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G. Pisent and F. Zardi : SQUARE WELL GENERALIZED OPTICAL MODEL. -

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1. - INTRODUCTION. -

A great deal of interest has been recently devoted to the coupled channels techniques, in order to describe the interaction of nucleons with "collective" nuclei.

Most of the literature on this subject refers to the excitation of collective states in medium-heavy nuclei, by means of high energy nucleons(1). In this kind of analysis, the model was mainly intended as an improvement on the DWBA.

A group of works deals with the systematic analysis of the effects of the channel coupling on the zero energy cross section behaviour, with particular regard to the strength function determination(2, 3).

Finally, the nature of the well separated resonances which appear in the low energy cross section, has been studied in the particular case of the n-^{12}C elastic scattering(4, 5).

The present research gives a general investigation on the low energy cross section behaviour in both energy regions of virtual and real excitation of the collective states.

In spite of the great amount of calculations carried out so far, the general features of the coupled channels model have not been stres-

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sed as deeply as desiderable. This is mainly due to the complicate structure of the system of differential equations, whose solution requires massive employement of electronic computers.

This situation is particularly troublesome in the case of resonant structures, where a detailed knowledge of the mechanisms involved, and a suitable method for a first order evaluation of the parameters is needed.

The problem can be faced by introduction of the square well radial potential, which leads to drastic semplifications, and allows the coupled system of equations to be solved exactly[6]. Since the main object of the research is the analysis of the effects due to the angular dependence of the potential, the choice of a potential with the simplest radial dependence, although somewhat crude, is nevertheless justified as a useful tool of investigation.

It is worthwhile to underline that the coupling channels formalism can be naturally framed in the more general problem of the intermediate structures[7]. Under this viewpoint the set up given in the present paper, can be also intended as a useful exemplification of this more fundamental approach.

The general formulas, in terms of the elastic channel logarit mic derivative, are given in paragraph 2. Paragraph 3 considers the interaction of neutrons with even-even nuclei, described by a square well spherical optical potential, taking into account the spin-orbit interaction and the effects of volume and surface absorption. In paragraph 4 the first 2^+ collective (rotational or vibrational) target level is taken into account, and the general formulas are given. Paragraph 5 deals with the problem of the growing up of the intermediate structures over the single particle cross section, as far as the coupling is switched on.

Finally, the particular problem of the zero energy and threshold cross sections behaviour is esamined in paragraphs 6 and 7.

2. - GENERAL FORMALISM. -

The purpose of what follows is to rederive some well known expressions, relative to the neutron-nucleus scattering, in a form suitable to be applied to both cases of single-channel and multi-channel interaction.

Following Lane and Thomas[8] we define the interaction radius R as the minimum projectile-target distance at which no interaction is sensitive.

Be u_c the radial wave function in the elastic channel (c being the channel index), and

(1) $$z_c \equiv x_c + iy_c = R\left(\frac{du_c}{dr}\right)_{r=R} / u_c(R) ,$$

the elastic channel logaritmic derivative.

Let us then describe the scattering process in terms of z_c, with particular reference to the resonant behaviour and zero energy behaviour of the cross sections.

The scattering matrix U_{cc} in the entrance channel is :

(2) $$U_{cc} = e^{-2i\phi_c} \frac{(x_c - S_c) + i(y_c + P_c)}{(x_c - S_c) + i(y_c - P_c)} ,$$

where ϕ_c, P_c and S_c are rigid sphere phase shift, penetrability and shift factor respectively, as defined in Ref. (8).

The shape elastic, compound nucleus and total cross section[9] can be put in the following form:

(3a) $$\sigma^{se} = \sum_c \sigma_c^{se} ,$$

(3b) $$\sigma^{cn} = \sum_c \sigma_c^{cn} ,$$

(3c) $$\sigma^{tot} = \sigma^{se} + \sigma^{cn} ,$$

(3d) $$\sigma_c^{se} = (\pi/k^2) g_c \left| 1 - U_{cc} \right|^2 = (\pi/k^2) g_c \left| A_c^p + A_c^r \right|^2 ,$$

(3e) $$\sigma_c^{cn} = (\pi/k^2) g_c (1 - \left| U_{cc} \right|^2) = (\pi/k^2) g_c \, Q_c^{cn} ,$$

k being the external momentum and g_c the spectroscopic factor.

The potential and resonant scattering amplitude A_c^p, A_c^r, the interference term $2\mathrm{Re}(A_c^p A_c^{r*})$, and the function Q_c^{cn} read :

(4a) $$A_c^p = e^{2i\phi_c} - 1 ,$$

(4b) $$A_c^r = \frac{-2iP_c}{(x_c - S_c) + i(y_c - P_c)} ,$$

(4c) $$2\mathrm{Re}(A_c^p A_c^{r*}) = 8P_c \sin\phi_c \frac{(x_c - S_c)\cos\phi_c + (y_c - P_c)\sin\phi_c}{(x_c - S_c)^2 + (y_c - P_c)^2}$$

(4d) $$Q_c^{cn} = \frac{-4P_c y_c}{(x_c - S_c)^2 + (y_c - P_c)^2}$$

In cases of well separated resonances the Breit-Wigner one level formula is :

$$(5a) \qquad A_c^r = \frac{i\,\Gamma_c^{se}}{E - E^r + i(\Gamma_c^{se} + \Gamma_c^{cn})/2} \quad ,$$

$$(5b) \qquad Q_c^{cn} = \frac{\Gamma_c^{se}\,\Gamma_c^{cn}}{(E - E^r)^2 + (\Gamma_c^{se} + \Gamma_c^{cn})^2/4} \quad ,$$

where the resonance energy E^r satisfies the following equation :

$$(6) \qquad (x_c - S_c)_{E = E^r} = 0 \quad (x) \; .$$

The Breit-Wigner parameters are :

$$(7a) \qquad \Gamma_c^{se} = -2\,P_c/(x_c - S_c)'_{E=E^r} \quad ,$$

$$(7b) \qquad \Gamma_c^{cn} = 2\,y_c/(x_c - S_c)'_{E=E^r} \quad .$$

Here and in the following the prime means derivative in respect of energy.

