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G. Passatore: ON THE ENERGY DEPENDENCE OF THE EMPIRICAL OPTICAL POTENTIAL. -

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ABSTRACT. -

The energy behaviour of the real part of the empirical optical potential for nucleons is derived starting from the energy dependence and the non-locality of the generalized optical potential defined in the many-body nuclear problem. The energy dependence of this one is evaluated by means of a dispersion relation and the effect of the non-locality is calculated in terms of a suitable form factor for the non-local kernel. A good over all agreement with experimental data for neutrons and protons from 10 MeV to 1 BeV is obtained and an estimate of about .8 f for the range of non-locality of the generalized optical potential is indicated.

1. INTRODUCTION. -

In a previous work[1] a dispersion relation derived by Feshbach[2, 3] for the energy dependence of the generalized optical potential was applied to the empirical optical model under suitable conditions. It was found that the experimental results derivated from the dispersion relation indicating a non-local character for the generalized optical potential.

Starting from this result in the present paper we try to obtain a formula for the energy dependence of the real part of the empirical potential and to get an estimate of the above non-locality.

The main idea is to extract from the generalized optical potential a contribution which becomes local as the energy increases and satisfies the dispersion relation. This is suggested both by the previous high energy results and by Feshbach's representation. The real part of the empirical potential can then be expressed as a sum of a local energy dependent term, which can be calculated from the dispersion relation, and a Fourier transform of a non-local kernel. This Fourier transform goes to zero for $E \to \infty$, but it is important at low and intermediate energies where it gives the additional energy dependence of the empirical potential due to its local character. If we describe the non-local kernel by a two-parameter form factor, we get a two-parameter formula for the real part of the empirical potential (a third parameter is brought by the dispersion relation, which has been used in a subtracted form). With a value of the non-locality parameter of approximately 0.8 f the experimental results are well reproduced in the whole range from about 10 MeV to 1 BeV. Below 10 MeV the present treatment does not apply and above some BeV it may become unreliable. Furthermore at these energies, while a theoretical potential in principle may be defined by means of an analytical continuation, it is not clear how to define an empirical potential to be compared with it.

Attempts of deriving the energy dependence of the empirical optical potential from properties suggested by the fundamental many-body approach have been made in the last few years[4, 10]. They are based on the non-locality of the fundamental potential. But its own energy dependence and a possible variation of the non-locality with the energy and the dispersion relation are completely disregarded. This paper tries to take them all into account as far as the potential depth in the interior of a heavy nucleus is concerned.

The approach developed in this paper rests on two points, a theoretical and a phenomenological one, which are at present only suggested but not yet proved in a decisive way.

The first one concerns the use of the dispersion relation itself. This relation is proved in the framework of Feshbach's unified theory of nuclear reactions and applies also when the exclusion principle is taken into account[11, 1]. It follows from a very general causal condition on the inelastic open channels. As Feshbach's theory is non-relativistic, the dispersion relation strictly applies to the non-relativistic generalized optical potential and to its analytical continuation. However its causal basis suggests its validity in a more complete theory which could take into account all the inelastic channels, including those with particle production. According to this assumption all these channels would contribute to the imaginary part within the dispersion integral.

The second point concerns the optical model analyses in the range from 500 MeV to 1 BeV. All they are consistent with the dispersion relation. This result suggests the formula obtained in this paper for the real part of the empirical potential. Unfortunately these high energy data mainly concern light nuclei, are very few and have large errors, so further measurements in this energy range would be important.

2. BASIC CONSIDERATIONS AND OUTLINE OF THE PROCEDURE. -

The amplitude for the elastic scattering of a nucleon by a nucleus is given by the scattering solution of the equation:

$$(1) \qquad \frac{\hbar^2}{2m} \Delta u(\underset{\sim}{r}) + E u(\underset{\sim}{r}) = -\int d^3r' \, \mathcal{U}(E; \underset{\sim}{r}, \underset{\sim}{r}') u(\underset{\sim}{r}')$$

The kernel $\mathcal{U}(E; \underset{\sim}{r}, \underset{\sim}{r}')$ is the well depth of the generalized optical potential[2] and has the following structure:

$$(2) \qquad \mathcal{U}(E; \underset{\sim}{r}, \underset{\sim}{r}') = V_o(\underset{\sim}{r}) \delta(\underset{\sim}{r} - \underset{\sim}{r}') + V(\underset{\sim}{r}, \underset{\sim}{r}') + \mathcal{V}(E; \underset{\sim}{r}, \underset{\sim}{r}')$$

where V_o and V are real, $\mathcal{V}$ complex. V_o is the expectation value of the many-body potential in the ground state of the target nucleus, V is due to the Pauli principle and $\mathcal{V}$ is a consequence of the many-body character of the interaction.

Let us consider the case of infinite nuclear matter. Here the generalized optical potential, eq. (2), takes the form:

$$(3) \qquad \mathcal{U}(E; s) = V_o \delta(\underset{\sim}{s}) + V(s) + \mathcal{V}(E; s)$$

where

(4) $\underset{\sim}{s} = \underset{\sim}{r}' - \underset{\sim}{r}$

and the dispersion relation becomes simply[2]:

(5) $$\mathrm{Re}\ \mathcal{U}(E;s) = \frac{1}{\pi} \mathcal{P} \int \frac{\mathrm{Im}\, \mathcal{U}(E', s)}{E' - E}\, dE'$$

Furthermore in infinite nuclear matter the non-local potential depth $\mathcal{U}(E;s)$ can be substituted by an "equivalent local potential depth"[4, 5, 6] $\mathcal{U}_L(E)$ given by:

(6) $$\mathcal{U}_L(E) = V_o + F(k^2) + \mathcal{F}(E;k^2)$$

where:

(7) $$F(k^2) = \int V(s)\, e^{i \underset{\sim}{k} \cdot \underset{\sim}{s}}\, d^3 s$$

(8) $$\mathcal{F}(E;k^2) = \int \mathcal{U}(E;s)\, e^{i \underset{\sim}{k} \cdot \underset{\sim}{s}}\, d^3 s$$

(9) $$k^2 = \frac{2m}{\hbar^2} \left[E + \mathcal{U}_L(E) \right]$$

A physical realization of the conditions of isotropy and uniformity belonging to the infinite nuclear matter is expected to be given by the interior of the heavy nuclei when the wavelength of the incident particle is small if compared with the nuclear dimensions. Furthermore under these conditions the equivalent local potential (6) can be identified with the empirical optical model potential[1]. So eq. (6) through eq. (5) contains a very general property of a directly measurable quantity. It is worthwhile to discuss the consequences which can be drawn from it.

If V(s) and $\mathcal{U}(E;s)$ were local, then $\mathcal{U}_L(E)$ itself would satisfy the dispersion relation. In ref. (1) it has been found that for 80 MeV $\lesssim$ E $\lesssim$ 600 MeV the experimental Re $\mathcal{U}_L(E)$ is in strong disagreement with that obtained from the subtracted in dispersion relation. On the other hand it was found that at both lower and higher energies the dispersion relation gave for Re $\mathcal{U}_L(E)$ a slope in agreement with the experiment.

In the present paper the previous calculation has been refined (see sect. 4) and we can conclude that

i) between 500 MeV and 1 BeV the calculated slope of Re $\mathcal{V}_L(E)$ agrees with the empirical one while at intermediate energies (100 MeV $\lesssim E \lesssim$ 500 MeV) it is in strong disagreement with the experiment, whatever absorption mechanism is chosen below 100 MeV. These results depend mainly on the dominant features of the absorption at high energies, i.e. the meson production which in turn determines the nucleon-nucleon total cross sections;

ii) Below 100 MeV the results for the slope of Re $\mathcal{V}_L(E)$ strongly depend on the absorption mechanism chosen at these energies.

