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V. Vanzani: MATHEMATICAL CORRESPONDENCE BETWEEN  
THE R-MATRIX THEORY OF NUCLEAR REACTIONS AND THE  
ELECTROMAGNETIC THEORY OF WAVEGUIDES AND RESONATORS. -

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V. Vanzani: MATHEMATICAL CORRESPONDENCE BETWEEN THE R-MATRIX THEORY OF NUCLEAR REACTIONS AND THE ELECTROMAGNETIC THEORY OF WAVEGUIDES AND RESONATORS.<sup>(x)</sup>

1. INTRODUCTION. -

The purpose of this paper is to outline a quantitative formulation of the mathematical correspondence existing between the R-matrix theory of nuclear reactions and the theory of waveguides and resonators. This problem is not new: in fact, such a correspondence has been very often invoked on the basis of pure intuition from the very beginning of the development of the theory of nuclear reactions<sup>(1)</sup> and more recently in order to stress some analogies between the behaviour of metastable particles and that of resonant cavities<sup>(2)</sup>.

Although the correspondence between the two theories has a formal character, it is nevertheless remarkable that both of them describe physical effects which are substantially similar; furthermore, such a correspondence discloses the possibility of an electromagnetic simulation of certain particular features of nuclear reactions.

The qualitative approach to the problem is based on the idea that the description of a nuclear reaction in terms of channels and interior region, can be visualized by a waveguide junction formed by a cavity connected with one or more guides; the channels correspond to guides and the nuclear interior region to the cavity. An incoming wave in the entrance guide is partly reflected because of the geometrical cross section limitations of the guide, partly because of the impedance mismatch at the entrance of the cavity; the surviving part of the wave enters the cavity, and, then, after de-excitation of the cavity, goes out through all the open guides. Similarly, an incoming wave in the entrance channel is partly reflected because of centrifugal and coulomb barriers in the chan

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nel, partly at the entrance of the interior region; the remainder is absorbed, and then, after de-excitation of the metastable structure, travels into all the open channels.

In both theories it is assumed that the exterior region (formed by the channels or guides) is well understood, while the interior region can be described only globally: the behaviour of the interior region in proximity of the external one is described by means of the R-matrix or the admittance Y-matrix, respectively.

We shall now outline a quantitative formulation of such a qualitative analogy.

The general features of the nuclear and junction theories will be derived by identical developments based upon the analogous quantities which will be put into correspondence. The procedures concerning the nuclear case are assumed to be known and the junction theory will be outlined on the basis of analogous arguments.

## 2. NUCLEAR CONFIGURATION SPACE AND WAVEGUIDE JUNCTION. -

The assumptions underlying the R-matrix theory<sup>(3)</sup> or the theory of waveguide junctions<sup>(4)</sup> are:

a) applicability of the stationary Schrödinger equation or of the stationary Maxwell equations in a homogeneous, isotropic, source-free medium; b) absence or unimportance of the radiative nuclear processes or of all processes involving losses through the guides and in the junction; c) existence, for any pair of fragments, of the channel radius, beyond which the nuclear interactions are negligible, or existence, for any guide of the terminal section, beyond which the evanescent modes originating in the cavity are negligible.

A reaction channel  $c$  is represented by the set of quantum numbers  $\{\alpha_1 \alpha_2 (I_1 I_2) s \nu, l m\}$  where  $\alpha_i$  specifies the nature and energy of the fragments<sup>(5)</sup>; a waveguide  $g$  by the set of numbers  $\{\beta, pq\}$ , where  $\beta$  specifies the shape of the cross section and the nature of the dielectric medium and  $pq$  the electromagnetic modes<sup>(6)</sup>.

We shall indicate with  $q_{\alpha_1}$  and  $q_{\alpha_2}$  the sets of internal co-ordinates of the two fragments and with  $r_{\alpha}$  the relative co-ordinate of their center of mass; we shall introduce for the waveguide the co-ordinate set  $x_{\beta} y_{\beta} z_{\beta}$  with the  $z_{\beta}$ -axis parallel to the tubular surface and directed towards the junction.

