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L. Drigo and G. Pisent: ANALYSIS OF THE p-He^3 LOW ENERGY INTERACTION. -

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L. Drigo and G. Pisent: ANALYSIS OF THE p-He^3 LOW ENERGY INTERACTION(x).

SUMMARY -

The elastic scattering of protons from ^3He nuclei is analyzed at the very low energies. A phase shift analysis on the cross section and polarization data leads to the determination of four solution sets. These mathematical solutions are examined in the light of a nucleon--nucleus potential model. The structure of the spin-dependent interaction which is necessary to assume in order to reproduce the experiments is widely discussed. Some elements on the discrimination between phase shift ambiguities are also given by the potential calculation, but, for a clear-cut conclusion on this point, further high energy analyses, and perhaps triple scattering experiments will be probably needed.

1 - INTRODUCTION -

A great deal of interest has been recently devoted to the Li^4 structure in connection with the general problem of the A = 4 nuclei(1). As is well known, the excited levels of Li^4 can be determined from the analysis of the p-He^3 interaction process, through evaluation of the phase shifts behaviour, and proper choice of the interaction radius. In spite of the big number of works dealing with the phase shifts problem, from both the theoretical and experimental viewpoint(2), only one analysis, due to Tombrello, can be considered complete(3). In fact, in previous

(x) - Work carried out under Contract EURATOM/CNEN-INFN.

papers, only central and spin-spin forces were considered, while recent polarization measurements[4] have shown that other types of spin dependent interactions are undoubtedly important.

The analysis of the scattering process between two particles of spin 1/2 is shown to be cumbersome because of the large number of states which are in general involved (four for each value of the angular momentum l). The extraction of phase shifts, discussion on the ambiguities and choice of the physical solution is not simple, even when both cross section and polarization are known at a given energy.

It is believed that a great deal of information for a general understanding of the interaction mechanism should be given by a deep analysis of the experimental data in the low energy region, where only few angular momenta are effective.

We have carried out such an analysis through the following steps:

1 - Polarization measurements below ~ 5 MeV, where no experiment was available, as to our knowledge. The measurements have been performed with the Legnaro Van de Graaff Accelerator, and the results are published elsewhere[5].

2 - Phase shift analysis and discussion of the ambiguities encountered. At the energies considered, only S and P waves are important, allowing a very particular and simple discussion of the ambiguities, as will be seen later.

Since the phase shifts given by Tombrello (Tab. II of ref. 3) account well for cross sections and polarizations at all the energies explored till now, our analysis will be devoted mainly (i) to improvement of this phase shift set at the very low energies, taking into account the polarization data, and (ii) to the research of possible alternative solutions, equally good from the viewpoint of single and double scattering experiments, but perhaps preferable for other reasons[x].

3 - Theoretical calculations on the ground of a nucleon-nucleus potential, taking into account the most general spin-dependent interaction. The purposes of these calculations are the following: (i) to obtain supplementary information for the discrimination of the phase shift sets; (ii) to establish which kind of splitting mechanism is responsible for the observed polarization, and in particular which is the comparative importance of the tensor and spin-orbit term. This is an important problem, which interests mainly the lighest nuclei. In fact, when the interaction of nucleons with heavier nuclei is dealt with, most spin effects are averaged on, and

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(x) - A similar analysis, at higher energies, is in progress at the Wisconsin group. We thank Prof. Haeberli for helpful correspondence on this subject.

the interaction is generally well accounted for by neglecting the target spin[x], [6] (iii) to obtain a reliable evaluation of the interaction radius; (iv) to try the extrapolation to higher energies.

The important, but limited and semiqualitative purposes of the calculations, justify some crude approximations which will be introduced in paragraph 3.

It is hoped that the general information obtained here on the average interaction of the incident proton with the target nucleus will constitute a useful background for further calculations, based on a nucleon-nucleon non central potential.