If one introduces the (complex) phase shift δ_c :

$$(8) \qquad U_{cc} = e^{2i\delta_c} \quad ,$$

it follows immediately from eq. (2) that :

$$(9) \qquad P_c \cot(\delta_c + \phi_c) = z_c - S_c \quad .$$

It is interesting to note that in the pure elastic case (δ_c real, $z_c \equiv x_c$), the resonance definition (6) gives : $(\delta_c + \phi_c)_{E=E^r} = \pi/2$.

The behaviour of the S wave cross sections in the limiting case $E \to 0$, can be characterized by the following parameters :

(x) - For a recent interpretation of the very well known condition (6) as the "natural boundary condition", see Ref. (10).

(10a) $$a_c = \lim_{E \to 0} \frac{1}{2k} \, \mathrm{Im}\,(1 - U_{cc}) \, ,$$

(10b) $$b_c = \lim_{E \to 0} \frac{1}{2k} \, \mathrm{Re}\,(1 - U_{cc}) \, .$$

The parameter a_c is the scattering length(x), while b_c is simply connected with the S wave strength function.

By introducing eq. (2) into eqs. (10) one obtains the following expressions :

(11a) $$a_c = R\left(1 - \frac{x_c}{x_c^2 + y_c^2}\right) ,$$

(11b) $$b_c = - R \, \frac{y_c}{x_c^2 + y_c^2} ,$$

which, in the pure elastic case become :

(12a) $$a_c = R\left(1 - \frac{1}{x_c}\right) ,$$

(12b) $$b_c = 0 \, .$$

3. - OPTICAL, SQUARE WELL, SPHERICAL POTENTIAL. -

Let us now specialize the above equations to the case of a square well spherical potential, and zero spin target. We assume the following hamiltonian, which takes into account the spin-orbit coupling and both surface and volume absorbtions.

(13) $$H_{\text{spheric}}(r\,\vartheta\varphi) = T(r\,\vartheta\varphi) - (V_o + iW_o) f(r) + \left[(\hbar/M_\pi c)^2 \, \bar{L}\cdot\bar{S} \, \frac{1}{r} V_s + i R_o W_s\right] (df/dr) \, ,$$

where T is the kinetic energy, and

(14a) $$f(r) = 1 \quad \text{for} \quad r < R_o \, ; \qquad f(r) = 0 \quad \text{for} \quad r > R_o \, ;$$

(x) - In the pure elastic case eq. (10a) assumes the very well known expression $a_c = - \lim_{E \to 0} (\delta_c / k)$.

(14b) $$df/dr = -\delta(r - R_o) .$$

The channel index is now $c \equiv jl$. In order to apply the formulas of the preceding paragraph, we may define the interaction radius as the square well radius ($R \equiv R_o$), provided the function z_{jl} be evaluated in the external region :

(15) $$z_{jl} \equiv x_{jl} + iy_{jl} = \lim_{r \to R^+} R\,(du_{jl}/dr)_{r=R} / u_{jl}(R) .$$

Owing to the surface effects, there is discontinuity between z_{jl} and the internal logaritmic derivative

(16) $$z_{jl}^{int} = x_{jl}^{int} + iy_{jl}^{int} = \lim_{r \to R^-} R(du_{jl}/dr)_{r=R} / u_{jl}(R) .$$

We can then define an effective internal logaritmic derivative Z_{jl}

(17) $$Z_{jl} \equiv X_{jl} + i\,Y_{jl} = z_{jl}^{int} + \mathcal{f}_{jl}$$

where the parameter $\mathcal{f}_{jl}$ takes into account the surface effects, namely:

(18a) $$\mathcal{f}_{jl} = \chi_{jl} + i\eta ,$$

(18b) $$\chi_{jl} = -V_s\,(2M/\hbar^2)(\hbar/M_\pi c)^2 \langle \bar{L} \cdot \bar{S} \rangle ,$$

(18c) $$\eta = -W_s\,(2M/\hbar^2)R^2 .$$

By equating the external and effective internal logaritmic derivative

(19) $$z_{jl} = Z_{jl}$$

we have finally the link between the function z_{jl} and the potential parameters.

Explicit expressions for the external functions $\phi_l(\rho_o)$, $P_l(\rho_o)$, $S_l(\rho_o)$ and for the internal functions $x_{jl}^{int}(\mu_o, \nu_o)$, $y_{jl}^{int}(\mu_o, \nu_o)$, are found in Ref. (9) and in the Appendix respectively. The arguments of

these functions are the external and internal momentum-times-radius ρ_o and $\mu_o + i\nu_o$:

$$\rho_o = R\left\{(2M/\hbar^2)E\right\}^{1/2} , \tag{20a}$$

$$\mu_o = R\left\{\left[M/\hbar^2\right]\left[\sqrt{(V_o+E)^2+W_o^2} + (V_o+E)\right]\right\}^{1/2} , \tag{20b}$$

$$\nu_o = R\left\{\left[M/\hbar^2\right]\left[\sqrt{(V_o+E)^2+W_o^2} - (V_o+E)\right]\right\}^{1/2} . \tag{20c}$$

The quantities A^p_{jl}, A^r_{jl}, Q^{cn}_{jl}, together with the resonant parameters and scattering length (eqs. 4...12), can be now calculated through the external logaritmic derivative (19), and the cross sections are given by eqs. (3) with a $g_c = (2j+1)/2$.

