \frac{(r_1/2)(R_o/2)}{1+(R_o/2)^2} ,$$

(8b) $$\Gamma_1 = \frac{r_1}{1+(R_o/2)^2}$$

We may still define the half width G_1 of the maximum (minimum) of $|A^r|^2$ in the case $|R_o| < 2$ ($|R_o| > 2$), namely:

(9) $$G_1 = \frac{r_1}{|1-(R_o/2)^2|}$$

Note that it is always $\Gamma_1 < r_1 < G_1$.

(c) - Two resonances without dynamical background.-

We get in this case(x):

(10a) $$\mathcal{E}_{1,2} = \frac{1}{2}\left[E_1+E_2 \mp \mathrm{ReK}\right] ,$$

(10b) $$\Gamma_{1,2} = \frac{r_1+r_2}{2} \pm \mathrm{ImK},$$

where

(11) $$K = \left[(E_2-E_1)^2 - \left(\frac{r_1+r_2}{2}\right)^2 - 2i\,\frac{r_2-r_1}{2}\right]^{1/2} .$$

Here and in the following we assume the conventions $E_2 > E_1$ and $\mathrm{ReK} > 0$. The sign of ImK is then determined by the sign of r_2-r_1(o).

The resonant cross section shows two peaks separated by a central valley(+).

(x) - It may be shown that $(r_1+r_2)/2 > |\mathrm{ImK}|$, so that Γ_1 and Γ_2 are both positive, as required.

(o) - Specifically, if $r_2-r_1 > 0$, then $\mathrm{ImK} < 0$, $\Gamma_2 - \Gamma_1 > 0$ and viceversa.

(+) - By reference to $|A^r|^2$, we observe that the position of the two maxima and of the minimum are E_1, E_2 and $(E_1/r_1+E_2/r_2)/(1/r_1+1/r_2)$ respectively. The position of the minimum is given by the centre of gravity of points E_1 and E_2, taken with weights $1/r_1$ and $1/r_2$ respectively.

It is convenient to introduce here, besides the half widths $G_{1,2}$ of the two peaks, the half width g of the central valley. One obtains:

(12a) $$G_{1,2} = \frac{r_1+r_2}{2} \mp (\sqrt{H_+} - \sqrt{H_-})$$

(12b) $$g = (\sqrt{H_+} + \sqrt{H_-}) - \frac{r_1+r_2}{2}$$

where

(13) $$H_{\pm} = \frac{1}{4}\left[(E_2-E_1)^2 + \left(\frac{r_1+r_2}{2}\right)^2 \pm 2(E_2-E_1)\frac{r_2-r_1}{2}\right]$$

From eqs. (10), (13) it is immediately seen that:

(i) The total width of the resonant structure is the same in the R matrix and S matrix formalism, and coincides with the sum of the empirical widths, namely:

(14) $$r_1+r_2 = \Gamma_1 + \Gamma_2 = G_1+G_2 = \Gamma .$$

(ii) The energies $\mathcal{E}_1$, $\mathcal{E}_2$ fall between E_1 and E_2, and are symmetric with respect to $E_1+E_2/2$.

(iii) The necessary and sufficient condition for $\Gamma_1 = \Gamma_2 = \Gamma/2$ is: $r_1=r_2=\Gamma/2$ and $E_2-E_1 \geq \Gamma/2$. In this case we have $G_1=G_2=\Gamma/2$ and $\mathcal{E}_{1,2} = 1/2\left[E_1+E_2 \mp \sqrt{(E_2-E_1)^2-(\Gamma/2)^2}\right]$.

(iv) The necessary and sufficient condition(x) for $\mathcal{E}_1 = \mathcal{E}_2 = (E_1+E_2)/2$ is: $r_1=r_2=\Gamma/2$ and $E_2-E_1 \leq \Gamma/2$. In this case we have $G_1=G_2=\Gamma/2$ and $\Gamma_{1,2} = \Gamma/2 \pm \sqrt{(\Gamma/2)^2-(E_2-E_1)^2}$.

Furthermore, when $E_2-E_1 \ll \Gamma/2$, one has $\Gamma_2 \simeq (E_2-E_1)^2/\Gamma \simeq g$. Although the empirical widths G_1 and G_2 are equal {as in case (iii)}, the two Γ_i are different, and, for E_2-E_1 sufficiently small, one of them (Γ_2 in our conventions) approaches to zero (and coincides with g), while the other approaches to the total width Γ.

(v) The collision of the poles is obviously realized by intersection of the conditions (iii) and (iv), namely:

(15) $$r_1 = r_2 = E_2 - E_1 = \Gamma/2 .$$

(x) - The case $E_1=E_2$ (double R matrix pole) is here excluded.