Unfortunately between 500 MeV and 1 BeV the empirical analyses are very few and the errors large (see Fig. 3). Only the analysis at 725 MeV[12] concerns heavy nuclei, while those at 625 MeV and at 970 MeV[13] concern C^{12}. On the other hand all these results are very similar, and so we accept all these data.

Point i) may be interpreted as saying that the additional energy dependence of Re $\mathcal{V}_L(E)$ due to its local character is confined to energies smaller than 500 MeV. Furthermore it suggests that the kernel $\mathcal{V}(E;s)$, eq. (8), must contain a term which behaves as local when the energy increases and satisfies the dispersion relation. This is also indicated by Feshbach's representation of the generalized optical potential as is shown in appendix 1.

These features can be easily expressed in a model suitable for calculations. We write $\mathcal{V}(E;s)$ as a superposition of a non-local and a local potential:

$$\mathcal{V}(E;s) = \frac{1}{4\pi s^2}\,\delta(s)\,\mathcal{V}_1(E;s) + \mathcal{V}_2(E;s) \tag{10}$$

Here $\mathcal{V}_2(E;s)$ is assumed to be regular function of s at any E important only within a "range of non-locality". Even if eq. (10) gives an oversimplified model for $\mathcal{V}(E;s)$, nevertheless it accounts for the property which is essential to the present application, i.e. the contribution of a local term which at high energies gives all the energy dependence of the r.h.s. in eq. (6). By using eq. (10), eq. (6) now becomes:

$$\mathcal{V}_L(E) = V_o + \mathcal{V}_1(E) + F(k^2) + \mathcal{F}_2(E;k^2) \tag{11}$$

where:

$$\mathcal{V}_1(E) = \mathcal{V}_1(E;s)_{s=o}\,, \tag{12}$$

$$\mathcal{F}_2(E;k^2) = \int \mathcal{V}_2(E;s)\, e^{i \underset{\sim}{k} \cdot \underset{\sim}{s}}\, d^3 s \tag{13}$$

In the following sections we discuss how eq. (11) may be used to obtain explicitely the energy dependence of the real part of the empirical potential. Let us summarize the steps of the procedure:

a): we discuss the high energy behaviour of the functions $F(k^2)$ and $\mathcal{F}_2(E;k^2)$ on the basis of the present data on $\mathcal{V}_L(E)$ and show that they are expected to vanish as $E \to \infty$;

b): this result allows us to calculate, even if with some uncertainty, the real part of $\mathcal{V}_1(E)$ by substituting the imaginary part of the empirical potential $\mathcal{V}_L(E)$ for the imaginary part of $\mathcal{V}_1(E)$ in the dispersion integral;

c): we construct a two parameter form factor for the non-local kernel $V(s) + \mathrm{Re}\, \mathcal{V}_2(E;s)$ which accounts for the discrepancy at intermediate energies between the quantity $V_o + \mathrm{Re}\, \mathcal{V}_1(E)$ obtained from the dispersion relation and the experimental $\mathrm{Re}\, \mathcal{V}_L(E)$.

3. THE HIGH ENERGY BEHAVIOUR OF THE FUNCTIONS $F(k^2)$ AND $\mathcal{F}_2(E;k^2)$. -

After making the angular integration in eqs. (7) and (13) we have:

$$F(k^2) = \frac{2\pi}{i}\, \frac{k_1 - ik_2}{k_1^2 + k_2^2} \int_{-\infty}^{\infty} ds\, s\, V(s)\, e^{-k_2 s}\, e^{ik_1 s} \tag{14}$$

$$\mathcal{F}_2(E;k^2) = \frac{2\pi}{i}\, \frac{k_1 - ik_2}{k_1^2 + k_2^2} \int_{-\infty}^{\infty} ds\, s\, \mathcal{V}_2(E;s)\, e^{-k_2 s}\, e^{ik_1 s} \tag{15}$$

Here $V(s)$ and $\mathcal{V}_2(E;s)$ have been defined for negative s as even functions and k_1 and k_2 are the real and imaginary parts of k. As this one depends on $\mathcal{V}_L(E)$, which is taken from relativistic analyses, it is more appropriate to use, when

$$E \gg \left| \mathcal{V}_L(E) \right| , \tag{16}$$

the relativistic expression:

$$k^2 = \frac{1}{\hbar^2 c^2} \left(E_t^2 - m^2 c^4 + 2 E_t\, \mathcal{V}_L(E) \right) \tag{17}$$

which at non-relativistic energies gives back eq. (9). Here E_t is the total energy. Then:

$$k_1 = \frac{1}{\hbar c}\left[(E_t^2 - m^2c^4 + 2E_t \mathrm{Re}\, \mathcal{V}_L)^2 + \right. \tag{18}$$

$$\left. + (2E_t \mathrm{Im}\, \mathcal{V}_L)^2\right]^{1/4} \cos\frac{1}{2}\mathrm{arc\,tg}\, \frac{2E_t \mathrm{Im}\, \mathcal{V}_L}{E_t^2 - m^2c^4 + 2E_t \mathrm{Re}\, \mathcal{V}_L}$$

$$k_2 = \frac{1}{\hbar c}\left[(E_t^2 - m^2c^4 + 2E_t \mathrm{Re}\, \mathcal{V}_L)^2 + \right. \tag{19}$$

$$\left. + (2E_t \mathrm{Im}\, \mathcal{V}_L)^2\right]^{1/4} \mathrm{sen}\frac{1}{2}\,\mathrm{arc\,tg}\, \frac{2E_t \mathrm{Im}\, \mathcal{V}_L}{E_t^2 - m^2c^4 + 2E_t \mathrm{Re}\, \mathcal{V}_L}$$

The asymptotic behaviour of k_1 and k_2 for large E is:

$$k_1 \sim \frac{1}{\hbar c}\,(E_t^2 - m^2c^4)^{1/2} \equiv \frac{1}{\lambda} \tag{20}$$

$$k_2 \sim \frac{\mathrm{Im}\, \mathcal{V}_L(E)}{\hbar c}\ , \tag{21}$$

if $\mathrm{Re}\, \mathcal{V}_L$ and $\mathrm{Im}\, \mathcal{V}_L$ can be neglected with respect to E.
This is suggested by the high energy empirical data and by the extrapolation formulae[14]:

$$\mathrm{Im}\, \mathcal{V}_L(E) = \frac{1}{2}\,\hbar v\,(\rho_n \sigma_{pn} + \rho_p \sigma_{pp}) \qquad \text{(protons)} \tag{22}$$

$$\mathrm{Im}\, \mathcal{V}_L(E) = \frac{1}{2}\,\hbar v\,(\rho_n \sigma_{nn} + \rho_p \sigma_{np}) \qquad \text{(neutrons)} \tag{23}$$

Here the ρ's are the neutron and proton densities, the σ's the nucleon-nucleon cross sections and v is the velocity of the incident nucleon in the laboratory. The asymptotic behaviour of the nucleon-nucleon total cross section is an open question[15]. If we assume as an upper bound the value indicated by all the recent measurements[15] between 5 and

25 BeV ($\sigma \approx 40$ mb), then eqs. (22) and (23) give an upper bound for Im ν_L as E goes to infinity.

From eqs. (20) and (21) it follows that:

$$\lim_{E \to \infty} F(k^2) = \lim_{E \to \infty} \mathcal{F}_2(E;k^2) = 0 \tag{24}$$

This would be an obvious result if the factors of $e^{ik_1 s}$ within the integrals in eqs. (14) and (15) did not depend on the energy. However it can be proved that eqs. (24) hold if the integrals (14) and (15) exist and under some very reasonable additional hypotheses on the energy dependence of $\nu_2(E;s)$, which, on the other hand, are implied by the dispersion relation. We refer to appendix 2 for a more detailed discussion.

