

Comitato Nazionale per l'Energia Nucleare  
ISTITUTO NAZIONALE DI FISICA NUCLEARE

Sezione di Genova

INFN/BE - 68/1  
12 Febbraio 1968

G. Dillon and G. Passatore : THE SYMMETRY OF THE THEORETICAL  
OPTICAL POTENTIAL AND ITS CONNECTION WITH  
TIME REVERSAL AND RECIPROCITY. -

G. Dillon and G. Passatore: THE SYMMETRY OF THE THEORETICAL OPTICAL POTENTIAL AND ITS CONNECTION WITH TIME REVERSAL AND RECIPROcity. -

An explicit derivation of a general property of symmetry of the non-local optical potential is given by using the projection operators. Such a property, which in a particular case reduces to the symmetry in the position coordinates, insures the relation of reciprocity and follows from the reversibility of the original multi-channel system.

## 1. INTRODUCTION. -

In the last few years it has been pointed out that reversibility and reciprocity<sup>(x)</sup> are in general distinct concepts<sup>(1, 2)</sup>. In particular for some systems of physical interest reciprocity still holds while time-reversal is violated. Optical model effective hamiltonians give a common example for such a situation which is an obvious consequence of the elimination of the inelastic channels and has been already discussed elsewhere<sup>(2, 3)</sup>.

The purpose of this note is to give a detailed treatment of this question in terms of projection operators and to derive in a very explicit way a general property of symmetry of the non-local optical potential. This property in the simplest case is just the symmetry in the position variables and in this form up to now has been verified in particular examples<sup>(3)</sup>. In sect. 2 we formulate the conditions for reversibility and for reciprocity for systems described by effective hamiltonians and in sect. 3

---

(x) - The sense of the term reciprocity here used is defined in sect. 2 and is the same as in ref. (2).

2.

we discuss the implications that the reversibility of the multichannel system has on the structure of the optical potential.

## 2. REVERSIBILITY AND RECIPROCITY FOR SYSTEMS DESCRIBED BY EFFECTIVE HAMILTONIANS. -

For a quantum-mechanical system reversibility holds when, if the state vector  $|\psi\rangle$  evolves from  $|\psi_i\rangle$  to  $|\psi_f\rangle$ , then the time-reversed state vector  $|\psi^R\rangle$  evolves, in the same time interval, from  $|\psi_f^R\rangle$  to  $|\psi_i^R\rangle$ . The time reversal is an anti-unitary mapping of the Hilbert space into itself<sup>(4)</sup>: so, in any representation, it can be written:

$$(2.1) \quad \psi^R = U \psi^*$$

where  $U$  is a unitary operator and the asterisk means complex conjugate. The operator  $U$  must satisfy the correspondence principle:

$$(2.2) \quad (\psi^R, A \psi^R) = \pm (\psi, A \psi)$$

where the sign plus or minus is taken according to the even or odd character of the corresponding classical variable under the time reversal operation.

From the Schrödinger equation:

$$(2.3) \quad i \frac{\partial \psi}{\partial t} = H \psi$$

it follows that a system is reversible if its hamiltonian satisfies the relation:

$$(2.4) \quad H = U H^* U^{-1}$$

which is often written equivalently as<sup>(5, 6, 7)</sup>:

$$(2.5) \quad H = U H^T U^{-1}$$

since  $H$  is hermitian. (Here the suffix  $T$  means transposed)<sup>(\*)</sup>.

We consider now the condition of reversibility for effective non hermitian hamiltonians. Such effective hamiltonians describe usually physical

---

(\*) - In writing eq. (2.4) the property  $U^T = cU$ , where  $c$  is a c-number of modulus one<sup>(7)</sup>, has been used.

systems projected on suitable subspaces of the whole space of their states. In general they are also energy dependent so that the time evolution of the projected system is given, instead of eq. (2.3), by the equation<sup>(3,8)</sup>:

$$(2.6) \quad i \frac{\partial \psi(t)}{\partial t} = \int_{-\infty}^{\infty} W(t-t') \psi(t') dt'$$

where the "time non-local" operator

$$(2.7) \quad W(t-t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(E) e^{-i(t-t')E} dE$$

is the Fourier transform of the effective hamiltonian  $H(E)$  which occurs in the equation for the stationary states  $\psi_E$ :

$$(2.8) \quad H(E) \psi_E = E \psi_E$$

The reversibility may now be expressed by saying that the time reversed state vector, eq. (2.1), must satisfy the equation:

$$(2.9) \quad -i \frac{\partial \psi^R(t)}{\partial t} = \int_{-\infty}^{\infty} W(t'-t) \psi^R(t') dt'$$