(d) - Two resonances with dynamical background. -

We have

(16a) $$\mathcal{E}_{1,2} = \frac{1}{2}\left[E_1+E_2-\frac{r_1+r_2}{2}\,\frac{R_o/2}{1+(R_o/2)^2} \mp \mathrm{Re}K\right],$$

(16b) $$\Gamma_{1,2} = \frac{(r_1+r_2)/2}{1+(R_o/2)^2} \pm \mathrm{Im}K ,$$

where

(17) $$K = \left[(E_2-E_1)^2 - \left(\frac{(r_1+r_2)/2}{1+iR_o/2}\right)^2 - 2i\,\frac{(E_2-E_1)(r_2-r_1)/2}{1+iR_o/2}\right]^{1/2}.$$

(i) The R matrix and S matrix total widths are now different, namely:

(18) $$\Gamma_1 + \Gamma_2 = \frac{r_1+r_2}{1+(R_o/2)^2} .$$

(ii) The energies $\mathcal{E}_1$ and $\mathcal{E}_2$ are no longer symmetric with respect to $(E_1+E_2)/2$.

4. - THE SQUARE WELL GENERALIZED OPTICAL MODEL. -

Let us consider now the square well generalized optical model, in the case of neutrons scattered by a zero spin "vibrational" target, without any absorption. The elastic channel logarithmic derivative {see eq. (36) of Ref. (6) and remember the hypothesis of no absorption} reads(x):

(19) $$x_c - S_c = X_c - S_c - \beta^2 \sum_\lambda \frac{\Omega^2_{c\lambda}}{X_\lambda - S_\lambda} .$$

It is readily seen from eq. (2) that the singularities of the R function correspond to the zeros of x_c-S_c.

In the weak coupling limit, the zeros of x_c-S_c (see footnote at pag. 320 of Ref. (6) may be found either near the zeros of X_c-S_c ("single particle" resonances), or near the zeros of X_λ-S_λ ("collective"

(x) - The coupled channels functions will be labelled by indices λ, μ, h, k. For the elastic channel see footnote at page 2.

$$R = \frac{R_{sp}(E-E_k^o)}{(1-\beta^2\omega_c)(E-E_k^o)-\beta^2\eta_{ck}} \tag{23}$$

where

$$\omega_c = \sum_{\lambda\neq k}' \frac{\Omega_{c\lambda}^2}{(X_c-S_c)(X_\lambda-S_\lambda)}, \tag{24a}$$

$$\eta_{ck} = \frac{\Omega_{ck}^2}{(X_c-S_c)X_k'}, \tag{24b}$$

$$R_{sp} = -2P_c/(X_c-S_c). \tag{24c}$$

Note that R_{sp} is the R function in a pure spherical potential. The collective resonance is characterized by a non null dynamical background, given by the non resonant single particle tail.

From eq. (23) the R matrix and S matrix resonance parameters are immediately obtained, namely:

$$R_o = \frac{R_{sp}}{1-\beta^2\omega_c}, \tag{25a}$$

$$E_1 = E_k^o + \beta^2\bar{\eta}_{ck}, \tag{25b}$$

$$\Gamma_1 = \beta^2 R_o \bar{\eta}_{ck}. \tag{25c}$$

$$\mathcal{E}_1 = E_k^o + \beta^2 \frac{\bar{\eta}_{ck}}{1+(R_o/2)^2}, \tag{26a}$$

$$\Gamma_1 = \beta^2 R_o \frac{\bar{\eta}_{ck}}{1+(R_o/2)^2}. \tag{26b}$$

In eqs. (25), (26) we have posed:

$$\bar{\eta}_{ck} = \frac{\eta_{ck}}{1-\beta^2\omega_c} \tag{27}$$

The empirical width (9) reads:

$$G = \beta^2 R_o \frac{\bar{\eta}_{ck}}{\left|1-(R_o/2)^2\right|} \tag{28}$$

An expansion of eqs. (25b)...(28) up to β^2 (which is simply obtained by replacing R_o and $\bar{\eta}_{ck}$ by R_{sp} and η_{ck} respectively), shows that all widths and shifts (with respect to the unperturbed energy E_k^o) are proportional to β^2, as expected.

If the single particle background is small, an expansion of the above formulas up to the first power in R_{sp} gives $r_1 \simeq \Gamma_1 \simeq G_1$ as in case (a). On the other hand, by taking $R_{sp}=0$ one obtains $r_1 = \Gamma_1 = G_1 = 0$, as obvious because no collective mechanism can be primed in absence of single particle scattering.

Since the dynamical background is absent in case (a), and always present in case (b), it is easily recognized that a shape analysis of the resonance may give a supplementary check in order to discriminate between "collective" and "single particle" states (see the second example of Section 6).