Eqs. (24) suggest then that if the energy is large enough:

$$\text{Im } \nu_L(E) \sim \text{Im } \nu_1(E) \tag{25}$$

$$\frac{\partial \text{ Re } \nu_L(E)}{\partial E} \sim \frac{\partial \text{ Re } \nu_1(E)}{\partial E} \tag{26}$$

The result discussed in sect. 2 about the slope of Re $\nu_L(E)$ given by the dispersion relation can be interpreted as an indication that eqs. (25) and (26) apply for $E \gtrsim 500$ MeV.

4. THE USE OF THE DISPERSION RELATION. -

We now assume that $\nu_1(E)$ given by eqs. (12), (10), satisfies the dispersion relation (5). We shall use this relation written in the subtracted form:

$$\text{Re } \nu_1(E) - \text{Re } \nu_1(E_o) = \frac{E-E_o}{\pi} \mathcal{P} \int_o^\infty \frac{\text{Im } \nu_1(E')}{(E'-E)(E'-E_o)} \, dE' , \tag{27}$$

to calculate the real part of $\nu_1(E)$ in terms of the imaginary part.

Of course it is impossible to obtain an explicit expression for Im $\nu_1(E)$ from any present theory. The best we can do is to identify, as suggested by eq. (25), Im $\nu_1(E)$ with Im $\nu_L(E)$ at high energies, and to take this one from the experiment. At lower energies we can give some general prescriptions on the behavior of Im $\nu_1(E)$, so that it is possible to use the dispersion relation to evaluate Re $\nu_1(E)$ - Re $\nu_1(E_o)$ with

an uncertainty which can be estimated.

To show this procedure in more detail we first consider the energy behaviour of the imaginary part of the empirical potential.

The results from the optical model analyses are shown in Figure 1. Below 100 MeV there is a strong ambiguity due to the assumption of volume or surface absorption. At high energies we use for both proton and neutrons[14]:

$$\text{Im } V_L(E) = \frac{1}{2} \hbar v \gamma \rho \bar{\sigma} \tag{28}$$

with

$$\rho = 2\rho_n = 2\rho_p \tag{29}$$

$$\bar{\sigma} = \frac{1}{2}(\sigma_{nn} + \sigma_{np}) = \frac{1}{2}(\sigma_{pp} + \sigma_{pn}) \tag{30}$$

and γ is the Goldberger's factor[16]. Formula (28) fits well into the results from the direct optical model analyses above 300 MeV. Differing from ref. (1), we here take into account the spatial dependence of ρ by assuming for it the distribution of the nuclear charge suggested by Hofstadter[17]:

$$\rho(r) = \frac{\rho_o}{1 + e^{(r-r_o)/a}} \tag{31}$$

with the parameters:

$$r_o = 1.07\, A^{1/3} \times 10^{-13} \text{ cm} \tag{32}$$

$$a = 5.46 \times 10^{-14} \text{ cm} \tag{33}$$

and normalization:

$$\int \rho(r)\, dV = A \tag{34}$$

The parameter r_o changes from 5.10 f to 6.65 f in passing from Ag to U and we have chosen the arithmetic average. At the centre of the nucleus the eq. (31) gives:

$$\rho(o) = 1.78 \times 10^{38} \text{ cm}^{-3} \tag{35}$$

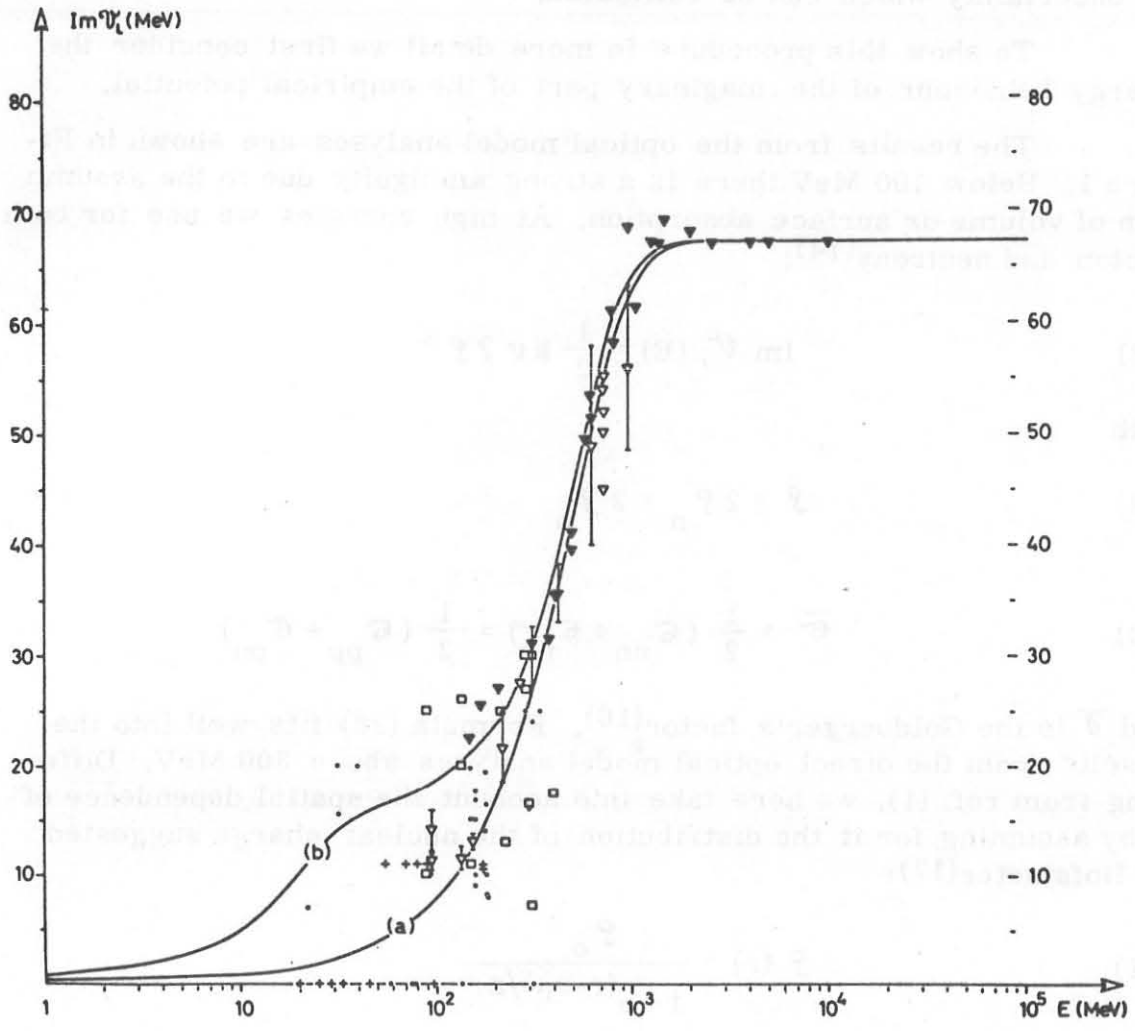

FIG. 1 - Curve (a): Im $\mathcal{V}_L(E)$ from eq. (36), case (a).
Curve (b): Im $\mathcal{V}_L(E)$ from eq. (36), case (b).

Experimental points from refs. (26), (25), (12), (13), (18), (15).