We conclude with some observations on the resonant behaviour of the cross section, starting from the spectrum which is observed in the pure elastic case ($W_o = W_s = 0$). In this case the position and width of the resonances are given by eqs. (6) and (7a) respectively, being $\Gamma^{cn}_c \equiv \equiv y_c = 0$. If a surface absorption is added ($W_o = 0$, $W_s \neq 0$), x_c remains unchanged, while $y_c \equiv \gamma$ becomes $\neq 0$.

Therefore the resonances are broadened but not shifted with respect to the elastic case. Finally, the introduction of a volume absorption too ($W_o \neq 0$, $W_s \neq 0$), modifies both x_c and y_c values, and causes enlargement and shifting of the levels.

It is easily seen that the presence of singularities in the behaviour of x^{int}_c vs. energy is a sufficient condition for the existence of resonances, independently on the values assumed by S_c and χ_c. It is shown in the Appendix that x^{int}_c has no singularities when $W_o \neq 0$.

Therefore, when the volume absorption becomes sufficiently large, some resonances can disappear.

4. - OPTICAL SQUARE WELL DEFORMED POTENTIAL. -

Let us consider now a collective target nucleus of spin $I = 0$, taking into account the first excited level, characterized by $I^\pi = 2^+$ and excitation energy $\epsilon_{I=2}$. The hamiltonian becomes :

$$H_{tot} = H_{spheric}(r\vartheta\varphi) + H_{target}(\xi) + H_{tensor}(\vartheta\varphi\xi) . \tag{21}$$

In addition to $H_{spheric}$, defined by eq. (13) we have now the target hamiltonian $H_{target}(\xi)$ (ξ standing for the internal coordinates of the target) and the tensorial interaction $H_{tensor}(\vartheta\varphi\xi)$ which is responsible for the coupling. The last term reads :

$$H_{tensor}(\vartheta\varphi\xi) = V_o \beta R \Sigma_m Y_2^{m*}(\vartheta\varphi) T_2^m(\xi) \frac{df}{dr} \quad . \tag{22}$$

The hamiltonian $H_{tensor}(\xi)$ and the second rank tensor $\overline{T}_2(\xi)$ which are esplicitely written in Ref. (11), depend on the rotational or vibrational nature of the 2^+ level.

By usual procedure, the decomposition of the total wave function in states Ijl leads (in the case of square well potential) to the following system of equations for the radial wave function components u_{Ijl} :

$$\begin{aligned} &\left[Z_c(\mu_o \nu_o) - z_c(\rho_o)\right] u_c + \Sigma_m \alpha_{c c_m} u_{c_m} = 0 \\ &\alpha_{c_n c} u_c + \Sigma_m \left\{ \left[Z_{c_n}(\mu_2 \nu_2) - z_{c_n}(\rho_2)\right] \delta_{mn} + \alpha_{c_m c_n} \right\} u_{c_m} = 0. \end{aligned} \tag{23}$$

As usual $c \equiv (I=0, jl)$ is the entrance channel index, while $c_m, c_n \equiv (I=2, jl)$ runs over all the channels which are coupled with the given c through conservation of parity and total angular momentum J.

The coupling matrix elements $\alpha_{I_1 j_1 l_1 ; I_2 j_2 l_2}$ can be written as:

$$\alpha_{I_1 j_1 l_1 ; I_2 j_2 l_2} = -\beta V_o R^2 (2M/\hbar^2) \omega_{I_1 j_1 l_1 ; I_2 j_2 l_2} \quad . \tag{24}$$

The pure geometrical factors $\omega_{I_1, j_1, l_1 ; I_2, j_2, l_2}$ are explicitely given in Ref. (5). It is worthwhile to remember only that

$$\alpha_{cc} = \alpha_{0jl;\,0jl} = 0 \quad , \tag{25}$$

and that the ω's relative to the vibrational and rotational case are related by the expression

$$\omega^{vibr}_{I_1 j_1 l_1 ; I_2 j_2 l_2} = \omega^{rot}_{I_1 j_1 l_1 ; I_2 j_2 l_2} (1 - \delta_{I_1 I_2}) . \tag{26}$$

The arguments $\rho_I \mu_I \nu_I$ of eq. (23) are given by expressions (20) provided one substitutes $E - \epsilon_I$ to E, and remembers that $\epsilon_o = 0$.

In eqs. (23) the effective internal logaritmic derivatives Z_c, Z_{cn} are calculated through the eq. (17) and the formulas of the Appendix. The external logaritmic derivative for anelastic channels z_{c_n} reads :

$$z_{c_n}(\rho_2) = S_{c_n}(\rho_2) + i P_{c_n}(\rho_2) \ , \tag{27}$$

where it is worthwhile to note that when ρ_2 is pure imaginary (below the 2^+ threshold) we have (see Ref. (9)) :

$$S_{c_n}(\rho_2) = S^{(-)}_{c_n}(\rho_2/i) \ ; \qquad P_{c_n}(\rho_2) = 0 \ . \tag{28}$$

The only unknown in the system (23) is then the external logaritmic derivative z_c in the elastic channel. This is determined from the condition of solubility of the linear homogeneous system (23) in the unknowns u; namely :

$$D \equiv \det \begin{vmatrix} Z_c - z_c & \alpha_{cc_1} & \cdots & \alpha_{cc_N} \\ \alpha_{c_1 c} & Z_{c_1} - z_{c_1} + \alpha_{c_1 c_1} & \cdots & \alpha_{c_1 c_N} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{c_N c} & \alpha_{c_N c_1} & \cdots & Z_{c_N} - z_{c_N} + \alpha_{c_N c_N} \end{vmatrix} = 0 \tag{29}$$

By solving eq. (29) one finds :

$$z_c = Z_c + M_c / D_{cc} \ , \tag{30}$$

where

$$M_c = \sum_{n=1}^{N} (-)^n \alpha_{cc_n} D_{cc_n} \tag{31}$$

and D_{cc_n} (D_{cc}) indicate the determinant obtained from (29) by dropping the c row and c_n (c) column.