(c) - Two interferring (single particle+collective) resonances: $\left[X_c - S_c\right]_{E_c^o} = 0$; $\left[X_k - S_k\right]_{E_k^o} = 0$.

We have(x):

$$R = \frac{\Gamma(E-E_k^o)}{(E-E_c^o)(E-E_k^o) - \beta^2 \eta_c (E-E_k^o) - \beta^2 \theta_{ck}} \tag{29}$$

where Γ is equal to Γ_1 of eq. (26b), and

$$\eta_c = \sum_{\lambda \neq k}' \frac{\Omega_{c\lambda}^2}{X_c'(X_\lambda - S_\lambda)} \tag{30a}$$

$$\theta_{ck} = \frac{\Omega_{ck}^2}{X_c' X_k'} \tag{30b}$$

(x) - Here and in case (d) all functions are intended to be evaluated at an energy intermediate between the two unperturbed resonances.

No dynamical background is present here, because the single particle resonant contribution is explicitly taken into account.

The resonance parameters are:

(31a) $$E_{1,2} = \frac{1}{2}\left[E_c^o + E_k^o + \beta^2 \eta_c \mp \sqrt{(E_c^o - E_k^o + \beta^2 \eta_c)^2 + 4\beta^2 \theta_{ck}}\right],$$

(31b) $$r_{1,2} = \left[\Gamma/2\right]\left[1 \mp \frac{E_c^o - E_k^o + \beta^2 \eta_c}{\sqrt{(E_c^o - E_k^o + \beta^2 \eta_c)^2 + 4\beta^2 \theta_{ck}}}\right].$$

(32a) $$\mathcal{E}_{1,2} = \frac{1}{2}\left[E_c^o + E_k^o + \beta^2 \eta_c \mp \mathrm{Re}K\right],$$

(32b) $$\Gamma_{1,2} = \Gamma/2 \pm \mathrm{Im}K,$$

where

(33) $$K = \left[(E_c^o - E_k^o + \beta^2 \eta_c)^2 + 4\beta^2 \theta_{ck} - (\Gamma/2)^2 - i\Gamma(E_c^o - E_k^o + \beta^2 \eta_c)\right]^{1/2}.$$

The empirical widths (12) are:

(34a) $$G_{1,2} = (\Gamma/2) \mp (\sqrt{H_+} - \sqrt{H_-}),$$

(34b) $$g = \sqrt{H_+} + \sqrt{H_-} - \Gamma/2,$$

where

(35) $$H_\pm = \frac{1}{4}\left[(E_c^o - E_k^o + \beta^2 \eta_c)^2 + 4\beta^2 \theta_{ck} + (\Gamma/2)^2 \pm \Gamma(E_c^o - E_k^o + \beta^2 \eta_c)\right].$$

First we observe that

$$r_1 + r_2 = \Gamma_1 + \Gamma_2 = G_1 + G_2 = \Gamma = \Gamma_1 \text{ of case b.}$$

In other words, the total width Γ is independent of β, and coincides with the single particle strength of the pure spherical potential.

Let us then analyze the behaviour of the resonance parameters, when the coupling strength β and the mutual position of the unperturbed energies $\Delta E^o = E^o_c - E^o_k$ is varied(x).

The condition for the S matrix poles to coincide (see eq. 15) is realized at the following critical values of ΔE^o and β :

(36a) $$\Delta E^o_{cr} = -\beta_{cr} \eta_c \quad ,$$

(36b) $$\beta_{cr} = \Gamma / (4 \sqrt{\theta_{ck}}) .$$

If we vary ΔE^o at constant β, the two peaks are seen to approach and depart one from the other and interchange their empirical widths, without coalescing. At $\Delta E^o = -\beta^2 \eta_c$ the peaks reach the minimum distance ($E_2 - E_1 = \min$), and exhibit equal empirical and R matrix widths ($G_1 = G_2$; $r_1 = r_2$).

From the S matrix viewpoint, the condition $\Delta E^o = -\beta^2 \eta_c$ corresponds to the maximum poles approach. This situation is realized in a quite different way, according as $\beta > \beta_{cr}$ or $\beta < \beta_{cr}$. In the first case we have $\mathcal{E}_2 - \mathcal{E}_1 = \min$ and $\Gamma_2 - \Gamma_1 = 0$, while in the second one it is $\mathcal{E}_2 - \mathcal{E}_1 = 0$ and $\Gamma_2 - \Gamma_1 = \min$. It is clear that the latter case corresponds to a sharp resonance over a resonant background. Since the two cases are indistinguishable in the R matrix picture, it is recognized again the misleading character of the R matrix widths, at least cases of maximum overlap.