2 - PHASE SHIFT ANALYSIS -

Let us write down the general form of the differential cross section σ and polarization P, of charged spin 1/2 particles, scattered by spin 1/2 targets. (Beam and target are supposed to be initially unpolarized):

$$(1) \qquad k^2\sigma(\theta) = \frac{1}{4}\sum_{s',s=0}^{1}\sum_{m'=-s'}^{s'}\sum_{m=-s}^{s}\left|f_{s'm',sm}(\theta)\right|^2,$$

$$k^2\sigma(\theta)P(\theta) = \frac{\sqrt{2}}{4}\,\mathrm{Im}\left\{e^{-i\phi}\left[(f_{10,11}(\theta)e^{2i\phi}-f_{11,10}(\theta))\cdot\right.\right.$$

$$(2) \qquad \cdot(f_{11,11}(\theta)+f_{10,10}(\theta)-f_{11,1-1}(\theta)e^{2i\phi})^* - 2f_{00,11}(\theta)\cdot$$

$$\left.\left.\cdot(f_{11,11}(\theta)+f_{00,00}(\theta)+f_{11,1-1}(\theta)e^{2i\phi})^*\right]\right\}, \quad (o)$$

where θ and ϕ are the scattering angles in the CM system. The matrix elements $f_{s'm',sm}(\theta)$ of eqs (1), (2) are given by:

(x) - Calculations of this kind (with a central plus spin-orbit potential, but assuming the target spin to be zero!) have been applied to the p-^{3}He scattering too[7]. The order of magnitude of depth and radius of the central potential of ref. (7) are in good agreement with ours. A detailed discussion on the spin dependent forces was there impossible, because of the lack of experimental information.

(o) - See for example refs. (8), (9).

$$f_{s'm',sm}(\theta) = f_c(\theta)\,\delta_{s's}\,\delta_{m'm} + i\sqrt{\pi}\sum_{l=0}^{\infty}\sqrt{2l+1}$$

$$(3)\qquad \sum_{j=|l-s|}^{l+s}\ \sum_{l'=|j-s'|}^{j+s'} (slm0\,|\,jm)(s'l'm', m-m'\,|\,jm)$$

$$\left\{\exp\left[i(\omega_l + \omega_{l'})\right]\delta_{s's}\,\delta_{l'l} - S^{j}_{s'l',sl}\right\} Y_{l'}^{m-m'}(\theta,\phi).$$

The symbols s'm' and sm indicate spin and projection of the output channel and entrance channel respectively, and j is the total angular momentum. In eq. (3) ω_l is the coulomb phase shift as defined in ref. (8) and $f_c(\theta)$ is the coulomb scattering amplitude, given by equation

$$(4)\qquad f_c(\theta) = -\frac{\eta}{(1-\cos\theta)}\exp\left(-i\eta\ln\frac{1-\cos\theta}{2}\right),$$

where $\eta = Z_1Z_2e^2/\hbar v$. In the most general case, if one takes into account the mixing parameters between singlet and triplet states (ε_j), and between states of the same parity (ε'_j), the relations between S matrix and phase shifts δ^{j}_{sl} read:

$$(5a)\qquad \underline{\Xi} = \underline{U}(\varepsilon_j)\,\underline{\Lambda}\,\underline{U}^{+}(\varepsilon_j),$$

$$(5b)\qquad \underline{\Xi}' = \underline{U}^{+}(\varepsilon'_j)\,\underline{\Lambda}'\,\underline{U}(\varepsilon'_j)$$

where $\underline{\Xi}$, $\underline{\Xi}'$, $\underline{\Lambda}$, $\underline{\Lambda}'$, $\underline{U}$, are two dimensional matrices, defined as follows (λ, μ = 0, 1):

$$(6a)\qquad \Xi_{\lambda\mu} = S^{j}_{\lambda j;\mu j'} \qquad \Lambda_{\lambda\mu} = \delta_{\lambda\mu}\exp 2i(\omega_j + \delta^{j}_{\lambda j}),$$

$$(6b)\qquad \Xi_{\lambda\mu} = S^{j}_{1,\,j-1+2\lambda\,;\,1,\,j-1+2\mu'} \qquad \Lambda'_{\lambda\mu} = \delta_{\lambda\mu}\exp 2i(\omega_{j-1+2\lambda} + \delta^{j}_{1,\,j-1+2\lambda}),$$

$$(6c)\qquad \underline{U}(x) \equiv \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}.$$

In S and P waves approximation, the S matrix elements are given by the following expressions ($\varepsilon_1 \equiv \varepsilon$):

$$(7a)\qquad S^{s}_{s0,\,s0} = \exp 2i\,\delta^{s}_{s0} \qquad (s = 0, 1),$$

(7b) $$S^{j}_{11,11} = \exp 2i(\omega_1 + \delta^{j}_{11}) \qquad (j = 0, 2),$$

(7c) $$S^{1}_{11,11} = (\cos^2\varepsilon \exp 2i\,\delta^{1}_{11} + \sin^2\varepsilon \exp 2i\,\delta^{1}_{01}) \exp 2i\,\omega_1,$$