The comparison between eqs. (19) and (30) clearly evidences the obvious fact that the term H_{tensor}, being a surface term, introduces a supplementary discontinuity between internal and external logaritmic derivatives. The occurrence of the intermediate structures is then determined by the behaviour of M_c/D_{cc}.

In the multichannel model, above the 2^+ threshold, one can calculate besides σ^{cn} the inelastic cross section σ^{in} defined as follows :

$$\sigma^{in} = \sum_c \sigma^{in}_c \ , \tag{32a}$$

$$(32b) \qquad \sigma_c^{in} = (\pi/k^2) g_c \sum_{c_n} \left| U_{cc_n} \right|^2 = (\pi/k^2) g_c Q_c^{in} .$$

The homogeneous system (23), together with the condition (29), gives the link between elastic and inelastic wave functions, namely

$$(33) \qquad u_{c_n} = (-)^n (D_{cc_n}/D_{cc}) u_c .$$

Since we have :

$$(34) \qquad u_c = I_c - U_{cc} O_c \; ; \quad u_{c_n} = - \sqrt{\rho_o/\rho_2}\; O_{c_n} U_{cc_n}$$

we finally obtain

$$(35) \qquad Q_c^{in} = \frac{4P_c}{(x_c - S_c)^2 + (y_c - P_c)^2} \sum_n P_{c_n} \left| D_{cc_n}/D_{cc} \right|^2 .$$

The difference between compound nucleus and inelastic cross sections defined above, correspondes to the compound elastic cross section. In the case $W_o = W_s = 0$, σ^{cn} and σ^{in} coincide, owing to the U matrix unitarity.

The formalism developed so far is rigorous within the limits of the chosen model.

In the next paragraphs some noticeable features of the channels coupling mechanism will be stressed by analysis of the following points:

- Genesis of the intermediate structures when the coupling is gradually switched on (limit $\beta \to 0$);
- Zero energy behaviour of the cross sections (limit $\rho_o \to 0$);
- Threshold behaviour of the cross section (limit $\rho_2 \to 0$).

5. - GENESIS OF THE INTERMEDIATE STRUCTURES FOR GRADUALLY INCREASING COUPLING. -

5a. - Vibrational model. -

The distinction between vibrational and rotational case, never stressed so far, becomes useful at this stage of specialization of the discussion. We start from analysis of the simpler vibrational model, and the concepts introduced this way will be then extended to the more complicated rotational case.

In the vibrational case the matrix of the system (23) in the subspace of the anelastic channels, is diagonal and independent on β (see eq. (26)), so that the logaritmic derivative (30) becomes :

$$z_c = Z_c - \beta^2 \sum_n \Omega^2_{cc_n} \frac{(X_{cn}-S_{cn}) - i(Y_{cn} - P_{cn})}{(X_{cn}-S_{cn})^2 + (Y_{cn} - P_{cn})^2} , \tag{36}$$

with the position

$$\Omega = - V_o R^2 (2M/\hbar^2) \omega . \tag{37}$$

From eq. (36) we have immediately the resonance condition for the general case :

$$(X_c-S_c) - \beta^2 \sum_n \Omega^2_{cc_n} \frac{X_{c_n}-S_{c_n}}{(X_{c_n}-S_{c_n})^2 + (Y_{c_n}-P_{c_n})^2} = 0 , \tag{38}$$

and for the pure elastic case :

$$(X_c - S_c) - \beta^2 \sum_n \frac{\Omega^2_{cc_n}}{X_{c_n} - S_{c_n}} = 0 . \tag{39}$$

We begin now the discussion on the growing up of the resonances by considering first eq. (39) in the limit of small β values.

Under this assumption the zeros of eq. (39) lie near points E^o_c and $E^o_{c_n}$, defined by the following equations(x) :

$$(X_c - S_c)_{E=E^o_c} = 0 , \tag{40a}$$

$$(X_{c_n} - S_{c_n})_{E=E^o_{c_n}} = 0 . \tag{40b}$$

It is immediately recognized that E^o_c is the unperturbed ($\beta = 0$) single particle resonance energy, whose width is given by

$$\Gamma^{se}_c = - \frac{2P_c}{(X_c - S_c)'} . \tag{41}$$

(x) - If β is small eq. (39) is satisfied either near E^o_c where $X_c - S_c$ becomes small, or near $E^o_{c_n}$ where the second term of the equation goes rapidly from ~ 0 to ∞ and from $-\infty$ to ~ 0. The case $E^o_c \sim E^o_{c_n}$ (i.e. the overlapping between single particle and collective resonances) will be not considered here.

On the other hand eq. (40b), where $(X_{c_n} - S_{c_n})$ depends on the negative energy $E - \epsilon_2$, is a bound state equation. This means that $E^o_{c_n}$ corresponds to a bound state of the incident particle in the excited target nucleus, and implies the existence of a single particle bound state in the ground state nucleus, at the energy $E^o_{c_n} - \epsilon_2$. We call $E^o_{c_n}$ "collective unperturbed resonance energy".