+ neutrons
● protons
□ protons, non-relativistic analyses
▽ protons, relativistic analyses
▼ from the nucleon-nucleon cross sections.

The nucleon-nucleon total cross sections at intermediate energies have been taken from refs. (18) and between 5 and 25 BeV from ref. (15). For the asymptotic behaviour the constant value of 40 mb, as suggested by the high energy measurements, had been assumed. Deviations from this assumption[15, 19] are expected to be unimportant in applying the dispersion relation in the subtracted form up to 1 BeV.

The curves (a) and (b) in Fig. 1 describe two possible behaviours of $\mathrm{Im}\, v_L(E)$ in the interior of a nucleus according to eqs. (28) and (35). Curve (a) corresponds to an "extreme surface absorption", curve (b) to an "extreme volume absorption". The appropriate behaviour of $\mathrm{Im}\, v_L(E)$ lies between these curves, curve (a) probably being the more realistic one. The two curves coincide at about 400 MeV where both show a rapid rise due to mesonic effects.

For the energy behaviour of $\mathrm{Im}\, v_1(E)$, eq. (27), we can give the following prescriptions:

i) For $E \to \infty$ $\mathrm{Im}\, v_1(E)$ has the same asymptotic behaviour as $\mathrm{Im}\, v_L(E)$ (eq. 25);
ii) $\mathrm{Im}\, v_1(E)$ is a monotonic increasing function of the energy from zero at $E = 0$;
iii) As the increase of $\mathrm{Im}\, v_1(E)$ is due to the threshold for inelastic processes which open at increasing energy, a rapid rise has to be expected when the meson production becomes important.

These prescriptions, together with the dispersion relation result for $[\partial\, \mathrm{Re}\, v_L(E)]/\partial E$, suggest to identify $\mathrm{Im}\, v_1(E)$ with $\mathrm{Im}\, v_L(E)$ for energies higher than 400 ÷ 500 MeV. At lower energies the above prescriptions leave $\mathrm{Im}\, v_1(E)$ uncertain to some extent. Below 100 MeV $\mathrm{Im}\, v_L(E)$ itself is very ambiguous. It seems reasonable to assume that $\mathrm{Im}\, v_1(E)$ lies between curves (a) and (b) of Fig. 1. This causes an uncertainty in $\mathrm{Re}\, v_1(E)$ which can be estimated from the change of the dispersion integral when $\mathrm{Im}\, v_1(E)$ changes from the shape (a) to (b).

Shapes (a) and (b) are described by the functions:

$$W^{(i)} = W_o + \sum_j \frac{A_j^{(i)}}{1 + e^{(E - B_j^{(i)})/C_j^{(i)}}} \qquad (i = a, b) \tag{36}$$

with (all quantities in MeV)

$$W_o = 67.2 \tag{37}$$

$$A_1^{(a)} = 92.4 \qquad A_2^{(a)} = 0$$

(38) $$B_1^{(a)} = 250$$

$$C_1^{(a)} = 256$$

$$A_1^{(b)} = 59.8 \qquad A_2^{(b)} = 15.0$$

(39) $$B_1^{(b)} = 400 \qquad B_2^{(b)} = 15.0$$

$$C_1^{(b)} = 201 \qquad C_2^{(b)} = 7.50$$

By using the functions (36) in the dispersion relation (27) we obtain:

(40) $$\mathrm{Re}\, \nu_1^{(i)}(E) - \mathrm{Re}\, \nu_1^{(i)}(E_o) = g^{(i)}(E) - g^{(i)}(E_o) \quad (i = a, b)$$

where:

(41) $$g^{(i)}(E) = -\frac{W_o}{\pi} \ln E - \sum_j \frac{A_j^{(i)}}{\pi} G_j^{(i)} \left(\frac{E}{C_j^{(i)}}\right)$$

(42) $$G_j^{(i)}(y) = \mathcal{P} \int_0^\infty \frac{f_j^{(i)}(x)}{x - y}\, dx$$

(43) $$f_j^{(i)}(x) = \frac{1}{1 + e^{x - \mu_j^{(i)}}} \; ; \; \mu_j^{(i)} = \frac{B_j^{(i)}}{C_j^{(i)}}$$

The functions $g^{(i)}(E)$ are plotted in Figs. 2a and 2.b, where they are also analyzed in the contributions coming from the various terms on the r.h.s. of eq. (41). The term $(A_2^{(b)}/\pi) G_2^{(b)}(E/C_2^{(b)})$ is completely negligible except at low energy. On the contrary, the terms $(A_1^{(i)}/\pi) G_1^{(i)}(E/C_1^{(i)})$ are important not only at low and intermediate energies, but also play an essential role at high energies in giving the slope of $g^{(i)}(E)$ from 500 MeV to 1 BeV. This slope is essentially determined by the fact that the above terms get their maximum around 500 MeV, and this is in turn a consequence of the step rise of Im $\nu_L(E)$ between 400 MeV and 1 BeV due to mesonic effects.

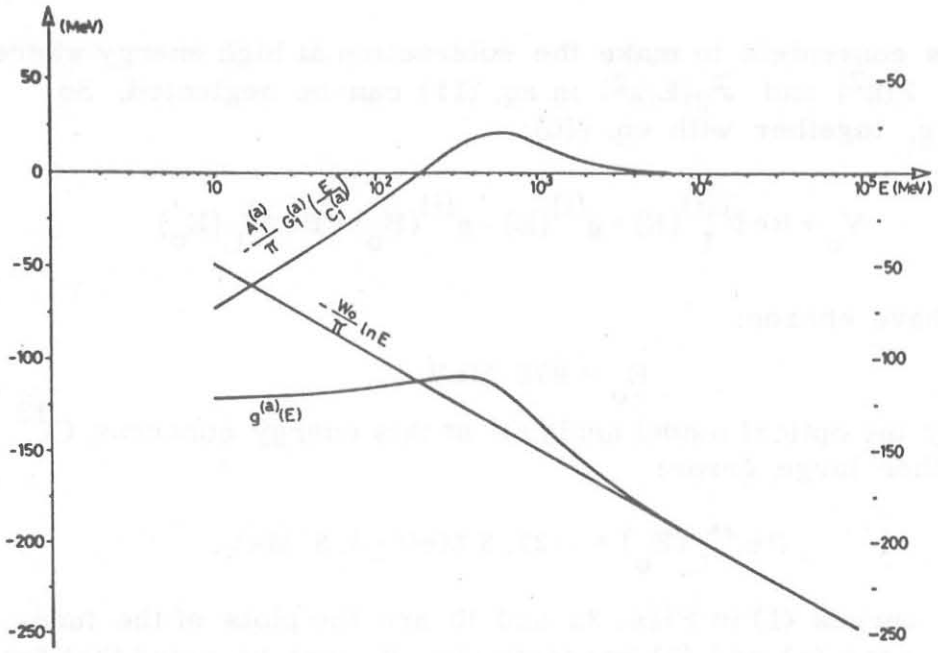

FIG. 2a - The functions $g^{(a)}(E)$, $-\frac{W_o}{\pi} \ln E$, $-\frac{A_1^{(a)}}{\pi} G_1^{(a)}\left(\frac{E}{C_1^{(a)}}\right)$