(d) - Two interferring collective resonances: $\left[X_h - S_h\right]_{E^o_h} = 0$; $\left[X_k - S_k\right]_{E^o_k} = 0$. -

The R function is

(37) $$R = \frac{R_{sp}(E - E^o_h)(E - E^o_k)}{(1 - \beta^2 \omega_c)(E - E^o_h)(E - E^o_k) - \beta^2 \eta_{ch}(E - E^o_k) - \beta^2 \eta_{ck}(E - E^o_h)}$$

where R_{sp} is given by eq. (24c) and where

(x) - See forward the example (c) of Section 6.

(38a) $$\omega_c = \sum_{\lambda \neq h,k} \frac{\Omega^2_{c\lambda}}{(X_c - S_c)(X_\lambda - S_\lambda)} ,$$

(38b) $$\eta_{c\mu} = \frac{\Omega^2_{c\mu}}{(X_c - S_c)X'_\mu} \qquad (\mu = h, k).$$

This case differs from case (c) for the two resonances are now of the same nature (note the symmetry of eq. (37) for the change $h \rightleftharpoons k$), and grow up over the single particle background.

The R matrix background R_o has the same expression as in case (b) (see eq. (25a)). By putting

(39) $$\bar{\eta}_{c\mu} = \frac{\eta_{c\mu}}{1 - \beta^2 \omega_c} \qquad (\mu = h, k),$$

the other R matrix parameters can be written as follows

(40a) $$E_{1,2} = \frac{1}{2}\Big[(E^o_h + E^o_k) + \beta^2(\bar{\eta}_{ch} + \bar{\eta}_{ck}) \mp \Big\{ (E^o_h - E^o_k)^2 + \beta^4 \times \\ \times (\bar{\eta}_{ch} + \bar{\eta}_{ck})^2 + 2\beta^2 (E^o_h - E^o_k)(\bar{\eta}_{ch} - \bar{\eta}_{ck}) \Big\}^{1/2} \Big],$$

(40b) $$r_{1,2} = \beta^2 \Big[R_o/2 \Big] \Big[(\bar{\eta}_{ch} + \bar{\eta}_{ck}) \mp \\ \mp \frac{(E^o_h - E^o_k)(\bar{\eta}_{ch} - \bar{\eta}_{ck}) + \beta^2 (\bar{\eta}_{ch} + \bar{\eta}_{ck})^2}{\Big\{ (E^o_h - E^o_k)^2 + \beta^4 (\bar{\eta}_{ch} + \bar{\eta}_{ck})^2 + 2\beta^2 (E^o_h - E^o_k)(\bar{\eta}_{ch} - \bar{\eta}_{ck}) \Big\}^{1/2}} \Big].$$

The S matrix parameters are:

(41a) $$\mathcal{E}_{1,2} = \frac{1}{2}\Big[E^o_h + E^o_k + \beta^2 \frac{\bar{\eta}_{ch} + \bar{\eta}_{ck}}{1 + (R_o/2)^2} \mp \mathrm{Re}K \Big],$$

(41b) $$\Gamma_{1;2} = \beta^2 (R_o/2) \frac{\bar{\eta}_{ch} + \bar{\eta}_{ck}}{1 + (R_o/2)^2} \pm \mathrm{Im}K ,$$

where

$$(42) \qquad K = \left[(E_h^o - E_k^o)^2 + \beta^4 \left(\frac{\bar{\eta}_{ch} + \bar{\eta}_{ck}}{1 + iR_o/2} \right)^2 + 2\beta^2 (E_h^o - E_k^o) \frac{\bar{\eta}_{ch} - \bar{\eta}_{ck}}{1 + iR_o/2} \right]^{1/2} .$$

The condition for the poles to coincide may be put into the form

$$(43a) \qquad \frac{2\sqrt{\eta_{ch}\eta_{ck}}}{(\eta_{ch} - \eta_{ck}) - \omega_c (E_h^o - E_k^o)} = \pm R_{sp}/2 ,$$

$$(43b) \qquad \beta^2 = - \frac{E_h^o - E_k^o}{(\eta_{ch} - \eta_{ck}) - \omega_c (E_h^o - E_k^o)} .$$

The equation (43a) refers to the parameters of the unperturbed problem. If this equation is satisfied, the critical coupling is given by eq. (43b).

Finally we observe that the interference of the two levels is effective only when $E_h - E_k$ is of the order of $r_1 + r_2 = \beta^2 R_o (\bar{\eta}_{ch} + \bar{\eta}_{ck})$ or smaller. Therefore, at least in the weak coupling limit, this case is endowed of a less phenomenological interest than case (c).

5. - NUMERICAL EXAMPLES. -

Cases (b) and (c), which bear more physical interest, are illustrated here by a numerical example.

By means of the square well generalized optical model, we build up a cross section exhibiting one or two interferring levels. This is then considered as an experimental cross section and analyzed in terms of the R and S matrix formulas. The behaviour of the parameters so obtained is finally discussed on the ground of the formalism developed in Section 4.