We shall call "nuclear interior region", the configuration space defined as  $r_{\alpha} < a_{\alpha}$  for all  $\alpha$  or "cavity", the junction with the waveguide portions defined as  $z_{\beta} > a_{\beta}$  for all  $\beta$ , where  $a_{\alpha}$  is the channel radius and  $a_{\beta}$  the longitudinal co-ordinate of the terminal section. The boundary

surface defined through the condition  $r_\alpha = a_\alpha$  or  $z_\beta = a_\beta$  separates the interior region from the exterior one. The latter is formed by all the cylinders representing the channels in configuration space or the guides for  $z_\beta < a_\beta$ . An element of the boundary surface is given by  $dS_\alpha = a_\alpha^2 d\Omega_\alpha dq_{\alpha 1} dq_{\alpha 2}$  or  $dS_\beta = dx_\beta dy_\beta$ .

### 3. THE WAVE EQUATION AND ITS SOLUTION IN A CHANNEL AND IN A WAVEGUIDE. -

Using standard techniques, we can separate in the Schrödinger equation for a channel the internal and angular variables from the radial one, and in the Maxwell equations for a waveguide the transverse variables from the longitudinal one<sup>(7)</sup>.

Solving the equations for the internal and angular wave functions or for the electromagnetic field depending on transverse variables, it is readily established that the quantum numbers  $\{\alpha s \nu, \ell m\}$  originate like the guide parameters  $\{pq\}$ .

It is convenient to single out the "nuclear surface factor"  $\phi_{\alpha s \nu, \ell m}^{(x)}$ , embodying the dependence on all the co-ordinates except the radial one  $r_c$ , from the radial factor  $u_c(r_c)$ , satisfying the Schrödinger equation

$$(1a) \quad \left\{ \frac{d^2}{dr_c^2} - \frac{2m_c}{\hbar^2} \frac{Z_{\alpha 1} Z_{\alpha 2} \ell^2}{r_c} + k_c^2 - \frac{\ell(\ell+1)}{r_c^2} \right\} u_c(r_c) = 0,$$

or to single out the "guide cross-section factors"  $\vec{H}_{Tpq}(x_g, y_g)$ ,  $\vec{E}_{Tpq}(x_g, y_g)$  (the T-index represents transverse components) from the longitudinal ones  $I_{pq}(z_g)$ ,  $V_{pq}(z_g)$ , which are called current and tension of the mode pq, because they satisfy equations identical to those called in engineering language generalized telegraphist equations,

$$(1b) \quad \frac{d^2 V_{pq}}{dz_g^2} + (\omega^2 \epsilon \mu - k_{Tpq}^2) V_{pq}(z_g) = 0,$$

$$\frac{d^2 I_{pq}}{dz_g^2} + (\omega^2 \epsilon \mu - k_{Tpq}^2) I_{pq}(z_g) = 0,$$

(x) - The factor  $\phi_{\alpha s \nu, \ell m}$  coincides with the one in ref. 5, when divided by  $r_c$ .

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where  $k_{Tpq}^2 = \omega^2 \epsilon \mu - k_z^2$  is the square of the transverse wave number and  $\omega^2 \epsilon \mu$  the total one. The asymptotic behaviour of the nuclear radial wave function and its first derivative, and the solutions of Eqs. (1b) read

$$u_c \sim v_c^{-\frac{1}{2}} (C_c e^{-i(k_c r_c - \ell\pi/2)} - D_c e^{i(k_c r_c - \ell\pi/2)}), \quad (2a)$$

$$\frac{du_c}{dr_c} \sim -ik_c v_c^{-\frac{1}{2}} (C_c e^{-i(k_c r_c - \ell\pi/2)} + D_c e^{i(k_c r_c - \ell\pi/2)});$$

$$I_g = A_g e^{-ik_z z_g} - B_g e^{ik_z z_g}, \quad (2b)$$

$$V_g = \mathcal{Z}_g (A_g e^{-ik_z z_g} + B_g e^{ik_z z_g}),$$

$\mathcal{Z}_g$  being the impedance of the waveguide  $g$ . The comparison of Eq. (2a) and (2b) shows that the radial factor  $u_c(r_c)$  and the waveguide current  $I_g(z_g)$  as well as  $du_c/dr_c$  and the tension  $V_g(z_g)$  are mathematically analogous: it should be noted that, apart from a numerical factor, the relation

$$V_g \propto (dI_g/dz_g) \quad (3)$$

holds (see ref. 7, Chapter 4, page 76). The analogous role played by the centrifugal and coulomb potential and by shape of the guide geometrical cross-section and the dielectric nature is evident, from Eqs. (1a), (1b).