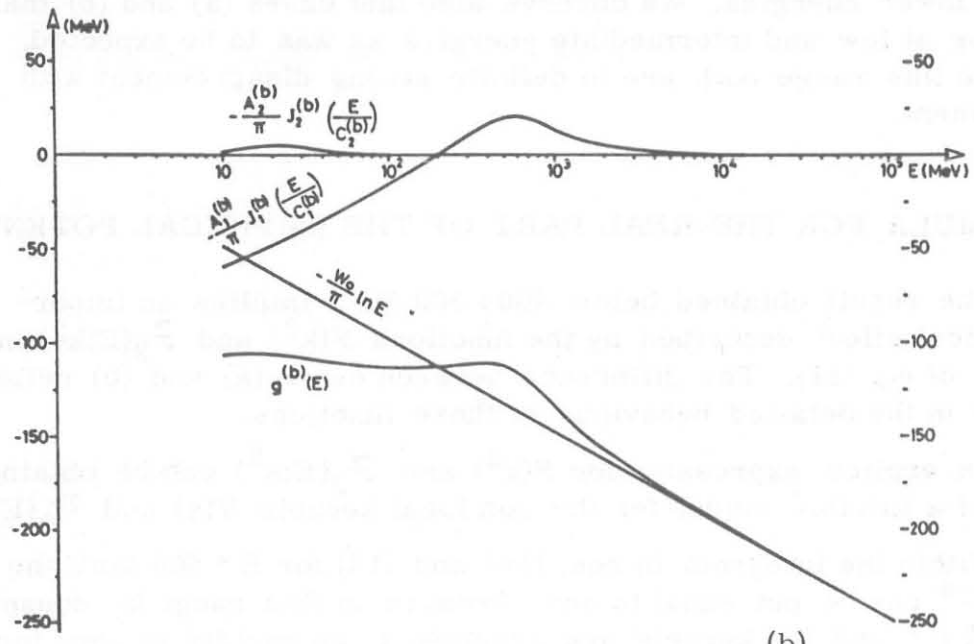

FIG. 2b - The functions $g^{(b)}(E)$, $-\frac{W_o}{\pi} \ln E$, $-\frac{A_j^{(b)}}{\pi} G_j^{(b)}\left(\frac{E}{C_j^{(b)}}\right)$.

It is convenient to make the subtraction at high energy where the functions $F(k^2)$ and $\mathcal{F}_2(E;k^2)$ in eq. (11) can be neglected. So eq. (11) gives, together with eq. (40):

$$V_o + \mathrm{Re}\, v_1^{(i)}(E) = g^{(i)}(E) - g^{(i)}(E_o) + \mathrm{Re}\, v_L(E_o) \tag{44}$$

For E_o we have chosen:

$$E_o = 970 \text{ MeV} \tag{45}$$

Unfortunately the optical model analysis at this energy concerns C^{12} and has a rather large error:

$$\mathrm{Re}\, v_L(E_o) = -22.3 \text{ MeV} \pm 4.3 \text{ MeV}. \tag{46}$$

The curves (1) in Figs. 3a and 3b are the plots of the function (44) for cases (a) and (b) respectively. It must be noted that for both cases this function fits well into the experimental values of $\mathrm{Re}\, v_L(E)$ in the energy range from about 400-500 MeV to E_o, as is implied by the present treatment. So we find that, after improving the function $\mathrm{Im}\, v_L(E)$ within the dispersion integral, the feature of the result found in ref. (1) suggesting eqs. (10) and (11) is confirmed and extended to somewhat lower energies. We observe also that cases (a) and (b) markedly differ at low and intermediate energies as was to be expected. However in this range both are in definite strong disagreement with the experiment.

5. A FORMULA FOR THE REAL PART OF THE EMPIRICAL POTENTIAL. -

The result obtained below 400÷500 MeV implies an important non-local effect described by the functions $F(k^2)$ and $\mathcal{F}_2(E;k^2)$ in the r.h.s. of eq. (11). The difference between cases (a) and (b) reflects itself only in the detailed behaviour of these functions.

An explicit expression for $F(k^2)$ and $\mathcal{F}_2(E;k^2)$ can be obtained in terms of a suitable model for the non local kernels $V(s)$ and $v_2(E;s)$.

Within the integrals in eqs. (14) and (15) for $E < 500$ MeV the factor $e^{-k_2 s}$ can be put equal to one, because in this range k_2 doesn't exceed $0.2\ f^{-1}$ and the kernels are assumed to go rapidly to zero for $s > 1f$. If we further neglect in eqs. (14) and (15) k_2 with respect to k_1, as is suggested by eqs. (18) and (19), we obtain:

$$\mathrm{Re}\left[F(k^2) + \mathcal{F}_2(E;k^2)\right] = \frac{2\pi}{k_1} \int_{-\infty}^{\infty} ds\ s\left[V(s) + \mathrm{Re}\, v_2(E;s)\right] \mathrm{sen} k_1 s \tag{47}$$

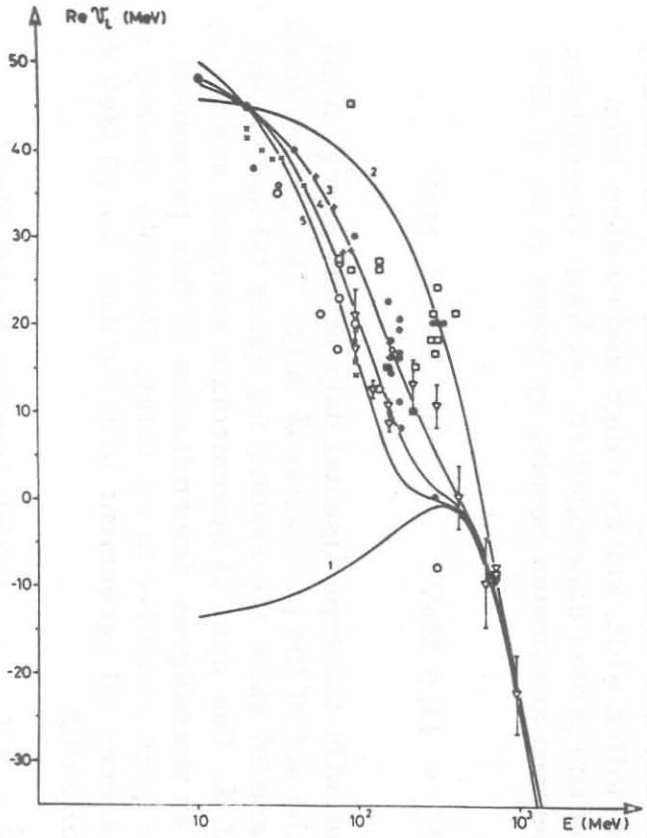

FIG. 3a - Curve (1): $V_o + \mathrm{Re}\, v_1^{(a)}(E)$ from eq. (44), Curves from (2) to (5): $\mathrm{Re}\, v_L(E)$ from eq. (51), case (a). (2): $\beta = .5$ f; (3): $\beta = .8$ f; (4) $\beta = 1.0$ f; (5) $\beta = 1.2$ f. Experimental points from refs. (26), (25), (12), (13), (20).