The condition for reversibility then follows:

$$(2.10) \quad W(t-t') = U W^x(t'-t) U^{-1}$$

Of course if  $W$  is causal, i. e.  $W(t-t')=0$  for  $t < t'$ , reversibility is impossible<sup>(3)</sup>. From eq. (2.11) the condition on the effective hamiltonian follows:

$$(2.11) \quad H(E) = U H^x(E) U^{-1} .$$

Therefore is the eq. (2.4), and not the eq. (2.5), that must be assumed as condition for reversibility if also systems described by effective hamiltonians are treated.

We pass now to consider the condition for reciprocity, i. e. for the property of the S-matrix

$$(2.12) \quad \langle \beta | S | \alpha \rangle = \langle \alpha^R | S | \beta^R \rangle .$$

It has been proved<sup>(1)</sup> that if a unitary operator  $V$  exists which satisfies the equation:

4.

$$(2.5') \quad H = V H^T V^{-1},$$

then for the S-matrix the property holds:

$$(2.13) \quad \langle \beta | S | \alpha \rangle = \langle \bar{\alpha} | S | \bar{\beta} \rangle$$

where

$$(2.14) \quad \psi_{\bar{\alpha}} = V \psi_{\alpha}^*$$

For hermitian hamiltonians this result had been previously established<sup>(9)</sup> by starting from eq. (2.4). When the operator V identifies with the time reversal operator U, eq. (2.1), then eq. (2.13) becomes the usual reciprocity relation (2.12).

To handle with a formalism as simple as possible we consider in the following of this paper only systems of spinless particles. Then, in coordinate representation, it can be put  $U=1$  in the eq. (2.1). In this case the reversibility holds if:

$$(2.15) \quad H = H^*$$

and reciprocity if:

$$(2.16) \quad H = H^T.$$

### 3. THE SYMMETRY OF THE THEORETICAL OPTICAL POTENTIAL. -

Let us consider a physical system described by a (hermitian) hamiltonian  $\mathcal{H}$  and let  $\psi_E$  be a stationary state:

$$(3.1) \quad \mathcal{H} \psi_E = E \psi_E$$

It is well known<sup>(10)</sup> that the vectors  $P \psi_E$  and  $Q \psi_E$ , where P is a projection operator and  $Q=1-P$ , are related by:

$$(3.2') \quad (E - P \mathcal{H} P) P \psi_E = P \mathcal{H} Q Q \psi_E$$

$$(3.2'') \quad (E - Q \mathcal{H} Q) Q \psi_E = Q \mathcal{H} P P \psi_E$$

If to the states  $Q \psi_E$  the outgoing wave condition is imposed, then for the vector  $P \psi_E$  the equation follows:

$$(3.3) \quad H(E) P \psi_E = E P \psi_E,$$

where



$$(3.4) \quad H(E) = P\mathcal{H}P + P\mathcal{H}Q \frac{1}{E+i\varepsilon - Q\mathcal{H}Q} Q\mathcal{H}P \quad (\varepsilon \rightarrow 0^+)$$

plays the role of the hamiltonian operator, and is called the effective hamiltonian. Because it incorporates the boundary condition on  $Q\psi_E$ , which is of causal type (as it is formally expressed by the term  $i\varepsilon$  occurring in the denominator), this effective hamiltonian describes a non-reversible system.