#### 4. THE WAVEFUNCTION AND ELECTROMAGNETIC FIELD IN THE EXTERIOR REGION. -

The nuclear surface functions  $\phi_c$  for all channels corresponding to the fragmentation  $\alpha$  and the guide cross-section field  $\vec{H}_{Tg}, \vec{E}_{Tg}$  for all modes in the guide of fixed shape  $\beta$  form a complete set(x).

Therefore, we can express the wavefunction for the fragmentation  $\alpha$  as a superposition of all its channel functions  $\phi_c u_c$  or the electroma-

(x) - In the following we shall call "material guide" a guide of fixed shape and dielectric medium nature  $\beta$ .

agnetic field for the material guide  $\beta$  as a superposition of all its modes  $I_g \vec{H}_{Tg}$ ,  $V_g \vec{E}_{Tg}$ .

Owing to the absence of spatial overlap between the configuration spaces corresponding to the various fragmentations or between the various material guides, the most general form of the total wavefunction or transverse field in the exterior region is

$$(4a) \quad \psi_{\text{ext}} = \sum_c u_c \phi_c$$

$$(4b) \quad \begin{cases} \vec{H}_{T, \text{ext}} = \sum_g I_g \vec{H}_{Tg} \\ \vec{E}_{T, \text{ext}} = \sum_g V_g \vec{E}_{Tg} \end{cases}$$

The energy dependence of the external wavefunction is contained in the radial factors and the frequency dependence of the external field in the currents or tensions, while the "nuclear surface factors" like the "guide cross-section factors" are independent.

The total energy in the channel  $c$  is the sum of the internal energy  $E_\alpha$  and the energy in the center-of-mass system  $E_c$ , and the total wave number in the guide  $g$  is the sum of the transverse and longitudinal ones:  $k_g^2 = k_{zg}^2 + k_{Tg}^2$ . Open ( $E_c > 0$ ) and closed ( $E_c < 0$ ) channels correspond to open ( $k_{zg}^2 > 0$ ) and closed ( $k_{zg}^2 < 0$ ) waveguides. In the latter cases we have exponentially decaying waves or evanescent regressive modes. The transverse dimensions of the material guide determine the minimum value allowed for  $k_{Tg}^2$ , so that for  $k_g^2 < k_{Tg}^2$ , minimum the guide refuses to transmit (high-pass filter behavior).

The relation  $E_c + E_\alpha = E_{c'} + E_{\alpha'}$  between the energies in the channels  $c$  and  $c'$  in the center-of-mass system can be transferred to the two guides  $g$  and  $g'$ :  $k_{zg}^2 + k_{Tg}^2 = k_{zg'}^2 + k_{Tg'}^2$ . We define as  $Q$ -value from the guide  $g$  to  $g'$  the quantity:  $Q_{gg'} = k_{Tg}^2 - k_{Tg'}^2$ , by analogy with the nuclear case, namely  $Q_{cc'} = E_\alpha - E_{\alpha'}$ : there is propagation in  $g'$  or  $c'$  provided that  $k_{zg}^2 \geq -Q_{gg'}$  or  $E_c \geq -Q_{cc'}$ .