(7d) $$S^{1}_{01,11} = \sin\varepsilon\cos\varepsilon\,(\exp 2i\,\delta^{1}_{11} - \exp 2i\,\delta^{1}_{01}) \exp 2i\,\omega_1,$$

(7e) $$S^{1}_{01,01} = (\cos^2\varepsilon \exp 2i\,\delta^{1}_{01} + \sin^2\varepsilon \exp 2i\,\delta^{1}_{11}) \exp 2i\,\omega_1.$$

It is well known (see eq. 1) that in general the cross section analysis leads to an indetermined problem, because the number of unknown phase shifts is higher than the number of angular distribution coefficients extracted from the experiment. On the other hand, contemporary analysis of cross section and polarization leads to overdetermination. In the first case the problem is unsolvable, while in the second one the conditioned best fit procedure for the extraction of the coefficients leads to a system of nonlinear equations. The case of a pure S and P waves interaction between charged particles without mixing terms is an important exception, because in this case the cross section depends on six phase shifts and six independent coefficients, while the polarization contains four phase shifts (i.e. it depends on the triplet states only) and four coefficients. This means that a separate analysis of cross section and polarization is expected to lead to unambiguous determination of the angular distribution coefficients for the two curves. Once these coefficients are known, all phase shift sets compatible with $\sigma(\theta)$ from one hand, and $P(\theta)$ from the other, should be calculated by solution of simple trigonometric equations (see Appendix). The problem of the consistency between cross section and polarization should be considered as a final stage.

Unfortunately, a direct application of the method is unwieldy in our case, because of the weakness of the interference between coulomb and nuclear scattering amplitude. In fact the best fit on experiments leads to interference coefficients which are completely meaningless. In spite of this, if one solution has been found by means of the usual variational procedure, the equations of the Appendix can be employed in order to derive all the other solutions, able to give the same cross section or the same polarization curves.

This way has been followed in present calculations, starting from the phase shifts of Table II, ref. (3).

First of all these phase shifts have been improved by minimization of the mean square error Δ defined as follows:

$$(8)\quad \Delta = \left\{ (2N)^{-1} \sum_{\lambda=1}^{N} \left[\left(\frac{\sigma_{exp}(\theta_\lambda) - \sigma(\theta_\lambda)}{\Delta\sigma(\theta_\lambda)} \right)^2 + \left(\frac{P_{exp}(\theta_\lambda) - P(\theta_\lambda)}{\Delta P(\theta_\lambda)} \right)^2 \right] \right\}^{\frac{1}{2}},$$

where the experimental polarizations P_{exp} are taken from ref. (5), while the cross sections σ_{exp} are obtained by interpolation of the results of Tombrello et al.[2] and McDonald et al.[4]. Symbols ΔP and $\Delta\sigma$ indicate experimental errors. The expressions of P and σ are given by eqs (1), (2), (3), (4), (7). The phase shifts and the mean square errors obtained are quoted in Table Ia. The mixing parameter ε has been found to be zero (within the limits of experimental errors), in the whole energy region here considered.

Figures 1 and 2 show that the phase shifts Ia match perfectly with those of ref. (3). Small differences are found only at very low energies, where the analysis by Tombrello was carried out without the experimental knowledge of the polarization. An example of the small discrepancies at the very low energies is given in Figs 3 and 4.

The phase shifts of solution (a) have been then transformed by means of the formulas of the Appendix, in order to find all solutions able to reproduce equally well the cross section (polarization). The phase shift sets leading to completely wrong polarization (cross section) were immediately rejected. The remaining solutions have been finally subjected to a new process of mean square error minimization (see eq. 8), in order to find the best values for contemporary reproduction of cross section and polarization data. The solutions (b), (c), (d) of Table I represent the final result of the procedure.

Solution (c) exhibits, as compared to solution (a), inversion of the S doublet except for some anomalous points, and different behaviour of the P singlet. Solutions (b) and (d) show inversion of the firsts two P triplet levels, while the remaining phase shifts are nearly equal to those of solutions (a) and (c) respectively.