The "true" collective resonance energy $E^r_{cc_n} = E^o_{c_n} + \Delta_c(c_n)$ which satisfies eq. (39), is characterized by the shift factor and width given below :

$$\Delta_c(c_n) = \beta^2 \frac{\Omega^2_{cc_n}}{\left[(X_c - S_c)(X_{c_n} - S_{c_n})'\right]_{E=E^o_{c_n}}} \qquad (42a)$$

$$\Gamma^{se}_c(c_n) = \beta^2 \left[\frac{-2P_c \Omega^2_{cc_n}}{(X_c - S_c)(X_{c_n} - S_{c_n})'}\right]_{E=E^o_{c_n}} \qquad (42b)$$ (x)

These expressions have been obtained by neglecting the non resonant inelastic channels and by expanding $(X_{c_n} - S_{c_n})$ to the first order in energy, near the point $E^o_{c_n}$. In eq. (42b) one term β^4 has been neglected.

In conclusion, when the coupling β is gradually switched on, a collective spectrum appears over the single particle background. In the limit $\beta \to 0$ this collective spectrum shows infinitely narrow resonances at the energies $E^o_{c_n}$, with degeneracy of all the elastic channels c which are coupled with the same c_n. When β is slightly increased the resonances are enlarged and shifted, so that the degeneracy is gradually removed.

The coupling between elastic and inelastic channels up to the G states is given in Table I for easy reference.

(x) - In connection with eqs. (42) it is worthwhile to observe that :
- Since we have in all cases $(X_{c_n} - S_{c_n})' < 0$, the sign of Δ is determined by the sign of the single particle function $X_c - S_c$.
- When a collective resonant state happens to be close to a single particle resonance, its coupling with the entrance channel is emphasized. The effect is clearly evidenced by term $X_c - S_c$ in eq. (42b), which is small in the neighbourhood of E^o_c.

TABLE I

c_n = I, j, l / c = I, j, l	2.1/2.0	2.1/2.1	2.3/2.1	2.3/2.2	2.5/2.2	2.5/2.3	2.7/2.3	2.7/2.4	2.9/2.4
0.1/2.0				+	+				
0.1/2.1			+			+			
0.3/2.1		+	+			+	+		
0.3/2.2	+			+	+			+	
0.5/2.2	+			+	+			+	+
0.5/2.3		+	+			+	+		
0.7/2.3			+			+	+		
0.7/2.4				+	+			+	+
0.9/2.4					+			+	+

The channels coupled together are signed by + in the table.

The more general case of absorbing potential and/or real excitation of the 2^+ level, is described by eq. (38).

It is easily seen that, for the single particle unperturbed resonances, there is no difference between this case and the elastic case discussed above. We have on the contrary quite a different situation as far as the collective structures are concerned. In fact, owing to the presence of the absorption, the second term of eq. (38) has no singularities, and below a certain value of β the collective resonances are cancelled out. In this case the cross sections are characterized by non resonant collective structure, as shown by the following expressions :

$$(43a)\quad \left|A_c^r\right|^2 = \frac{4P_c^2}{(X_c-S_c)^2+(Y_c-P_c)^2}\left[1 - 2\beta^2 \sum_n \Omega^2_{cc_n}(f_c f_{c_n} - g_c g_{c_n})\right], \quad (x)$$

(x) - We have considered for the sake of semplicity the only resonant part of the elastic cross section. When ϕ_e is not negligible the interference between scattering and potential amplitude must be obviously taken into account, but this complication does not influence the general conclusions drawn below.

$$Q_c^{cn} = \frac{-4P_c Y_c}{(X_c-S_c)^2 + (Y_c-P_c)^2} \cdot$$

(43b)

$$\cdot \left[1 + \beta^2 \Sigma_n \Omega^2_{cc_n} \left\{ (f_{c_n}/Y_c) - 2 (f_c f_{c_n} - g_c g_{c_n}) \right\}\right],$$

(43c) $$Q_c^{in} = 4\beta^2 \frac{P_c}{Y_c - P_c} f_c \Sigma_n \Omega^2_{cc_n} \frac{P_{c_n}}{Y_{c_n} - P_{c_n}} f_{c_n} ,$$

where

(44a) $$f_i = \frac{Y_i - P_i}{(X_i - S_i)^2 + (Y_i - P_i)^2} ,$$

(44b) $$g_i = \frac{X_i - S_i}{(X_i - S_i)^2 + (Y_i - P_i)^2} .$$

Eqs. (43) have been obtained by power expansion up to β^2. Note that the expansion fails to hold in the pure elastic case, so that eqs. (43) cannot be used in the limit $W_o \to 0$, $W_s \to 0$ under the threshold.

In the neighbourhood of the energy $E^o_{c_n}$, by assuming (as in the elastic case) $(X_{c_n} - S_{c_n})$ linear in energy, and all other functions constant in energy, f_{c_n} and g_{c_n} behave as shown in Fig. 1. These functions fall rapidly to zero outward of an interval of the order of magnitude of $2(Y_{c_n} - P_{c_n})$, which represents the half width for f_{c_n} and the maximum-minimum distance for g_{c_n}. We can say therefore, as a qualitative statement, that the cross sections contain a single particle background plus collective structures (in correspondence to energies E^o_c) of height proportional to β^2 and pseudo-width independent of β, given by :

(45) $$\gamma_{c_n} = \left| 2 \frac{Y_{c_n} - P_{c_n}}{(X_{c_n} - S_{c_n})'} \right|_{E=E^o_{c_n}} .$$

In the particular case of σ^{in}, the single particle background is obviously zero.

In other words the collective structures (otherwise than the collective resonances in the pure elastic case), born with constant width and infinitely small height, as the coupling is switched on.