The probability fluxes and transferred powers in the entrance channel or respectively in the entrance guide are defined as

$$(5a) \quad N = - \int \text{Re} \left( \psi^* \frac{\hbar}{im} \vec{\nabla} \psi \right) \cdot \vec{n} r_0^2 d\Omega,$$

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$$(5b) \quad W = -\frac{1}{2} \int \operatorname{Re} (\vec{H}_T^* \times \vec{E}_T) \cdot \vec{u}_z \, dS,$$

where the notation is conventional. Assuming  $A_{pq} = 1$ ,  $\eta_{pq} = B_{pq}$  {cfr. (2a)}, it is found that

$$(6b) \quad W = \frac{1}{2} \sum_{pq} \mathcal{L}_{pq} (1 - |\eta_{pq}|^2).$$

This formula corresponds to the well know relation

$$(6a) \quad N = \frac{\pi}{K^2} \sum_{\ell} (2\ell + 1) (1 - |\eta_{\ell}|^2),$$

used in the nuclear case. The analogy between N and W is more clearly shown by writing Eqs. (5) in the following form

$$(7a) \quad N = - \sum_{\ell} \operatorname{Re} \left\{ u_{\ell}^* \frac{1}{im} \frac{du_{\ell}}{dr} \right\},$$

$$(7a) \quad W = \frac{1}{2} \sum_{pq} \operatorname{Re} \left\{ I_{pq}^* V_{pq} \right\}.$$

From Eq. (6) it is seen that the transferred probability integral fluxes, relative to the various partial waves, are mutually independent like the transferred powers relative to the various guide modes.

The similar role played by  $u_{\ell}$  and  $I_{pq}$ ,  $du_{\ell}/dr$ , and  $V_{pq}$ , brings out the correspondence between the absorption total cross section (integral over entrance channel cross-section of the current vector) and total power transferred into the junction (integral over guide cross-section of the Poynting vector). In conclusion, Eqs. (5) disclose the following correspondence

$$(8) \quad \vec{H}_T \rightarrow \psi, \quad \vec{E}_T = (\vec{\nabla} \times \vec{H}) / i\omega \epsilon \rightarrow (\vec{\nabla} \psi)_n.$$

## 5. BOUNDARY BEHAVIOUR OF THE INTERIOR WAVEFUNCTION OR FIELD AND DISPERSION FORMULA. -

The nuclear and junction scattering matrices express the outgoing (regressive) wave amplitudes as linear combinations of incoming (progressive) ones. It follows that the correspondence.

$$(9) \quad u_c \rightarrow I_g, \quad \frac{du_c}{dr_c} \rightarrow V_g$$

ensures in a straightforward way the correspondence between the R-matrix and junction admittance Y-matrix: the former expresses the radial factors on the boundary surface S as linear combinations of their derivatives on S<sup>(8)</sup>, the latter the currents as linear combinations of the tensions on the guide terminals<sup>(4)</sup>. It has to be noted that the correspondence exists also between the nuclear diagonal matrices L and  $\Omega_c = e^{-2i\phi_c}$  representing the external interaction and the diagonal characteristic impedance matrix  $\mathcal{Z}$  and the matrix  $\Omega_g = e^{-2i\phi_g}$ ; in the latter case  $\phi_g$  is the phaseshift corresponding to the guide g with open terminal at  $z_g = a_g$ , whereas in the former one  $\phi_c$  is the scattering phaseshift by a hard-sphere of radius  $a_c$ .

The common mathematical background of the R and Y matrices, the deduction of which differs in several details, can be brought into view by comparing the results obtained using the Green scalar theorem for reactions and the Green vector theorem for junctions, namely ( $S = \sum_c S_c$ )

$$(10a) \quad \frac{\hbar^2}{2m} \int_{\tau} (\psi_2^x \nabla^2 \psi_1 - \psi_1 \nabla^2 \psi_2^x) d\tau = \sum_c \frac{\hbar^2}{2m_c} \int_{S_c} (\psi_2^x \vec{\nabla} \psi_1 - \psi_1 \vec{\nabla} \psi_2^x) \cdot \vec{n} dS$$

$$(10b) \quad \int_{\tau} (\vec{H}_1 \cdot \vec{\nabla}_x \vec{\nabla}_x \vec{H}_2^x - \vec{H}_2^x \cdot \vec{\nabla}_x \vec{\nabla}_x \vec{H}_1) d\tau = \int_S (\vec{H}_2^x \vec{\nabla}_x \vec{H}_1 - \vec{H}_1 \vec{\nabla}_x \vec{H}_2^x) \cdot \vec{n} dS.$$

Equations (10a) and (10b) show that a similar role is played by the gradient and the curl operator,  $\nabla^2$  and  $\vec{\nabla}_x \vec{\nabla}_x$ , and by products between scalars and dot-products between vectors; this comparison suggests the correspondence on the boundary surface expressed by Eq. (8). The correspondence (8) unifies the derivation and the development of the R-matrix and Y-matrix theory; for instance, the Lane-Thomas reactions theory<sup>(3)</sup> can be readily translated into the language of the junction theory, and viceversa.