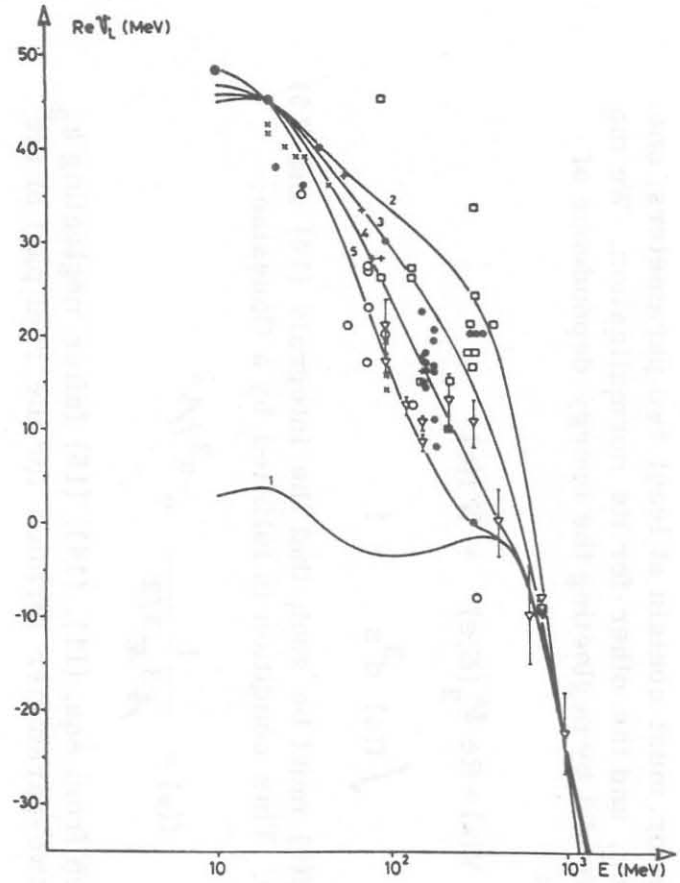

FIG. 3b - Curve (1): $V_o + \mathrm{Re}\, v_1^{(b)}(E)$ from eq. (44), case (b). Curves from (2) to (5): $\mathrm{Re}\, v_L(E)$ from eq. (51), case (b). (2): $\beta = .3$ f; (3) $\beta = .5$ f; (4) $\beta = .7$ f; (5) $\beta = 1.0$ f.

+ neutrons, volume absorption; ✖ neutrons, surface absorption; ● protrons, volume absorption; ○ protons, surface absorption; □ protons, non relativistic analyses; ▽ protons, relativistic analyses; ⊠ average for neutrons and protons from ref. (20).

So only a model for the quantity $V(s) + \mathrm{Re}\, \mathcal{V}_2(E;s)$ is needed here.

This form factor must contain at least two parameters: one for its spatial extension, and the other for its normalization. We make an oversimplified model by neglecting the energy dependence of $\mathcal{V}_2(E;s)$. So we write:

$$V(s) + \mathrm{Re}\, \mathcal{V}_2(E;s) = N f(s) \tag{48}$$

$$\int f(s)\, d^3s = 1 \tag{49}$$

The function f(s) must be such that the integrals (14) and (15) exist (see appendix 2). This condition is fulfilled by a Gaussian:

$$f(s) = \frac{1}{\beta^3 \pi^{3/2}}\, e^{-s^2/\beta^2} \tag{50}$$

We obtain then from eqs. (11), (14), (15) (after neglecting k_2 in these last ones) a two-parameter formula for the real part of the empirical potential:

$$\mathrm{Re}\, \mathcal{V}_L(E) = V_o + \mathrm{Re}\, \mathcal{V}_1(E) + N e^{-k_1^2 \beta^2/4} \tag{51}$$

The parameter N can be determined as a function of β by matching $\mathrm{Re}\, \mathcal{V}_L(E)$ given by eq. (51) to the accurate low energy data. The most recent systematic analyses are those by Rosen et al.[20]. As we are concerned in this work with a wide energy range and so have been obliged to keep neutron and proton data together, we have chosen here the average between neutron and proton results by Rosen et al. at E = = 20 MeV:

$$\mathrm{Re}\, \mathcal{V}_L(E) = 45.0 \text{ MeV} \qquad \text{for } E = 20 \text{ MeV} \tag{52}$$

The agreement with the experimental data at any energy must then be obtained with the aid of the parameter β only. The curves from (2) to (5) in Figs. 3a and 3b show the results for cases (a) and (b) and for various choices of β. The data at intermediate energies are not so sharp as to allow for an unambigous determination of this parameter, even if the results are quite sensitive to its value. Possible choices of β which lead to a good over all agreement with the data for 10 MeV $\lesssim$ $\lesssim E \lesssim$ 1 BeV are the following:

(53 a) case (a) $\beta = .8$ f (which implies N = 96 MeV)

(53 b) case (b) $\beta = .7$ f (which implies N = 61 MeV)

6. CONCLUDING REMARKS. -

Eq. (51) makes a clear distinction between the two sources of the energy dependence of the real part of the empirical potential, i.e. the energy dependence of the generalized optical potential itself and the non-local effect. The first one, which is dominant at high energies, is given by the term $V_o + \mathrm{Re}\ \mathcal{V}_1$ and can be calculated by the dispersion relation, eq. (44). The second one, which is important at low and intermediate energies, is given by the term $F(k^2) + \mathcal{F}_2(E;k^2)$ and can be obtained from a suitable form factor for the non-local kernels. Of course this description is oversimplified, nevertheless it may expresse the essential and general features of the energy dependence of the empirical potential. The values of the parameters β and N have only a semi-quantitative meaning.

Eq. (51) does not apply at low energies (i.e. of some MeV) mainly because there the optical model potential is a quantity quite different from the generalized optical potential and furthermore because the approximations used in writing the dispersion relation in simple form and in deriving the equivalent local potential may not apply.

Also at very high energies (i.e. $E \gtrsim 10$ BeV) eq. (51) must be considered with caution. First of all at these energies the significance of the quantities involved in the dispersion relation should be carefully discussed. A way for defining a potential could be the dispersion relation itself, after defining the imaginary part in terms of the nucleon-nucleon cross sections, eqs. (22), (23). But then it is not clear how to define an empirical potential to be compared with this one. Furthermore the asymptotic behaviour:

$$\mathrm{Re}\ \mathcal{V}_L(E) \sim -\frac{W_o}{\pi} \ln E + \text{constant} \tag{54}$$

indicated by the eq. (51) may be incorrect, because it follows from the asymptotic behaviour assumed for $\mathrm{Im}\ \mathcal{V}_L$, which is very uncertain. As the nucleon-nucleon total cross sections have been explored only up to about 25 BeV, we feel that the above results on $\mathrm{Re}\ \mathcal{V}_L(E)$ may be reliable only up to some BeV. Measurements of proton scattering at 3 BeV/c on carbon indicate a repulsive interaction[21], in agreeement with eq. (51). A similar indication comes from recent measurements and analyses of differential and total cross sections of 19.3 BeV protons by a range of nuclei[22, 23, 24]. It must be noted, however, that the value of $|\mathrm{Re}\ \mathcal{V}_L|$ obtained from the analysis at 19.3 BeV[24] is much lower than that obtained by extrapolating the present dispersion relation

result and does not suggest any logarithmic divergence.

Therefore we think that eq. (51) may be used in the range from about 10 MeV to some BeV. Here it can resolve some optical model analysis ambiguities. An example is given by the analysis by Roos and Wall[25] at 160 MeV. The lower values for Re $V_L(E)$ are suggested by the present calculation. Another example is shown by the systematic discrepancy above 100 MeV between the relativistic and non-relativistic analyses[26]. Only the first ones are consistent with the dispersion relation.