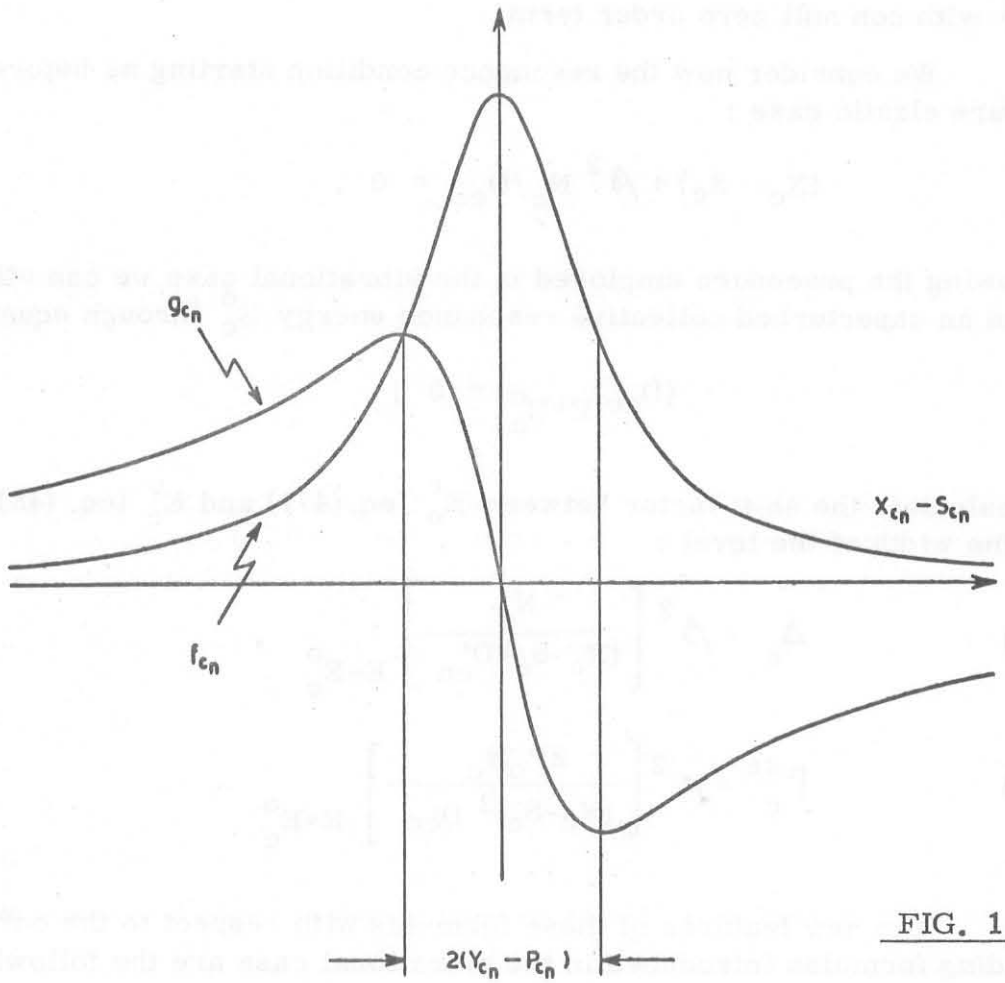

FIG. 1

When β increases and becomes sufficiently large with respect to the absorbing effects, the structures are gradually transformed into true resonances.

We note finally that each structure in the anelastic channel c_n, above threshold, implies the existence of a single particle resonance below threshold (at the unperturbed energy $E^o_{c_n} - \epsilon_2$), in the elastic channel $c \equiv c_n$.

5b. - Rotational model. -

In spite of the presence of the non diagonal terms in the anelastic subspace, the logaritmic derivative (30) can be still written in a form similar to that of eq. (36) namely:

(46) $$z_c = Z_c + \beta^2 N_c / D_{cc} ,$$

where it is immediately recognized that $N_c = M_c/\beta^2$ is a polynomial in β with non null zero order term.

We consider now the resonance condition starting as before from the pure elastic case :

$$(47) \qquad (X_c - S_c) + \beta^2 N_c/D_{cc} = 0 \ .$$

Following the procedure employed in the vibrational case we can still define an unperturbed collective resonance energy E_c^o through equation

$$(48) \qquad (D_{cc})_{E=E_c^o} = 0 \ ,$$

and calculate the shift factor between E_o^r (eq. (47)) and E_c^o (eq. (48)), and the width of the level :

$$(49a) \qquad \Delta_c = \beta^2 \left[\frac{-N_c}{(X_c - S_c) D'_{cc}} \right]_{E=E_c^o} \ ,$$

$$(49b) \qquad \Gamma_c^{se} = \beta^2 \left[\frac{2 P_c N_c}{(X_c - S_c)^2 D'_{cc}} \right]_{E=E_c^o} \ .$$

The new features of these formulas with respect to the corresponding formulas introduced in the vibrational case are the following:

- E_c^o is a function of β.
- The unperturbed resonance is determined by the cooperative contribution of all the coupled channels.
- The functions N_c and D_{cc} contain both even and odd powers of β, as obvious because in this case the prolate or oblate form of the deformed nucleus comes into play.

In the limit $\beta \to 0$ the coupling is destroyed and a certain channel c_n becomes predominant in the resonance equation. Consequently both energies E_c^o and E_c^r tend to the unperturbed energy defined in the vibrational case (eq. (40a)).

It is interesting to observe that eq. (48) gives the eigenenergies relative to the closed channels subspace. Therefore our definition of the unperturbed resonances coincides with the definition given in the Feshbach's picture[12]. It could be also demonstrated that widths and shift factors given by us are easily related to the expressions foundby Feshbach.

In the case of absorbing potentials and/or real level excitation, the cross sections evaluation in the limit $\beta \to 0$ is straigthforward. In fact, being $(Y_{c_n} - P_{c_n}) \neq 0$, a power expansion in β allows complete channels separation and leads to the formulas (43) already obtained for the vibrational case.

6. - ZERO ENERGY CROSS SECTION. -

We consider here the S wave scattering length in the pure elastic case (eq. (12a)).

It is immediately seen (Table I) that three channels only come into play: $c = 0,\ 1/2,\ 0;\ c_1 = 2.3/2.2;\ c_2 = 2.5/2.2$.