Because of the continuity conditions on the boundary surface, the internal fields  $\vec{H}_{Tint}$ ,  $\vec{E}_{Tint}$  can be developed in the  $\vec{h}_{Tg}$ ,  $\vec{e}_{Tg}$  sets, just as the nuclear wave function  $\psi_{int}$  is usually written in terms of the  $\psi_{c,s}$ <sup>(3)</sup>. It follows that from (10b) the Green vector theorem is obtained in the form

$$(11b) \quad (k_2^2 - k_1^2) \int \vec{H}_2^x \cdot \vec{H}_1 d\tau = i \omega \epsilon \sum_g (I_{2g}^x V_{1g} + I_{1g}^x V_{2g})_{z_g = a_g}$$

which corresponds to the relation



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$$(11a) \quad (E_2 - E_1) \int \psi_2^* \psi_1 d\tau = \sum_c \frac{\hbar^2}{2m_c} \left\{ u_{2c}^* \frac{du_{1c}}{dr_c} - u_{1c} \left( \frac{du_{2c}}{dr_c} \right)^* \right\}_{r_c = a_c}$$

In order to find a set of interior modes and express the Y-matrix in terms of them, let us impose on the wave equation for  $\vec{H}_{int}$  the boundary condition that the tension for each guide vanishes on the terminal section. This is analogous to the nuclear condition that the derivative of each radial factor vanishes on  $S(\mathbf{x})$ . It is easy to identify this condition with the usual one assumed for junctions<sup>(4)</sup>. These boundary problems generate a set of eigenvalues  $E_\lambda$  or  $k_\lambda$  and orthonormal eigenfunctions  $\chi_\lambda$  or  $\vec{b}_\lambda$ .

The internal wave function  $\psi_{int}(E)$  and the field  $\vec{H}_{int}(k)$  can be expanded in terms of the sets  $\chi_\lambda$  and  $\vec{b}_\lambda$  respectively; the expansion coefficients are then determined by applying the Green scalar and vector theorem. On the basis of the correspondence previously outlined, one readily obtains from well known expression of the nuclear R-matrix elements,

$$(12a) \quad R_{cc'} = \sum_\lambda \frac{\gamma_{\lambda c'} \gamma_{\lambda c}}{E_\lambda - E},$$

where

$$(13a) \quad \gamma_{\lambda c} = \left( \frac{\hbar^2}{2m_c a_c} \right)^{\frac{1}{2}} \int_{S_c} \phi_c^* \chi_\lambda dS,$$

the dispersion formula for the Y-matrix, namely ( $\eta = \sqrt{\epsilon/\mu}$ )

$$(12b) \quad Y_{gg'} = ik\eta \sum_\lambda \frac{\gamma_{\lambda g'} \gamma_{\lambda g}}{k_\lambda^2 - k^2},$$

where

$$(13b) \quad \gamma_{\lambda g} = I_{\lambda g}(a_g) = \int_{\Sigma_g} \vec{H}_{Tg} \cdot \vec{b}_\lambda dS.$$

The quantities  $\gamma_{\lambda c}$  are the coupling coefficients between the waves in the channels and the internal eigenstates  $\chi_\lambda$ , whereas the quantities  $\gamma_{\lambda g}$  are the coupling coefficients between the guide and cavity modes; these coefficients are independent of the energy  $E$  or the wave number  $k$  respectively.

## 6. COMMON MATHEMATICAL PROPERTIES AND THE PHYSICAL CONTENT OF THE R-MATRIX AND Y-MATRIX THEORIES. -

It is well known that the unitarity property of the nuclear or junction scattering matrices is a consequence of the conservation principle of

(x) - This is the simplest kind of boundary condition assumed<sup>(8)</sup>.

the probability and respectively energy flux<sup>(9)</sup>. In the nuclear case this property is usually established by applying the Green scalar theorem, which contains implicitly the conservation principle: it follows that the vector Green theorem must be used in order to establish the same property for the junctions case.