The solution sets (b), (c) and (d) have been extrapolated at the energies of 5.51 and 6.82 MeV, using the experimental data of McDonald et al.[4], and introducing the mixing parameter ε and two D phase shifts ($\delta^2_{02} \equiv \delta_{02}$, $\delta^1_{12} = \delta^2_{12} = \delta^3_{12} \equiv \delta_{12}$). The results are listed in Table II, together with extrapolation of solution (a), taken from ref. (3). All solutions seem to be continuous in energy up to the values here considered. The D phase shifts have, almost in all cases, negative values.

For solutions (a) and (c) we have calculated the effective range parameters relative to the singlet and triplet S wave (a_s, r_{os}, a_t, r_{ot}), and the scattering lengths corresponding to a pure nuclear interaction (a^n_s, a^n_t). (For details see refs (10) and (11)).

The behaviour of the S phase shifts relative to solutions (b) and (d) is very similar to that of solution (a) and (c) respectively.

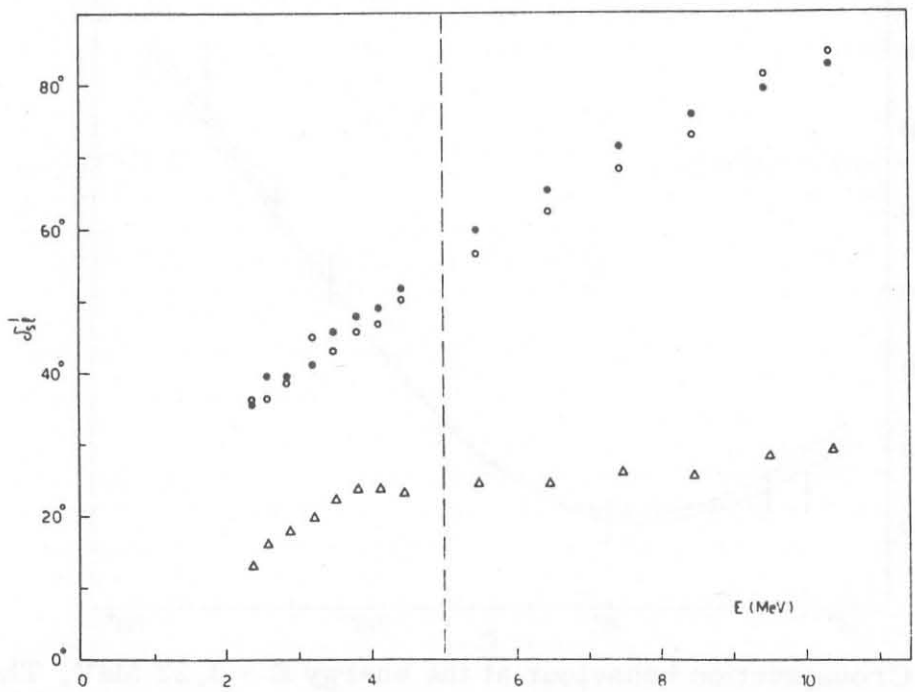

FIG. 1 - Behaviour of the phase shifts δ^{0}_{00}(o), δ^{1}_{10}(•) and δ^{1}_{01}(Δ) vs. energy. The phase shifts relative to energies lower (higher) than 5 MeV, are those of Table Ia (Table II of Ref. [3]).

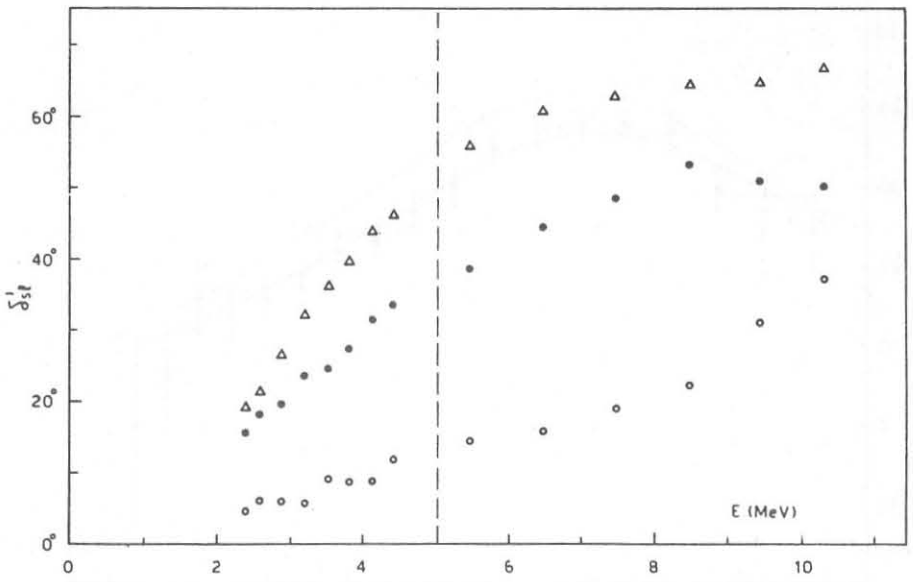

FIG. 2 - Behaviour of the phase shifts δ^{0}_{11}(o), δ^{1}_{11}(•) and δ^{2}_{11}(Δ) vs. energy.