The resonance condition ($a_c \to \infty$; $x_c = 0$) is then written as follows :

$$X_c - \beta^2 \frac{\Omega^2_{cc_1} X_{c_2} + \Omega^2_{cc_2} X_{c_1} - 2\beta \Omega_{cc_1} \Omega_{cc_2} \Omega_{c_1 c_2}}{X_{c_1} X_{c_2} - \beta^2 \Omega^2_{c_1 c_2}} = 0. \tag{50}$$

where all functions X are calculated in the point E=0.

Eq. (50) is identical to the general resonance condition (47) specialized to the particular channel $c = 0,\ 1/2,\ 0$ with the difference that the variable is in this case the radius R rather than the energy E. One can therefore translate to this case most of the concepts introduced in the preceding paragraph, and in particular one can define, for sufficiently small β values an unperturbed resonance condition

$$X_{c_1} X_{c_2} - \beta^2 \Omega^2_{c_1 c_2} = 0 . \tag{51}$$

It immediately follows that the resonances of the scattering length a_c, can be only due to the elastic channel S or to the anelastic channels D.

In the vibrational case eq. (50) becomes :

$$X_c - \beta^2 \left(\frac{\Omega^2_{cc_1}}{X_{c_1}} + \frac{\Omega^2_{cc_2}}{X_{c_2}} \right) = 0 \tag{52}$$

and the unperturbed collective resonances are due to separate contribution of the D wave elastic channels.

7. - THRESHOLD EFFECTS.

This paragraph deals with the behaviour of the cross sections σ^{se} and $\sigma^{cn} = \sigma^{in}$ at the 2^+ threshold, in the case of real potentials ($W_o = W_s = 0$).

Let us write down for easy reference the detailed expressions of functions (4b, c, d) :

$$A_c^r = \frac{-2\,i\,P_c}{\left[X_c+\mathrm{Re}(M_c/D_{cc})-S_c\right] + i\left[\mathrm{Im}(M_c/D_{cc}) - P_c\right]} \tag{53a}$$

$$2\mathrm{Re}\left(A_c^p A_c^{r\,*}\right) = 8\,P_c \sin\phi_c \cdot$$

$$\cdot \frac{\left[X_c+\mathrm{Re}(M_c/D_{cc})-S_c\right]\cos\phi_c + \left[\mathrm{Im}(M_c/D_{cc}) - P_c\right]\sin\phi_c}{\left[X_c+\mathrm{Re}(M_c/D_{cc})-S_c\right]^2+\left[\mathrm{Im}(M_c/D_{cc})-P_c\right]^2} \tag{53b}$$

$$Q_c^{cn} = Q_c^{in} = \frac{-P_c\,\mathrm{Im}(M_c/D_{cc})}{\left[X_c+\mathrm{Re}(M_c/D_{cc})-S_c\right]^2+\left[\mathrm{Im}(M_c/D_{cc})-P_c\right]^2} \tag{53c}$$

In eqs. (53) $X_c S_c P_c \phi_c$ are regular functions of the energy through the threshold. The only source of anomalies may be M_c/D_{cc}, which contains the effects of the inelastic channels. Let us analyze the structure of the determinants D_{cc_n} (see eq. (31)) and D_{cc}.

Besides the determinant (29) which writes in this case

$$D = \det \begin{vmatrix} Z_c - z_c & \alpha_{cc_1} & \alpha_{cc_2} & \cdots \\ \alpha_{c_1 c} & (X_{c_1} - S_{c_1} + \alpha_{c_1 c_1}) - iP_{c_1} & \alpha_{c_1 c_2} & \cdots \\ \alpha_{c_2 c} & \alpha_{c_2 c_1} & (X_{c_2} - S_{c_2} + \alpha_{c_2 c_2}) - iP_{c_2} & \\ & & & \cdots\cdots \\ \cdots & \cdots & \cdots & \end{vmatrix} \tag{54}$$

it is convenient to consider the real determinant d, defined as follows :

$$(55)\quad d = \det \begin{vmatrix} X_c - x_c & \alpha_{cc_1} & \alpha_{cc_2} \\ \alpha_{c_1c} & X_{c_1} - S_{c_1} + \alpha_{c_1c_1} & \alpha_{c_1c_2} \\ \alpha_{c_2c} & \alpha_{c_2c_1} & X_{c_2} - S_{c_2} + \alpha_{c_2c_2} \end{vmatrix}$$

Since functions P_{c_n} are infinitesimal in threshold :

$$(56)\quad \lim_{E \to \epsilon_2^+} P_{c_n} = \lim_{E \to \epsilon_2^+} \rho_2^{2l+1} / \left[(2l-1)!!\right]^2 = 0,$$

we can expand the determinant (54) to the first order in the P_{c_n}'s, by employment of the Leibnitz theorem for determinant derivation. The following expressions are obtained :

$$(57a)\quad D_{cc} = d_{cc} - i \sum_m P_{c_m} d_{cc}^{c_m c_m},$$

$$(57b)\quad D_{cc_n} = d_{cc_n} - i \sum_m P_{c_m} d_{cc_n}^{c_m c_m} (1 - \delta_{mn}),$$

where d_{ij} is as usual the minor of order ij of d, and d_{ij}^{lm} is the determinant obtained from d by dropping the rows i, l and the columns j, m.

From eqs. (57), to the first order in the P_{c_n}'s we obtain :

$$(58)\quad \frac{D_{cc_n}}{D_{cc}} = \frac{d_{cc_n}}{d_{cc}} \left[1 + i \sum_m P_{cm} \left\{ \left(d_{cc}^{c_m c_m} / d_{cc} \right) - \left(d_{cc_n}^{c_m c_m} / d_{cc_n} \right) (1 - \delta_{mn}) \right\} \right] .$$

Owing to eq. (56) the functions P_{c_n} present infinite slope in energy, in threshold, for S wave inelastic channels only. We have consequently that only the part $\mathrm{Im}(M_c/D_{cc})$ shows anomalies in threshold, and this happens when S wave inelastic channels are implied.