The symmetry property of the nuclear scattering matrix is derived from the time-reversibility principle, by imposing that the time-reversed wavefunction be obtained from the general expression for the wavefunction<sup>(8)</sup>. The same argument applied to the time-reversed field (where the progressive and regressive waves are mutually exchanged) gives the symmetry property for the junction case.

The symmetry property for the R and Y matrices is implicit in their resonant formulas, but it can be also established by using the Green theorems<sup>(9)</sup>.

The fundamental connection between the nuclear scattering and R-matrix or junction scattering and Y-matrix follows at once in both cases simply by handling in a convenient way the R or Y matrix, and is expressed in matrix notation by the relations

$$(14a) \quad U = (ka)^{\frac{1}{2}} 0^{-1} (1-RL)^{-1} (1-RL^{\mathbf{x}}) I (ka)^{-\frac{1}{2}},$$

$$(14b) \quad S = (e^{ik_z a})^{-1} (1+Y\mathcal{Z})^{-1} (1-Y\mathcal{Z}) e^{-ik_z a},$$

where L-matrix and  $\mathcal{Z}$ -matrix play the same role<sup>(x)</sup>

The power transferred from one guide to another is proportional to the square of the corresponding scattering element as well as the cross section from one channel to another, and can be calculated following the procedures used in the nuclear case owing to the analogy between Poynting vector and the probability current vector: so one obtains

$$(15) \quad W_{gg'} = \frac{\mathcal{Z}_{g'}}{\mathcal{Z}_g} |S_{g'g}|^2.$$

Finally, by using the Eqs. (15), the cross sections can be formally expressed in terms of the R-matrix elements and the transferred powers in terms of the Y-matrix elements.

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(x) - I and 0 are diagonal matrices corresponding to the incoming and outgoing waves (see ref. 8, Chap. 16, page 487).

## 7. CONCLUDING REMARKS. -

It has been shown that the formulation of the R-theory of nuclear reactions and the Y-theory of waveguides and resonators are based on a common mathematical ground. A fundamental role is played by the Green theorem in deriving the resonant expressions of the R and Y matrices, and the unitarity and symmetry properties in the framework of the conservation of probability or energy flux, time reversibility and the causality prescriptions.

As an illustrative example of application of the general correspondence, we shall compare the S-wave elastic scattering and the wave propagation in a guide short-circuited at one end, and we shall mention the analogy between resonance widths and reciprocals of the quality-factors, life-times of the metastable nuclear states and delay-times of a wave into a resonant cavity.

We consider the S-wave elastic scattering of neutral spinless particles by a square potential well and the propagation of the dominant-mode in a waveguide formed by two regions with different dielectric and short-circuited at one end by a metal plate. The results for the nuclear case are surveyed in the Lane-Thomas review article<sup>(3)</sup>. The tension and current in the guide propagate as sinusoidal waves in quadrature. Therefore, the impedance has the form

$$Z = -i \mathcal{Z}' \operatorname{tg} K_z a = -i \mathcal{Z}' K_z a \sum_0^\infty \lambda \frac{2}{\pi^2 (\lambda + \frac{1}{2})^2 - K_z^2 a^2},$$

where  $\mathcal{Z}'$  and  $K_z$  are the interior characteristic impedance and wave number, respectively. If we put into correspondence the radial function  $u$  and the tension,  $(du/dr)$  and current, we can see that the R-quantity<sup>(3)</sup>

$$R = \frac{\operatorname{tg} Ka}{Ka} = \sum_0^\infty \lambda \frac{2}{\pi^2 (\lambda + \frac{1}{2})^2 - K^2 a^2}$$

coincides with  $Z/i\mathcal{Z}'K_z a$ . By connecting  $R$  with  $U = e^{2i\delta_0}$ , and  $Z$  with  $S = e^{2i\delta_0}$  one obtains, respectively,

$$\delta_0 + ka = \operatorname{arctg} \left( \frac{k}{K} \operatorname{tg} Ka \right),$$