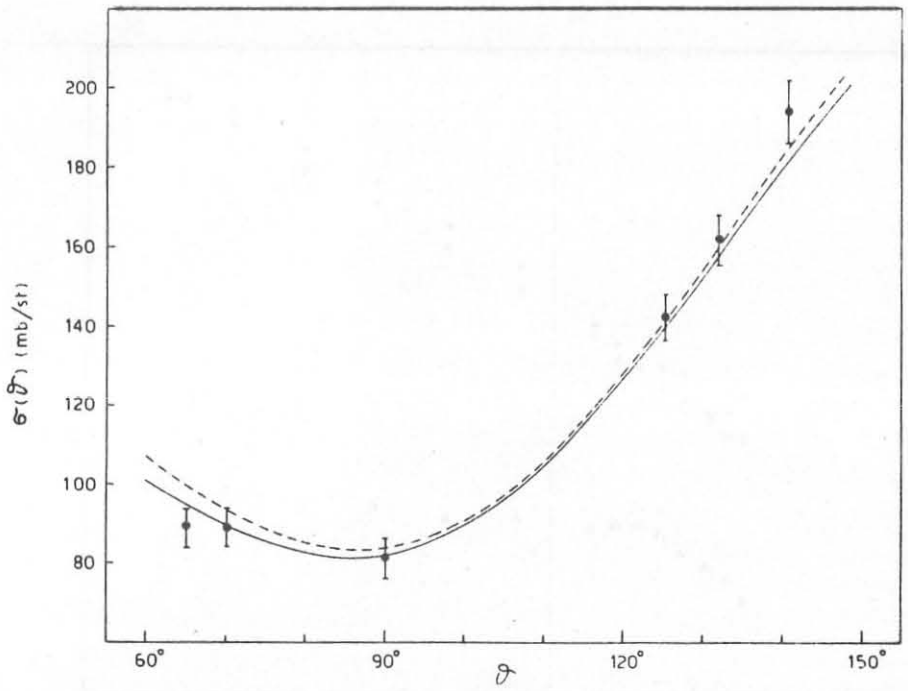

FIG. 3 - Cross section behaviour at the energy E = 3.22 MeV. The experimental points (•) obtained by interpolation of the data given by Tombrello (Ref.[2]), are compared with the curves calculated by phase shifts of Table Ia (———), and by phase shifts of Table II Ref. (3) (------) respectively.