The following optical model parameters (for the symbols' meaning see Ref. (6)) have been choosen:

$$V_o = 53.25 \text{ MeV}, \qquad V_S = 24 \text{ MeV}, \qquad R_o = 3.2057 \text{ f}.$$

In the unperturbed case (spherical potential) these parameters give rise to a $D_{3/2}$ single particle resonance at the energy $E_c = 5.29$ MeV and to an $S_{1/2}$ bound state at -1.34 MeV (in the centre of the mass system). When a coupling $\beta \neq 0$ is introduced, a proper choice of the excitation energy of the first target level allows to obtain the collective resonance $D_{3/2}$, daughter of the $S_{1/2}$ bound state, either well separated from {case (b) $\mathcal{E}$ =9 MeV}, or overlapped with {case (c), $\mathcal{E}$ ranging between 6 and 6.5 MeV} the single particle $D_{3/2}$ resonance.

Case (b): $D_{3/2}$ collective resonance.-

Figure 1 shows how the collective resonance develops as β grows up.

In Fig. 2 the parameters extracted from the cross sections (called shortly "experimental"), are compared with those calculated by means of the approximate expressions of Section 4 ("theoretical").

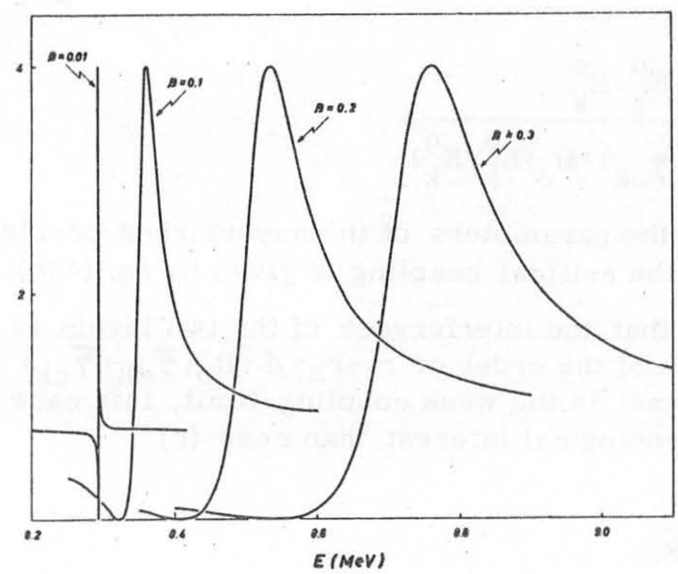

FIG. 1

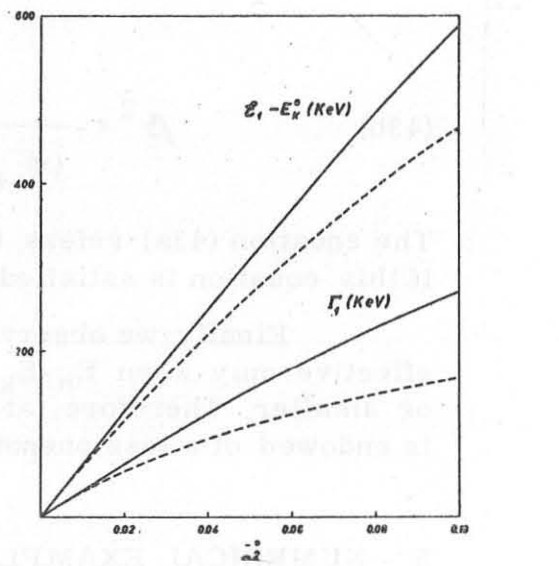

FIG. 2

FIG. 1 - Behaviour of the $D_{3/2}$ collective resonance for different β values.

FIG. 2 - The shift factor $\mathcal{E}_1 - E_k^0$ and the width Γ_1 are plotted vs β^2. Dotted and solid lines represent "experimental" and "theoretical" values respectively.

From figure inspection one immediately sees that the theoretical parameters are nearly linear in β^2 (the corrections in β^4 being unimportant), while the experimental ones show an evident discrepancy, except for very small β's.

Among the functions kept constant in Section 4 and responsible for the observed discrepancy, we have verified that X_c plays the principal role. This numerical experiment may test the limits of validity of the expressions derived, here and in all the theories which make use of approximations of the same nature.

Case (c): Two interferring $D_{3/2}$ resonances.-

In order to enlight the formal discussion of Sect. 4, we have analyzed the double $D_{3/2}$ resonance evolution, when β or ε are varied (note that in our model ε variations amount to ΔE^{o} variations).