$$\delta_0 + k_z a = \operatorname{arctg} \left( \frac{\mathcal{Z}'}{\mathcal{Z}} \operatorname{tg} K_z a \right),$$

where  $k$ ,  $k_z$ ,  $\mathcal{Z}$  are the exterior wave numbers and characteristic impedance, respectively, and  $K$  the interior wave number. We have seen

that  $\mathcal{Z}(\mathcal{Z}')$  corresponds to the nuclear  $L(L')$  quantity. In our case we have  $L/L'=k/K$ . The considered waveguide is equivalent to a transmission lossless line, short-circuited in  $z=0$  and with the characteristic impedance discontinuous at  $z=a$ . It is well known<sup>(6)</sup> that the circuit equivalent to a transmission line portion of characteristic impedance  $\mathcal{Z}'$ , and, therefore to a cavity  $\epsilon'$ ,  $\mu'$ , is a ladder network with as many antiresonant circuits  $L_\lambda C_\lambda$  (where  $L_\lambda$  is in parallel to  $C_\lambda$ , and  $L_\lambda$ ,  $C_\lambda$  are determined by  $\mathcal{Z}'$ ) in series among them as the transmission line or cavity modes are. In this way we have obtained the electrical circuit equivalent to our scattering problem: the nuclear potential determines the quantity  $L'=aK$  and, thus, the circuit characteristics.

Now we mention the analogy between resonance widths and reciprocals of the quality factors. We know that the lack of energy definition of metastable structure level is expressed quantitatively by the resonance width  $\Gamma_\lambda$ , relative to that level. The indetermination in the frequency definition of a cavity proper mode can be expressed quantitatively by the quality-factor  $Q_\lambda = \omega_\lambda \bar{W}_\lambda / \bar{P}_\lambda$ , where  $\omega_\lambda$  is the resonant frequency,  $\bar{W}_\lambda$  the total time-average energy stored in the cavity for  $\omega = \omega_\lambda$  and  $\bar{P}_\lambda$  the time-average power lost through the guides<sup>(4)</sup>. The ratio  $\bar{W}/\bar{P}$  represents the retaining-time  $\tau$  of a pulse into the cavity<sup>(9)</sup> and  $\bar{W}$  reaches a maximum in correspondence to each resonance frequency<sup>(10)</sup>. Therefore, the retaining-time becomes maximum at the resonance frequency. Thus, for  $\omega = \omega_\lambda$ ,  $\tau = \tau_\lambda = Q_\lambda / \omega_\lambda$  represents the cavity de-excitation-time relative to its  $\lambda$ -mode, as well as  $\hbar/\Gamma_\lambda$  represents the life-time of the  $\lambda$ -state of the metastable structure.

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## REFERENCES. -

- (1) - E. P. Wigner, Am. J. Phys. 17, 99 (1949).  
T. Teichmann, J. Appl. Phys. 23, 701 (1952);  
T. Teichmann and E. P. Wigner, J. Appl. Phys. 24, 262 (1952);  
see also J. M. Blatt and V. F. Weisskopf, Theoretical Nuclear Phys  
(J. Wiley and Sons, New York, 1952);
- (2) - G. F. Chew, M. Gell-Mann and A. H. Rosenfeld, Sci. American  
210, 74 (1964).
- (3) - A. M. Lane and R. G. Thomas, Revs. Mod. Phys. 30, 257 (1958).
- (4) - F. E. Borgnis and C. H. Papas, PH 16, 285 (1958).
- (5) - J. Humblet and L. Rosenfeld, Nuclear Phys. 26, 529 (1961).
- (6) - G. Toraldo di Francia, Onde Elettromagnetiche (Zanichelli, Bologna  
1953).
- (7) - R. N. Ghose, Microwave Circuit Theory and Analysis, (McGraw-Hill,  
New York, 1963);
- (8) - M. A. Preston, Phys. of the Nucleus (Addison-Wesley, Reading  
1962).
- (9) - C. G. Montgomery, R. M. Dicke and E. M. Purcell, Principles of  
Microwave Circuits (McGraw-Hill, New York 1948).
- (10) - H. A. Atwater, Introduction to Microwave Theory (McGraw-Hill,  
New York 1962).