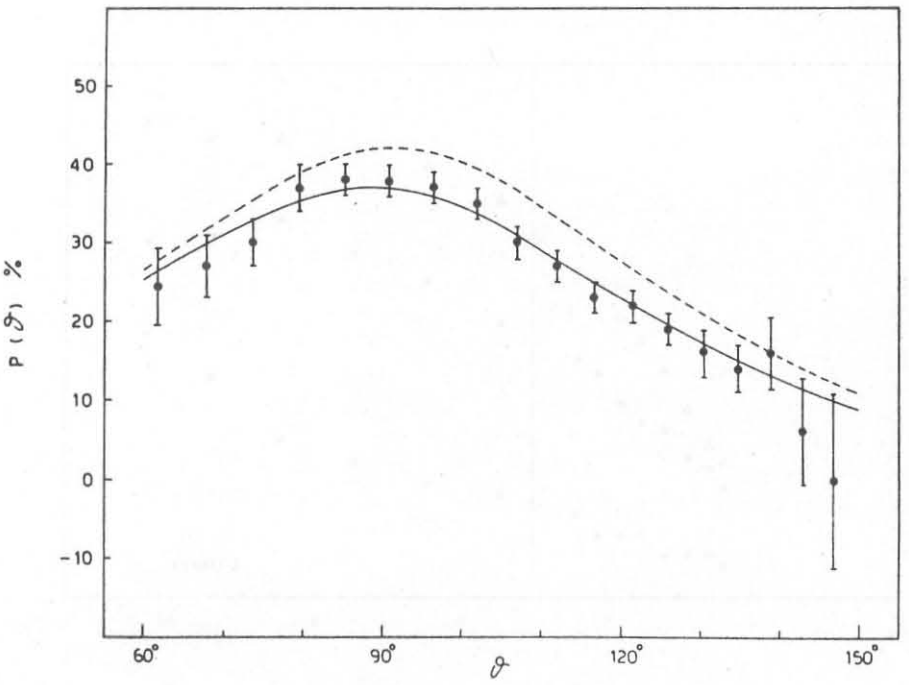

FIG. 4 - Polarization at the energy E = 3.22 MeV. The experimental points of Ref. (5) are compared with the curves calculated from the phase shifts. Continuous and dotted lines are defined as in Fig. 3.

TABLE I

Phase shift sets extracted from cross section and polarization experiments below 5 MeV.

	E (MeV)	$-\delta^0_{00}$	$-\delta^0_{10}$	δ^1_{01}	δ^0_{11}	δ^1_{11}	δ^2_{11}	Δ
(a)	2.38	36.0°	35.4°	12.8°	4.4°	15.3°	18.8°	1.89
	2.61	36.2°	39.6°	15.8°	5.9°	18.1°	21.1°	0.56
	2.89	38.8°	39.3°	17.6°	5.7°	19.5°	26.4°	0.87
	3.22	44.6°	41.0°	19.7°	5.7°	23.4°	32.0°	0.46
	3.54	43.1°	45.5°	22.0°	8.9°	24.4°	36.1°	1.00
	3.84	45.5°	47.6°	23.4°	8.2°	27.2°	39.5°	0.98
	4.15	46.8°	48.8°	23.4°	8.8°	31.3°	43.7°	0.85
	4.46	49.8°	51.4°	22.8°	11.6°	33.3°	45.8°	1.82
(b)	2.38	38.0°	33.4°	14.8°	21.2°	8.2°	21.0°	1.88
	2.61	37.2°	38.6°	16.8°	20.2°	9.5°	22.0°	0.62
	2.89	40.8°	37.3°	19.6°	23.5°	8.7°	28.0°	0.92
	3.22	44.6°	39.0°	21.7°	28.5°	11.5°	34.0°	0.56
	3.54	42.1°	44.5°	23.0°	28.0°	13.0°	37.0°	1.08
	3.84	42.5°	44.6°	26.4°	33.5°	13.0°	42.0°	1.08
	4.15	44.8°	46.8°	25.4°	36.4°	15.7°	46.0°	1.14
	4.46	45.8°	49.4°	24.8°	42.0°	16.0°	48.0°	2.15
(c)	2.38	36.2°	34.7°	19.1°	4.3°	14.7°	18.5°	1.83
	2.61	41.3°	37.9°	20.1°	5.0°	16.1°	19.8°	0.55
	2.89	39.6°	39.1°	24.5°	7.6°	14.6°	24.7°	0.86
	3.22	38.2°	43.8°	29.3°	4.3°	18.8°	28.0°	0.48
	3.54	46.7°	44.3°	33.2°	8.4°	19.3°	32.6°	0.94
	3.84	47.7°	45.6°	37.5°	3.4°	23.1°	36.0°	0.70
	4.15	52.8°	46.8°	43.1°	5.3°	23.6°	38.1°	0.73
	4.46	55.2°	49.6°	46.7°	7.0°	26.0°	38.8°	1.73
(d)	2.38	37.1°	33.7°	20.1°	17.7°	7.2°	19.5°	1.89
	2.61	42.3°	36.9°	21.1°	19.2°	8.2	20.8°	0.57
	2.89	40.6°	38.1°	25.5°	17.0°	10.0°	25.7°	0.90
	3.22	36.2°	43.8°	29.3°	23.5°	9.7°	30.0°	0.52
	3.54	45.7°	43.3°	34.2°	20.5°	12.0°	33.6°	0.98
	3.84	49.7°	41.6°	41.5°	30.0°	8.0°	38.0°	0.74
	4.15	51.8°	43.8°	46.1°	28.0°	10.2°	39.1°	0.68
	4.46	58.2°	46.6°	49.7°	26.5°	15.5°	39.8°	1.41

In the best fit procedure, experimental data of Tables Ia, Ic, IIa, IIc have been considered. For higher energies the behaviour of the experimental phase shifts is no longer accounted for by the effective range formula.