Since the inelastic S channel is coupled only with elastic D channels, the threshold behaviour of σ^{in} and thre shold anomalies of σ^{se} are driven essentially by the D wave elastic channels.

In particular the threshold inelastic cross section reads :

$$\sigma^{in} \simeq (\pi/k^2)(2Q^{in}_{0,3/2,2} + 3Q^{in}_{0,5/2,2}) . \tag{59}$$

Let us then consider in detail the elastic channel $c = 0, j, 2$, which is coupled with $c_1 = 2, 1/2, 0$ and with D and G inelastic states c_n.

For this particular channel (to the first order in ρ_2) eqs. (57) and (58) become :

$$D_{cc} \simeq d_{cc} - i\rho_2 d^{c_1c_1}_{cc} , \tag{60a}$$

$$D_{cc_n} \simeq d_{cc_n} - i\rho_2 d^{c_1c_1}_{cc_n} (1 - \delta_{1n}) , \tag{60b}$$

$$\frac{D_{cc_n}}{D_{cc}} \simeq \frac{d_{cc_n}}{d_{cc}} \left[1 + i\rho_2 \left\{ (d^{c_1c_1}_{cc}/d_{cc}) - (d^{c_1c_1}_{cc_n}/d_{cc_n})(1 - \delta_{1n}) \right\} \right] . \tag{61}$$

And finally :

$$\mathrm{Im}(M_c/D_{cc}) \simeq \rho_2 \Sigma_n (-)^n (d_{cc_n}/d_{cc}) \cdot$$
$$\cdot \left[(d^{c_1c_1}_{cc}/d_{cc}) - (d^{c_1c_1}_{cc_n}/d_{cc_n})(1 - \delta_{1n}) \right] \tag{62}$$

which shows that $\mathrm{Im}(M_c/D_{cc})$ presents in this case infinite slope in energy as forseen.

In the vibrational case the contribution of the individual inelastic channels are well separated, and eq. (62) becomes :

$$\mathrm{Im}(M_c/D_{cc}) =$$
$$= - \frac{\rho_2}{X_{c_1} - S_{c_1} + \alpha_{c_1c_1}} \Sigma_n \frac{\alpha^2_{cc_n}}{X_{c_n} - S_{c_n} + \alpha_{c_nc_n}} (2 - \delta_{1n}) . \tag{63}$$

APPENDIX. -

The very well known expression of the internal wave function and logaritmic derivative are :

$$F_l(\mu+i\nu) = \sqrt{\pi(\mu+i\nu)/2}\; J_{l+1/2}(\mu+i\nu) , \tag{A1}$$

$$z_o^{int} = x_o^{int} + i\, y_o^{int} = (\mu+i\nu)\cot(\mu+i\nu) ,$$

$$z_l^{int}(l\neq 0) = x_l^{int} + i\, y_l^{int} = (\mu+i\nu)\,\frac{F_{l-1}(\mu+i\nu)}{F_l(\mu+i\nu)} - 1 . \tag{A2}$$

By the position

$$F_l(\mu+i\nu) = F_l^R + i\, F_l^I , \tag{A3}$$

one obtains, from eqs. (A2) :

$$x_o^{int} = \left[\mu \sin 2\mu + \nu \sinh 2\nu\right]\left[\cosh 2\nu - \cos 2\mu\right]^{-1} ,$$

$$y_o^{int} = \left[\nu \sin 2\mu - \mu \sinh 2\nu\right]\left[\cosh 2\nu - \cos 2\mu\right]^{-1} , \tag{A4}$$

$$x_l^{int}(l\neq 0) = \frac{\mu\,(F_l^R F_{l-1}^R + F_l^I F_{l-1}^I) - \nu\,(F_l^R F_{l-1}^I - F_l^I F_{l-1}^R)}{(F_l^R)^2 + (F_l^I)^2} - 1 ,$$

$$y_l^{int}(l\neq 0) = \frac{\nu\,(F_l^R F_{l-1}^R + F_l^I F_{l-1}^I) + \mu\,(F_l^R F_{l-1}^I - F_l^I F_{l-1}^R)}{(F_l^R)^2 + (F_l^I)^2} . \tag{A5}$$

and

The real/imaginary part of F_l can be calculated by the following recursion formulas :

$$F_o^R = \sin\mu \cosh\nu ,$$

$$F_o^I = \cos\mu \sinh\nu , \tag{A6}$$

$$F_1^R = \frac{\mu F_o^R + \nu F_o^I}{\mu^2 + \nu^2} - \cos\mu \cosh\nu , \qquad ./.$$

$$F_1^I = \frac{\mu F_o^I - \nu F_o^R}{\mu^2 + \nu^2} + \sin\mu \sinh\nu \quad , \tag{A6}$$

$$F_1^R(1>1) = \frac{21-1}{\mu^2+\nu^2}\left[\mu F_{1-1}^R + \nu F_{1-1}^I\right] - F_{1-2}^R \quad ,$$

$$F_1^I(1>1) = \frac{21-1}{\mu^2+\nu^2}\left[\mu F_{1-1}^I - \nu F_{1-1}^R\right] - F_{1-2}^I \quad . \tag{A7}$$

Since functions F_1 (eq. (A1)) are known to have no complex zero it follows (eq. (A2)) that in presence of volume absorption, z_e^{int} has no singularity for real energies. This is obviously no more true in the case of elastic scattering or pure surface absorption.

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