Figure 3 shows the evolution of the resonant structure when the collective unperturbed energy changes. Two quasi-asymptotic cases are shown, together with the case of maximum overlap (curve II).

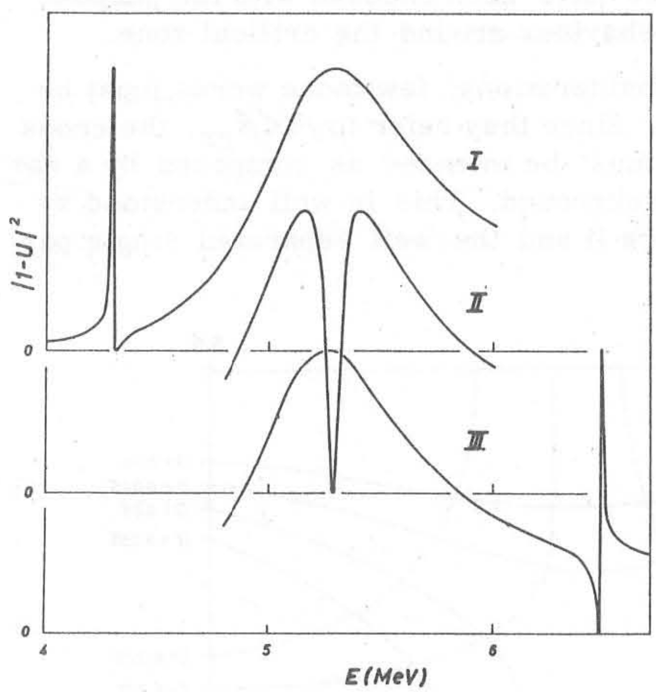

FIG. 3 - Behaviour of the $D_{3/2}$ double resonance, for three different values of the excitation energy $\{\varepsilon = 5.25$ (I), 6.25(II) and 7.25(III) MeV$\}$. The coupling strength is kept constant ($\beta = 0.025$).

Figure 4 shows the S matrix poles trajectories for β = constant (dotted lines) and ε = constant (solid lines). In each ε = constant (β = constant), curve two branches relative to the two poles of the resonant structure are distinguished.

When $\beta \approx 0$, the two poles are placed in the unperturbed (collective and single particle) positions. When β is increased (at constant ε), a pole's collision is observed. In particular, the collective pole is scattered to the left or to the right, depending upon whether $\varepsilon < \varepsilon_{cr}$ or $\varepsilon > \varepsilon_{cr}$. The critical value ε_{cr} is defined through ΔE^{o}_{cr} $\{$see eq. (36a)$\}$

(44) $$E^{o}_{c} - E^{o}_{k}(\varepsilon_{cr}) = \Delta E^{o}_{cr} \; .$$

A similar collision path is observed in the β = constant branches, when ε is increased. In the neighbouring of the scattering region (i.e. the region of minimum poles distance), the colliding trajectories are either horizontal ($\Gamma \sim$ constant) or vertical ($\varepsilon \sim$ constant), depending upon whether $\beta < \beta_{cr}$ or $\beta > \beta_{cr}$. The physical meaning of the critical values has been widely discussed in Section 4 (c).

The critical values derived either from the figure or from eqs. (36) are:

$$\beta_{cr} \simeq 0.047 \; ; \qquad \mathcal{E}_{cr} = 6.25 \text{ MeV} .$$

The β and $\mathcal{E}$ values of the figure have been choosen with the purpose of evidencing the trajectories behaviour around the critical zone.

In the light of these considerations, few more words must be devoted to the curves of Fig. 3. Since they refer to $\beta < \beta_{cr}$, the cross section II (relative to $\mathcal{E} \sim \mathcal{E}_{cr}$) must be intended as composed by a narrow resonance on a resonant background. This is well understood by comparison between the structure II and the well separated single particle resonances I and III.