The results are shown in Table III, where a_s^n and a_t^n are also compared with values obtained by Szydlik[12], for the n - ^{3}H interaction. The overall error relative to the evaluation of the effective range parameters is presumably higher than the small discrepancies observed in the Table. It is however worthwhile to note that solution (c) agrees better with ref. (12), because it gives $a_s^n > a_t^n$.

Finally, an S, P and D waves analysis of the experimental data has been performed by means of eq. (8), assuming as starting values the S and P phase shifts of Table I, and zero D phase shifts. Very small D phase shifts ($< \sim 1^o$) have been obtained in the whole energy region. This result confirms "a posteriori" the assumption of paragraph 2, and justifies the approximations which will be introduced in paragraph 3.

3 - POTENTIAL CALCULATIONS -

Let us consider a nucleon-nucleus potential of the following form:

$$V = V_0(r) + V_{\sigma\sigma}(r)\,\vec{\sigma}_1 \cdot \vec{\sigma}_2 + V_T(r) S_{12} + V_{ls}(r)\,\vec{l}\cdot\vec{s}, \tag{9}$$

where

$$V_0(r) = -U_0 g(r) + V_c(r), \tag{10a}$$

$$V_{\sigma\sigma}(r) = -U_{\sigma\sigma} g(r), \tag{10b}$$

$$V_T(r) = -U_T g(r), \tag{10c}$$

$$V_{ls}(r) = U_{ls} \left(\frac{\hbar}{M_\pi c}\right)^2 \frac{1}{r}\frac{\partial g}{\partial r}. \tag{10d}$$

In eqs. (10) M_π is the pion mass, g(r) gives the radial wave function which will be defined later, and V_c is the usual coulomb potential, namely

$$V_c(r) = \left[Z_1 Z_2 e^2/2R\right]\left[3-(r/R)^2\right] \quad \text{for } r \le R, \tag{11a}$$

$$V_c(r) = Z_1 Z_2 e^2/r \quad \text{for } r > R. \tag{11b}$$

TABLE II

Extrapolation of the phase shifts of Table I to higher energies.

	E MeV	$-\delta^0_{00}$	$-\delta^1_{10}$	δ^1_{01}	δ^0_{11}	δ^1_{11}	δ^2_{11}	δ_{02}	δ_{12}	ε	Δ
(a)	5.51	53.2°	59.9°	24.3°	13.8°	38.6°	54.4°	-5.4°	-1.2°	-0.9°	0.56
	6.82	59.0°	67.5°	23.1°	16.7°	46.2°	59.8°	-9.9°	-0.3°	-4.7°	0.67
(b)	5.51	53.5°	57.0°	25.5°	53.0°	20.0°	57.0°	-3.0°	-2.0°	6.0°	1.13
	6.82	62.0°	65.0°	27.0°	61.0°	27.0°	63.0°	-4.0°	-2.0°	8.0°	1.98
(c)	5.51	61.0°	60.0°	57.0°	14.0°	31.0°	40.0°	0.0°	-2.0°	8.0°	0.51
	6.82	70.5°	65.0°	67.0°	16.5°	34.0°	45.0°	-2.0°	-2.0°	8.0°	0.71
(d)	5.51	62.0°	59.0°	57.0°	35.0°	22.0°	42.0°	0.0°	-2.0°	8.0°	0.46
	6.82	69.0°	66.0°	63.0°	42.0°	24.5°	48.0°	1.0°	-3.0°	10.0°	0.69

TABLE III

S-wave effective range parameter (in fermis).

	a_s	r_{0s}	a_t	r_{0t}	a^n_s	a^n_t
p - ^{3}He (sol. a)	8.45	1.38	8.45	1.62	3.50	3.72
p - ^{3}He (sol. c)	7.94	1.96	7.89	1.62	3.89	3.60
n - ^{3}H (ref. 12)	--	--	--	--	3.38	3.25

Since we are interested mainly in a qualitative discussion on the effect of the various potential terms on the interaction, we start from a simple square well radial shape, which leads to straight-forward integration of the Schrödinger equation. Namely

(12a) $g(r) = 1$ for $r \leq R$

(12b) $g(r) = 0$ for $r > R$.

At the end of the paragraph the effect of a more realistic diffused potential will be shorthly discussed.