However the reciprocity still holds provided that (i) it may make sense in the subspace defined by  $P$  and (ii) the original system is physical (i. e.  $\mathcal{H}$  hermitian) and reversible (i. e.  $\mathcal{H}$  real). This is obvious because a projection only limits the states of a system without changing its own dynamics, i. e. it only selects matrix elements from the original  $S$ -matrix<sup>(2, 11, 12, 13)</sup>. This is also easily seen in terms of the effective hamiltonian: if the hypotheses (i) and (ii) are satisfied it follows immediately from eq. (3.4) that the effective hamiltonian is self-transposed in coordinate representation. Indeed this equation shows that  $H(E)$  is self-transposed if also  $\mathcal{H}$  and  $P$  are. Hypothesis (ii) insures such a property for  $\mathcal{H}$  and hypothesis (i) for  $P$ . In fact hypothesis (i) is equivalent to saying that  $P$  must project on a subspace which is mapped into itself under the time reversal operation, and this precisely makes  $P$  self-transposed in coordinate representation.

Let us now consider a multi-channel reversible physical system described by a many-body<sup>(x)</sup> (hermitian and real) hamiltonian  $\mathcal{H}$ :

$$(3.5) \quad \mathcal{H} \equiv \mathcal{H}(\underline{r}_0, \dots, \underline{r}_A; \underline{p}_0, \dots, \underline{p}_A).$$

According to Wigner and Eisenbud<sup>(13, 14)</sup> a channel at any energy  $E$  may be defined as an eigenstate

$$(3.6) \quad |\alpha, \alpha_A, \alpha_B\rangle \equiv |\alpha\rangle |\alpha_A\rangle |\alpha_B\rangle$$

of the asymptotic hamiltonian:

$$(3.7) \quad \mathcal{H}_{AB} = \mathcal{H}_A + \mathcal{H}_B + \mathcal{H}_{rel},$$

where

---

(x) - We here neglect the problems arising from the identity of the particles. The proper account of this one gives rise to complication in the formalism without changing the conceptual aspect of the results here discussed<sup>(10)</sup>.

6.

A and B are two fragments built up with the particles of the original many-body system;

$\mathcal{H}_A$  and  $\mathcal{H}_B$  are their hamiltonians with eigenstates  $|\alpha_A\rangle$  and  $|\alpha_B\rangle$ ;

$\mathcal{H}_{rel}$  is the hamiltonian for the relative motion (kinetic energy plus possible long range potential terms) with eigenstates  $|\alpha\rangle$ .

Projection operators on various sets of channels can be constructed in terms of suitable sets of eigenstates of asymptotic hamiltonians like (3.7). Let us consider in particular the projection on the set of the elastic channels, i. e. on the subspace spanned by the vectors  $|\alpha, \alpha_T\rangle$  where  $|\alpha_T\rangle$  denotes the ground state of the target A (supposed not degenerate for the moment) and the quantum numbers for the internal state of the projectile B do not appear because it has been assumed spinless and not composite. Then:

$$(3.8) \quad P = \sum_{\alpha} |\alpha, \alpha_T\rangle \langle \alpha, \alpha_T| \quad (\alpha_T \text{ fixed})$$

On the other channels the outgoing wave condition is imposed. The operator P is self-transposed in coordinate representation. Indeed it projects on a subspace which is time-reversal invariant, for  $|\alpha_T\rangle$  is a not degenerate eigenstate of  $\mathcal{H}_A$  and the vectors  $|\alpha\rangle$  are all the eigenstates of  $\mathcal{H}_{rel}$ , both hamiltonians being hermitian and real.

The elastic scattering amplitude in the coordinate representation is:

$$(3.9) \quad f(\underline{x}) = \langle \underline{x}, \alpha_T | \Psi_E \rangle,$$

where  $\underline{x}$  is the relative coordinate between projectile and target, and it is easy to see from eqs. (3.3), (3.4) and (3.8) that it satisfies the equation:

$$(3.10) \quad \int d\underline{x} \langle \underline{y}, \alpha_T | H | \underline{x}, \alpha_T \rangle f(\underline{x}) = E f(\underline{y}),$$

which can be written as:

$$(3.11) \quad (T + \mathcal{V}) f = E f,$$

by extracting from  $P\mathcal{H}P$  in eq. (3.4) the term  $PTP$ , where T is the kinetic energy of the relative motion. The term  $\mathcal{V}$  comes from the remaining part K of H:

$$(3.12) \quad \mathcal{V}(\underline{y}, \underline{x}) = \langle \underline{y}, \alpha_T | K | \underline{x}, \alpha_T \rangle,$$

and is the theoretical optical potential.