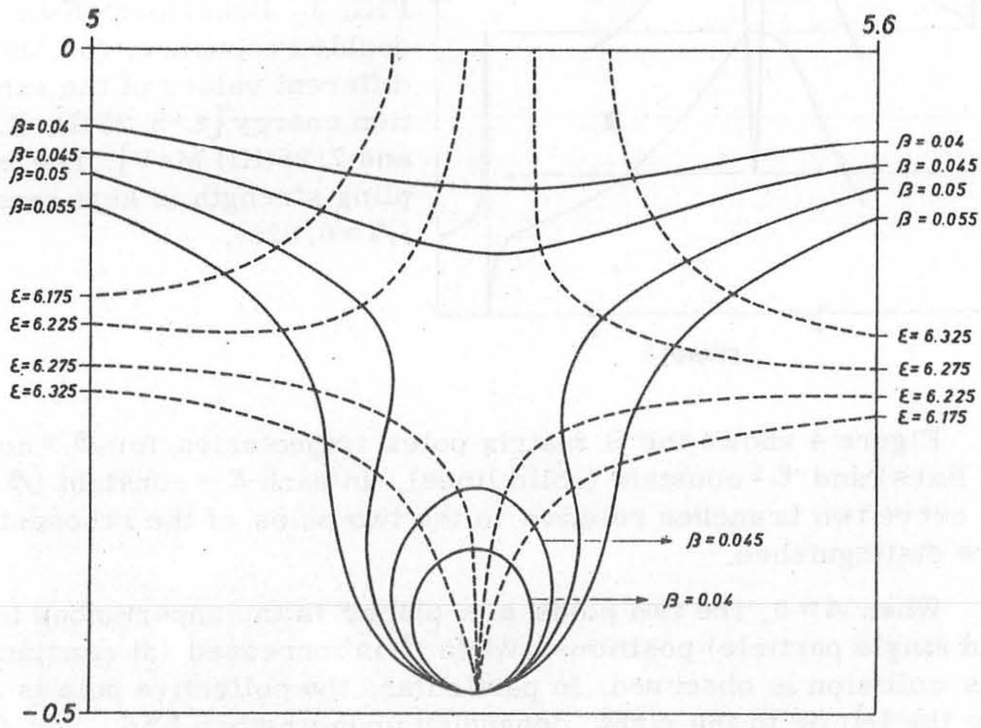

FIG. 4 - Trajectories of the S matrix poles ($\mathcal{E}_h - i\Gamma_n/2$) in the variable $\mathcal{E}$ (solid lines) and β (dotted lines).

6. - SHAPE ANALYSIS EXAMPLES. -

After the numerical example, a shape analysis in a concrete physical case, is believed of interest.

It is well known that the low energy cross section in the neutron-^{12}C elastic scattering has been successfully analyzed in terms of the generalized optical model(9)(x). In particular, the $D_{3/2}$ respnant structure at $E \simeq 3$ MeV has been interpreted as a single particle+collective resonance, while the $D_{5/2}$ peak at $E \simeq 2$ MeV is of collective nature. These resonant structures are therefore ideal examples of the cases (c) and (b) discussed above.

Analysis of the double $D_{3/2}$ resonance.-

From a best fit on the experimental $D_{3/2}$ phase shifts(10, 11) in terms of eqs. (6), the parameters of Table I have been obtained.

TABLE I

	First resonance		Second resoncance	
	energy (MeV)	width (KeV)	energy (MeV)	width (KeV)
R matrix	E_1=2.956	r_1=105	E_2=3.621	r_2=926
S matrix	$\mathcal{E}_1$=3.138	Γ_1=292	$\mathcal{E}_2$=3.439	Γ_2=738

The results of the fit are shown in Fig. 5.

We observe in first place that the peak separation E_2-E_1=675 KeV is only slightly exceeding the crytical value $(\Gamma_1+\Gamma_2)/2$ = 515 KeV, so that the two level formula is in this case necessary for a correct derivation of the resonance parameters. This is also evidenced by the appreciable difference between the S and R matrix widths, the latter being closer to the literature values of 100 and 1100 KeV.

The hard sphere phase shift obtained from the fit is ϕ =4.47°, which corresponds to an interaction radius $\mathcal{R}_0 \simeq 3.8$ f.

(x) - The interest of this scattering process, in connection with the generalized optical model, is explained as follows: the ^{13}C system shows a low neutron separation energy, and the ^{12}C target shows a high excitation energy of the 2^+ level, as compared to other light nuclei. It follows that the n-^{12}C elastic scattering interests a wide energy interval under the first anelastic threshold, and explores a low-lying region of the ^{13}C compound system, characterized by a spectrum of well defined levels.

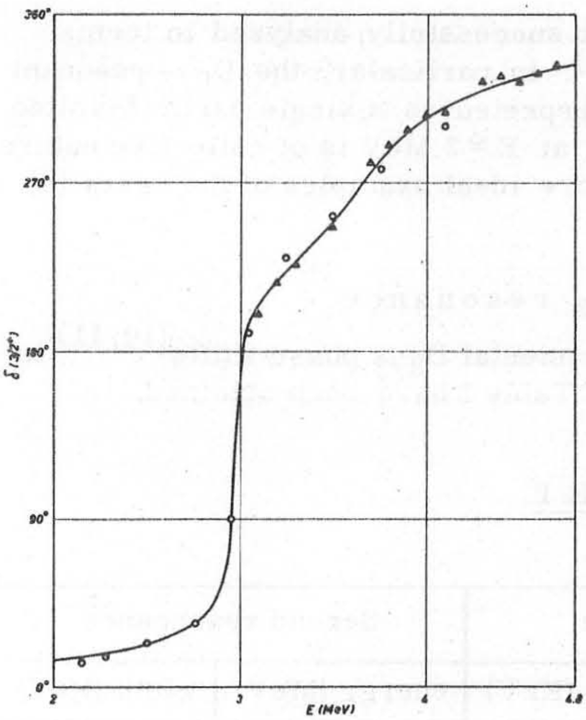

FIG. 5 - Plot of the $D_{3/2}$ phase shift in the n-^{12}C elastic scattering. The experimental points taken from Ref. (10) (o) and (11) (Δ), are fitted by the resonance formulas (6) (solid line).