Finally let us compare eq. (51) with other formulae proposed to describe the energy dependence of the real part of the empirical potential. There are empirical formulae which apply in narrow energy ranges[26]. Furthermore several attempts have been made to derive the energy dependence of the empirical potential from an energy-independent non-local model. This procedure has been applied at low energies by Perey and Buck[4] and then by various authors[8,9]. Very recently it has been extended to a wider energy range by Engelbrecht and Fiedeldey[10]. The model developed by the latter authors gives a very accurate description of the energy behaviour of the real part of the empirical potential between zero and 160 MeV. All the energy dependence comes from the relation connecting the non-local (energy-independent) potential with the equivalent local potential. As this model specifically concerns neutrons, it gives:

(55) $$\text{Re } V_L(E) = 39.7 \text{ MeV} \qquad \text{for } E = 20 \text{ MeV}$$

instead of eq. (52). If the parameter N in eq. (51) is determined according to eq. (55), the values obtained for Re $V_L(E)$ fit in with those given by Engelbrecht and Fiedeldey up to 200 MeV both for cases (a) and (b) and with $\beta = .8$ f and $\beta = .7$ f respectively. At higher energies Re $V_L(E)$ given by eq. (51) assumes lower values and changes its sign at about 400 MeV (v. Fig. 4). On the other hand, the model by Engelbrecht and Fiedeldey claims its validity up to about 160 MeV and cannot give any change in sign.

All the models published up to now which derive all the energy dependence of the empirical potential from a non-local energy-independent potential cannot account for a transition from an attractive to a repulsive interaction around 400 MeV, which seems to be indicated by the experiment.

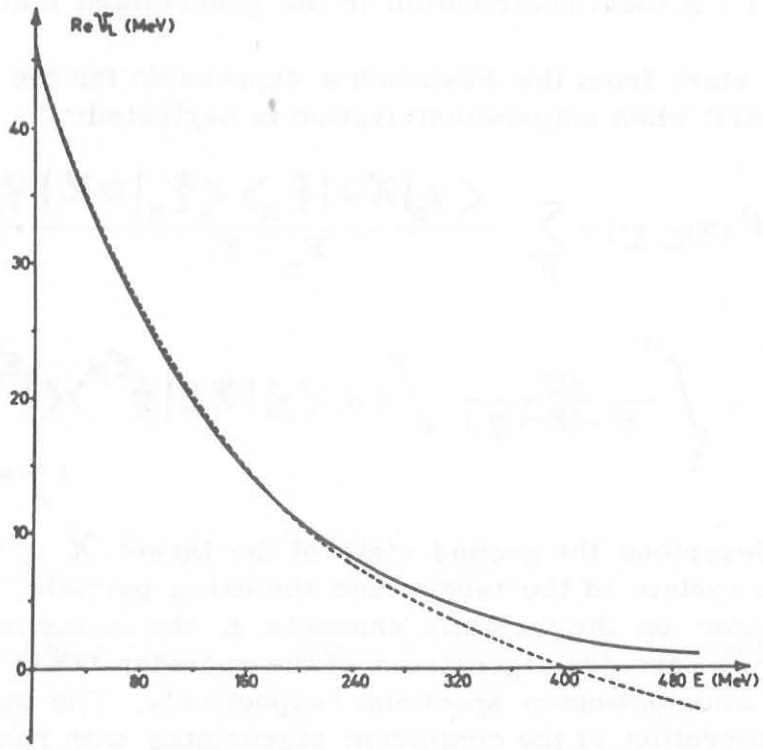

FIG. 4 - Unbroken line : Re $v_L(E)$ from ref. (10)
Dotted line: Re $v_L(E)$ from eq. (51) normalized at 39.7 MeV for E = 20 MeV. (Case (a), b = .8 f).

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APPENDIX 1 - A local contribution to the generalized optical potential. -

We start from the Feshbach's expression for the generalized optical potential when antysimmetrization is neglected:

$$V(E;\underset{\sim}{r},\underset{\sim}{r}') = \sum_n \frac{\langle \varphi_o | \mathcal{H} Q | \Phi_n \rangle \langle \Phi_n | Q \mathcal{H} | \varphi_o \rangle}{E_n - E}$$

(A1.1)
$$+ \int_{\mathcal{E}}^{\infty} \frac{dE'}{E' - (E + i\eta)} \int d\alpha \langle \varphi_o | \mathcal{H} Q | \Phi^{E',\alpha} \rangle \langle \Phi^{E',\alpha} | Q \mathcal{H} | \varphi_o \rangle \qquad (\eta \to 0^+)$$

where φ_o describes the ground state of the target, $\mathcal{H}$ is the hamiltonian of the system of the target and the extra particle, Q is the projection operator on the inelastic channels, $\mathcal{E}$ the inelastic threshold, Φ_n and $\Phi^{E,\alpha}$ are the eigenstates of the operator $Q\mathcal{H}Q$ belonging to the discrete and continuum spectrum respectively. The suffix α describes the degeneration of the continuum eigenstates with respect to the energy. The proper account of the antisymmetrization does not change the essential character of the following discussion.

Let us consider the quantity

(A1.2)
$$\int d\alpha \langle \varphi_o | \mathcal{H} Q | \Phi^{E,\alpha} \rangle \langle \Phi^{E,\alpha} | Q \mathcal{H} | \varphi_o \rangle$$

appearing in eq. (A1.1). The eigenfunction $\Phi^{E,\alpha}$ describes a state of the system of A+1 particles which contains at least two fragments with any amount of relative kinetic energy E_{rel} allowed by the total energy E of the state. The suffix α describes, then, the various possible partitions of the system of A+1 particles at the given energy E and also the quantum numbers for any partition.

Among the states described by $\Phi^{E,\alpha}$ there are those where a free fragment is the incident particle itself. Let us refer to the system of the other A nucleons as to the "residual nucleus". If we neglect long range interactions the wave function for the relative motion can be assumed to be a plane wave

(A1.3)
$$\psi_{rel} = e^{i\underset{\sim}{k} \cdot \underset{\sim}{r}}$$

where $\underset{\sim}{r}$ is the coordinate of the incident particle relative to the centre of mass of the "residual nucleus" and $\underset{\sim}{k}$ is the wave number of the

relative motion. This description of the relative motion can be assumed to be correct everywhere if the relative energy is large enough, i.e. much larger than the potential experienced by the incident particle from possible bound systems of other nucleons.

Let us now consider in particular those states $\Phi^{E,\alpha}$ where the energy of the "residual nucleus" itself belongs to the continuum. These states exist if the total energy E is large enough. The summation over these states, which is implied by the integration over α in (A1.2) gives rise to the integration over the vector $\underset{\sim}{k}$, eq. (A1.3), within a sphere Σ whose radius $\bar{k}$ is the maximum wave number allowed for the relative motion at a given total energy E. Of course the wave number $\bar{k}$ goes to infinite with E. We are interested in the contribution of such states to the expression (A1.2). Let us write for these states:

$$\Phi^{E,\alpha} = f^{E'(k),\beta}(\underset{\sim}{r}_1, \dots \underset{\sim}{r}_A)\, e^{i \underset{\sim}{k} \cdot \underset{\sim}{r}} \tag{A1.4}$$

where the function f describes the state of the "residual nucleus", β denotes its quantum numbers besides the energy and E'(k) its energy.