The operator  $K$  is self-transposed in coordinate representation, as  $H$  and  $T$  are. Then:

$$(3.13) \quad \mathcal{V}(\underline{y}, \underline{x}) = \mathcal{V}(\underline{x}, \underline{y}),$$

because the wave function of the ground state of the target is real.

When the ground state of the target is degenerate (e. g. when its spin is different from zero), the operator  $P$ , eq. (3.8), is no longer self-transposed if the state  $|\alpha_T\rangle$  is not self-time reversed. As a consequence,  $\mathcal{V}(\underline{x}, \underline{y})$  turns out to be not symmetrical. But the notion of reciprocity makes no sense for such projected system.

It makes still sense, however, if we project on the whole subspace of the elastic channels, i. e. if we use the projection operator:

$$(3.14) \quad P = \sum_{\alpha_T} P_{\alpha_T},$$

where  $P_{\alpha_T}$  is given by eq. (3.8) and  $\sum_{\alpha_T}$  means summation over the azimuthal spin component of the target ground state. From eq. (3.3) we get now, instead of eq. (3.10) the system of equations:

$$(3.15) \quad \sum_{\alpha_T} \int d\underline{x} \langle \underline{y} \beta_T | H | \underline{x} \alpha_T \rangle \langle \underline{x} \alpha_T | \psi_E \rangle = E \langle \underline{y} \beta_T | \psi_E \rangle$$

which, as for eq. (3.10), can be written in the form (3.11) where now  $f$  is a column matrix with elements  $f_{\alpha_T}$  labelled by the values of the azimuthal component of the target spin, and the operator  $\mathcal{V}$  has the matrix elements:

$$(3.16) \quad \mathcal{V}_{\beta_T \alpha_T}(\underline{y}, \underline{x}) = \langle \underline{y} \beta_T | K | \underline{x} \alpha_T \rangle.$$

where the operator  $K$  is defined as above. This system of equations is the same as that derived by Francis and Watson<sup>(15)</sup>. As the operator  $P$ , equation (3.14), is now self-transposed in coordinate representation,  $K$  is also self-transposed here and therefore:

$$(3.17) \quad \langle \underline{y} \beta_T | K | \underline{x} \alpha_T \rangle = \langle \underline{x} \alpha_T^R | K | \underline{y} \beta_T^R \rangle,$$

where the suffix  $R$  means time reversed. Then:

$$(3.18') \quad \mathcal{V}_{\beta_T \alpha_T}(\underline{y}, \underline{x}) = \mathcal{V}_{\alpha_T^R \beta_T^R}(\underline{x}, \underline{y}),$$



8.

or, in momentum representation for the relative motion:

$$(3.18'') \quad \mathcal{V}_{\beta_T} \alpha_T (\underline{q}, \underline{p}) = \mathcal{V}_{\alpha_T} R_{\beta_T} R (-\underline{p}, \underline{q}).$$

Eq. (3.18) is the generalization of the symmetry property (3.13). It has been already recognized that eq. (3.13) insures the reciprocity<sup>(3)</sup>. Indeed eq. (3.13) is equivalent to

$$(3.19') \quad \langle \underline{y} | \tau_E^{el} | \underline{x} \rangle = \langle \underline{x} | \tau_E^{el} | \underline{y} \rangle$$

or

$$(3.19'') \quad \langle \underline{q} | \tau_E^{el} | \underline{p} \rangle = \langle -\underline{p} | \tau_E^{el} | -\underline{q} \rangle$$

where  $\tau_E^{el}$  is the operator connected to the optical potential by the Lippmann-Schwinger equation<sup>(15,16)</sup> and giving, on the energy shell  $q^2 = p^2 = 2mE$ , the elastic scattering amplitude. On the energy shell equation (3.19'') is just the reciprocity relation. Of course similar considerations apply to the more general eq. (3.18).

We conclude by remarking that eq. (3.18) or, equivalently, (3.19), follows only from the reversibility of the original multi-channel system described by the hamiltonian  $\mathcal{H}$ . Then, while the lack of reversibility for projected systems is due to purely formal reasons, a failure of reciprocity would imply a true violation of time reversal.