Finally, in Fig. 6 the experimental total cross section[(12)] is compared with that calculated by means of the parameters of Table I, all non resonant phase shifts being kept constant. By taking into account the experimental errors of Ref. (12), the agreement is seen to be very good. This result becomes noticeable if one considers the wide energy interval involved.

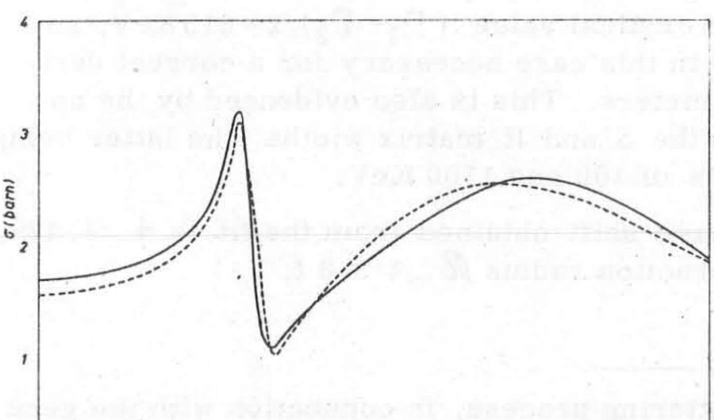

FIG. 6 - Total cross section of n-^{12}C elastic scattering. The experimental curve (dotted line) is compared with the curve calculated from the parameters of Table I (solid line), all non resonant phase shifts being kept constant.

Analysis of the "collective" $D_{5/2}$ resonance.-

From a best fit on the experimental $D_{5/2}$ phase shift[(13)], the following quantities have been obtained:

$$\theta = 7.3^{o}; \qquad \mathcal{E}_1 = 2.080 \text{ MeV}; \qquad \Gamma_1 = 7.15 \text{ KeV}.$$

The results of the fit are shown in Fig. 7.

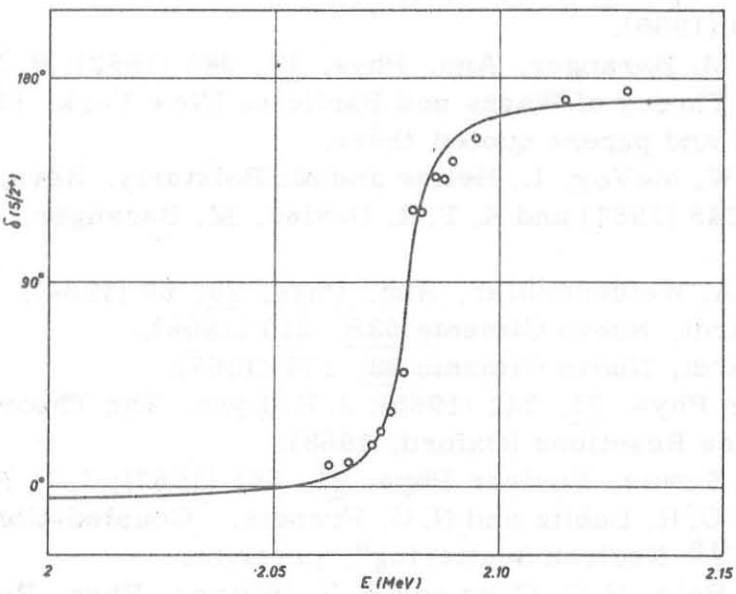

FIG. 7 - Plot of the $D_{5/2}$ phase shift in the n-^{12}C elastic scattering. The experimental points taken from Ref. (13) are fitted by the resonant formulas (6) (solid line).

From these parameters, and from the interaction radius found above, one gets:

$$\phi = 1.4^{o}; \qquad R_o = 0.2066; \qquad E_1 = 2.080 \text{ MeV}; \qquad r_1 = 7.31 \text{ KeV}.$$

The dynamical background is shown to be very small, and no appreciable difference exists among S and R matrix parameters and literature values (resonance energy = 2.076 MeV, width = 7 KeV). In spite of this, the presence of a dynamical background, and consequently the collective nature of the level is clearly evidenced by the shape analysis, as shown by comparison between θ and ϕ..

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