The contribution of these states to the quantity (A1.2) can be written as:

$$\langle \varphi_0 | \mathcal{H} Q | \int_\Sigma d^3k\, g^{E'(k)}(\underset{\sim}{r}'_1 \dots \underset{\sim}{r}'_A \underset{\sim}{r}_1 \underset{\sim}{r}_A)\, e^{i\underset{\sim}{k}(\underset{\sim}{r} - \underset{\sim}{r}')} | Q \mathcal{H} | \varphi_0 \rangle \tag{A1.5}$$

where

$$g^{E'(k)}(\underset{\sim}{r}'_1 \dots \underset{\sim}{r}'_A \underset{\sim}{r}_1 \dots \underset{\sim}{r}_A) = \tag{A1.6}$$
$$= \sum_\beta f^{*E'(k)}(\underset{\sim}{r}'_1 \dots \underset{\sim}{r}'_A) f^{E'(k)}(\underset{\sim}{r}_1 \dots \underset{\sim}{r}_A)$$

The expression (A1.5) has been obtained after observing that, for the states (A1.4):

$$\int d\alpha \to \sum_\beta \int d\Omega_{\underset{\sim}{k}}\, dE_{rel}\, \rho(E_{rel}) = \sum_\beta \frac{1}{(2\pi)^3} \int_\Sigma d^3k \tag{A1.7}$$

where $\rho(E_{rel})$ is the number of final states for the unit energy interval and unit solid angle in the relative motion, i.e.:

$$\rho(E_{rel}) = \frac{mk}{(2\pi)^3 \hbar^2} \tag{A1.8}$$

To write the quantity (A1.5) more explicitely, we use a definite expression for the projection operator Q:

$$Q = \sum_{i>o} |\varphi_i\rangle\langle\varphi_i| \tag{A1.9}$$

where $|\varphi_i\rangle$, $i>o$, describe the excited states of the target. Strinctly speaking, this expression is correct only if the rearrangement collisions can be neglected. Even if eq. (A1.9) does not give the more general expression for Q, it is to be expected that the present considerations will retain their validity in a more general approach. From the representation (A1.9) it follows that the quantity (A1.5) contains terms like:

$$V_{oi}(\underset{\sim}{r})\, V^{*}_{oj}(\underset{\sim}{r}') \int_{\Sigma} d^3k\, a_{ij}(E'(k))\, e^{i\underset{\sim}{k}\cdot(\underset{\sim}{r}-\underset{\sim}{r}')} \tag{A1.10}$$

where:

$$a_{ij}(E'(k)) = \tag{A1.11}$$

$$= \langle \varphi_i(\underset{\sim}{r}'_1, \ldots \underset{\sim}{r}'_A) \,|\, g^{E'(k)}(\underset{\sim}{r}'_1, \ldots \underset{\sim}{r}'_A, \underset{\sim}{r}_1, \ldots \underset{\sim}{r}_A) \,|\, \varphi_j(\underset{\sim}{r}_1, \ldots \underset{\sim}{r}_A)\rangle$$

The expression (A1.10) has been obtained after writing:

$$\mathcal{H} = T_o + H_A + V(\underset{\sim}{r}, \underset{\sim}{r}_1, \ldots \underset{\sim}{r}_A) \tag{A1.12}$$

where T_o is the kinetic energy operator for the extra particle, H_A the hamiltonian of the target, V the interaction energy between the target and the extra particle, and:

$$V_{oi}(\underset{\sim}{r}) = \langle \varphi_o | V | \varphi_i \rangle = \langle \varphi_o | \mathcal{H} | \varphi_i \rangle \tag{A1.13}$$

For $E \to \infty$, the sphere Σ in (A1.10) covers all the momentum space. Furthermore when in the integrand k goes to infinity the energy E'(k) of the system of the "residual nucleus" goes to the threshold for its continuum, i.e. to the separation energy S of a nucleon. If the quantity $a_{ij}(E')$, eq. (A1.10), calculated for $E' = S$ doesn't vanish, the expression (A1.10) contains, in the limit for $E \to \infty$, a Fourier transform of a function which goes to a non zero limit for k going to infinite, i.e. a delta function $\delta(\underset{\sim}{r} - \underset{\sim}{r}')$. Furthermore, the contribution (A1.10) to the quantity (A1.2) satisfies itself the dispersion relation, as it is evident from (A1.2) and (A1.1).

It must be remarked that, as this discussion concerns the generalized optical potential as derived from a non-relativistic theory, the non-relativistic kinematics has been used throughout. By using the relativistic kinematics, into the integrand on the last side in eq. (A1.7) the factor $1/\sqrt{\hbar^2c^2k^2+m^2c^4}$ appeared, thus preventing the above conclusion.

We think, however, that the use of the relativistic kinematics in the framework of a full non-relativistic theory would be quite inconsistent and would lead to deceptive conclusions. A relativistic generalization cannot concern the kinematics only. This is intuitively seen in the present case by observing that a relativistic theory must account for the particle production, and so more and more new states, which may play the role of those discussed above, would enter as the energy increases.

APPENDIX 2 - The functions $F(k^2)$ and $\mathcal{F}_2(E;k^2)$. -

To discuss the high energy behaviour of the functions $F(k^2)$ and $\mathcal{F}_2(E;k^2)$ defined in eqs. (14) and (15) we must consider an integral of the type:

$$I(E) = \int_{-\infty}^{\infty} ds\, s\, f(E;s)\, e^{iks} \tag{A2.1}$$

where $f(E;s)$ is an even function of s and k is given by eqs. (18), (19), (20), (21). Here the relativistic expression for k is used, because now the quantity to be discussed is determined by the empirical quantity $V_L(E)$ which would contain all the physical effects. First of all, the integral $I(E)$ exists if

$$\lim_{s\to\infty} \frac{s\, f(E;s)}{e^{-\bar{k}_2 s}} = 0 \tag{A2.2}$$

where $\bar{k}_2$ is the upper bound for the absolute value of k_2, eq. (19). The eqs. (19) and (21) for k_2, together with the experiment values for $V_L(E)$ and the extrapolation formulae (22) and (23), show that this bound exists. So the condition (A2.2) can be satisfied by suitable shapes of the function $f(E;s)$. A gaussian function of s is allowed, while a rational function, such as a laurentzian, is ruled out. Exponentials and Yukawa functions can be used only if they go to zero for $s\to\infty$ according to eq. (A2.2). As for $\bar{k}_2^{-1}$ a value of some fermis is indicated, i.e. a value not much larger than the expected non-locality, these last functions would be emploied with care.

We discuss now the high energy behaviour of the integral I(E), eq. (A2.1). As the integrand contains the oscillating factor $e^{ik_1 s}$, with $k_1 \to \infty$ as E, (see eq. (20)), if the other terms are not dependent on E the limit of I(E) for $E \to \infty$ would be zero according to the Riemann--Lebesgue lemma.

Under very reasonable conditions on the kernel f(E;s) such result applies also in this case. Let us assume that the function f(E, s) has derivatives of any order with respect to s which vanish for $s \to \pm \pm \infty$. Then after partial integration we obtain:

$$I(E) = \frac{(-1)^n}{(ik_1)^n} \sum_{p=0}^{n} \binom{n}{p} (-1)^p \, k_2^{p-1} \, I^{(np)}(E) \tag{A2.3}$$

with

$$I^{(np)}(E) = \int_{-\infty}^{\infty} (k_2 s + p)\, e^{-k_2 s} f^{(n-p)}(E, s)\, e^{ik_1 s}\, ds \tag{A2.4}$$

Here n is an arbitrarily large integer and $f^{(n)}(E, s)$ is the n-th derivative of f(E, s) with respect to s. The integrals $I^{(np)}(E)$ exist if the eq. (A2.2) is satisfied. From eqs. (A2.3), (22), (23) it follows that

$$\lim_{E \to \infty} \frac{I(E)}{k} = 0 \tag{A2.5}$$

provided that the integrals $I^{(np)}(E)$ have a limit for $E \to \infty$ and don't diverge faster than E^n.

If, for example, we assume for the kernel f(E, s) a separable model:

$$f(E;s) = \varphi(E)\ \chi(s) \tag{A2.6}$$

where Re $\varphi(E)$ and Im $\varphi(E)$ have a limit for $E \to \infty$, the above condition requires that these functions do not go to infinity, for $E \to \infty$, faster than any power. On the other hand, if this condition is not satisfied, it would be impossible to write for $\varphi(E)$ a dispersion relation with a finite number of subtractions.

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