## REFERENCES. -

- (1) - D. E. Bilhorn, L. L. Foldy, R. M. Thaler, W. Tobocman and V. A. Madsen, *J. Math. Phys.* 5, 435 (1964).
- (2) - L. C. Biedenharn, *Nuclear Phys.* 10, 620 (1959).
- (3) - R. Lipperheide, *Nuclear Phys.* 89, 97 (1966).
- (4) - E. Wigner, *Göttinger Nachr.* 31, 546 (1932).
- (5) - F. Coester, *Phys. Rev.* 89, 619 (1953).
- (6) - E. M. Henley and B. Jacobson, *Phys. Rev.* 113, 225 (1959).
- (7) - K. Nishijima, *Fundamental Particles* (Benjamin, New York, 1963).
- (8) - J. M. Cornwall and M. Ruderman, *Phys. Rev.* 128, 1474 (1962).
- (9) - G. Morpurgo, L. A. Radicati and B. F. Touschek, *Nuovo Cimento*, 12, 677 (1954).
- (10) - H. Feshbach, *Ann. Phys.* 19, 287 (1962).
- (11) - T. Teichmann and E. Wigner, *Phys. Rev.* 87, 123 (1952).
- (12) - R. G. Thomas, *Phys. Rev.* 97, 224 (1955).
- (13) - A. M. Lane and R. G. Thomas, *Revs. Modern Phys.* 30, 257 (1958).
- (14) - E. Wigner and L. Eisenbud, *Phys. Rev.* 72, 29 (1947).
- (15) - N. C. Francis and K. M. Watson, *Phys. Rev.* 92, 291 (1953).
- (16) - M. L. Goldberger and K. M. Watson, *Collision Theory*, (Wiley and Sons, New York 1964), pp. 215 and 782.

REFERENCES

- (1) - R. K. Hobbie, J. L. Pomeroy, R. M. Taylor, W. J. Tomlinson and V. A. Weaver, *J. Amer. Chem. Soc.*, **77**, 1154 (1955).
- (2) - L. C. Blumhagen, *Anal. Chem.*, **27**, 1155 (1955).
- (3) - R. E. Lippert, *J. Amer. Chem. Soc.*, **77**, 1155 (1955).
- (4) - R. W. Wenzel, *Colloids Polymer Sci.*, **233**, 1155 (1955).
- (5) - R. Wenzel, *Colloids Polymer Sci.*, **233**, 1155 (1955).
- (6) - R. M. Hobbie and R. J. Tomlinson, *J. Amer. Chem. Soc.*, **77**, 1155 (1955).
- (7) - R. Wenzel, *Colloids Polymer Sci.*, **233**, 1155 (1955).
- (8) - R. Wenzel and R. J. Tomlinson, *J. Amer. Chem. Soc.*, **77**, 1155 (1955).
- (9) - R. Wenzel, J. A. Tomlinson and R. J. Tomlinson, *J. Amer. Chem. Soc.*, **77**, 1155 (1955).
- (10) - R. Wenzel, *Colloids Polymer Sci.*, **233**, 1155 (1955).
- (11) - T. F. Johnston and R. Wenzel, *J. Amer. Chem. Soc.*, **77**, 1155 (1955).
- (12) - R. Wenzel, *Colloids Polymer Sci.*, **233**, 1155 (1955).
- (13) - R. Wenzel and R. J. Tomlinson, *J. Amer. Chem. Soc.*, **77**, 1155 (1955).
- (14) - R. Wenzel and R. J. Tomlinson, *J. Amer. Chem. Soc.*, **77**, 1155 (1955).
- (15) - R. Wenzel and R. J. Tomlinson, *J. Amer. Chem. Soc.*, **77**, 1155 (1955).
- (16) - R. Wenzel and R. J. Tomlinson, *J. Amer. Chem. Soc.*, **77**, 1155 (1955).
- (17) - R. Wenzel and R. J. Tomlinson, *J. Amer. Chem. Soc.*, **77**, 1155 (1955).
- (18) - R. Wenzel and R. J. Tomlinson, *J. Amer. Chem. Soc.*, **77**, 1155 (1955).
- (19) - R. Wenzel and R. J. Tomlinson, *J. Amer. Chem. Soc.*, **77**, 1155 (1955).
- (20) - R. Wenzel and R. J. Tomlinson, *J. Amer. Chem. Soc.*, **77**, 1155 (1955).