By considering first only central, spin-spin and spin-orbit forces, the radial part of the Schrödinger equation for the state $|jsl\rangle$ reads:

$$\begin{aligned}(13)\qquad &\left\{(\hbar^2/2M)\left[-d^2/dr + l(l+1)/r^2\right] + V_c(r) - \right.\\ &- \left[U_0 + U_{\sigma\sigma}\,(2s(s+1)-3)\right] g(r) +\\ &+(\hbar/M_\pi c)^2 (U_{ls}/2)(j(j+1)-l(l+1)-s(s+1))(1/r)(\partial g/\partial r) -\\ &\left.-(M/M_p)E\right\}\, u^j_{sl}(r) = 0,\end{aligned}$$

where M_p is the proton mass, M the reduced mass of the system and E is the energy in the laboratory frame. Introducing eqs (12) into eq. (13), and approximating the coulomb potential inside the nucleus by the constant $V_c(R) = Z_1 Z_2 e^2/R$(*), the connection between phase shifts and potential parameters is simply given by the following equation:

$$\begin{aligned}(14)\qquad &P^c_l(\rho)\cot\left[\delta^j_{sl} + \phi^c_l(\rho)\right] = \mu_s F_{l-1}(\mu_s)/F_l(\mu_s) -\\ &-l-S^c_l(\rho) - (M/\hbar^2)(\hbar/M_\pi c)^2 U_{ls}(j(j+1)-l(l+1)-s(s+1)),\end{aligned}$$

where F_l is the regular solution of the Schrödinger equation for positive energy free neutral particles, P^c_l, S^c_l, ϕ^c_l are penetrability, shift factor and hard sphere phase shift for charged particles[8]. The external and internal momenta are defined as follows:

$$(15a)\qquad \rho = \left[(2M^2/\hbar^2 M_p)R^2 E\right]^{1/2},$$

$$(15b)\qquad \mu_s = \left[(2M/\hbar^2)R^2\left\{U_0 + \left[2s(s+1)-3\right] U_{\sigma\sigma}\right\} + \rho^2 - 2\eta\rho\right]^{1/2}.$$

Let us consider finally the tensorial term V_T. It is well known that the tensor interaction leads to coupling of triplet states with same parity and same j. Therefore, a rigorous approach to the problem would

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(*) - It can be easily shown that the error introduced by this approximation is irrelevant. In any case, after the potential parameters were obtained under this assumption, all phase shifts have been recalculated by numerical integration, using eqs (11) for V_c and a Wood's and Saxon form with very small diffuseness for g(r). The phase shifts obtained were in agreement with those derived from eq. (14) within less than 1^o.

require new phase shifts analyses, taking into account higher angular momenta and mixing coefficients ε_j' (see eq. 5b). Nevertheless, in the framework of our crude model, at the very low energies considered here, a useful approximation devoted to evaluate the order of magnitude of the effects could be that of neglecting couplings, and considering only the diagonal terms of the tensor operator.

Equation (13) is then modified by taking into account the diagonal terms of the operator S_{12}, which are given below for easy reference:

(16a) $\langle j, 1, j-1 | S_{12} | j, 1, j-1 \rangle = -2(j-1)/(2j+1)$,

(16b) $\langle j, 1, j | S_{12} | j, 1, j \rangle = 2$,

(16c) $\langle j, 1, j+1 | S_{12} | j, 1, j+1 \rangle = -2(j+2)/(2j+1)$.

The analysis of all the solutions of Table I in the ligth of the theoretical approach outlined above, leads to the following conclusions:

- All solutions show a very small splitting of the S doublet. The spin-spin interaction must than be small, and will be neglected throughout.
- In solution (c) and (d) the phase shift δ^1_{01} is in general higher than δ^2_{11}. Now, if we assume $V = V_c$ we have obviously $\delta^1_{01} = \delta^2_{11}$. If V_T is added, it can be easily seen from eq. (16) that δ^2_{11} is only slightly varied (unless exceedingly large V_T values be introduced), while the introduction of V_{ls} (of the usual sign), can only rise the phase shift δ^2_{11}. The conclusion is that solutions (c) and (d) cannot be reproduced by our model. This conclusion is confirmed by actual calculations, and is widely independent on the radial potential form.

Let us then consider solutions (a) and (b) from a more quantitative viewpoint:

- Solution (a) shows the P triplet sequence as expected from a spin-orbit interaction, but it can be easily seen that a pure spin-orbit force is not able to reproduce the experimental splittings. In fact, from eq. (14) one can derive the following relation among the phaseshifts of the P triplet:

(17) $$2\cot(\delta^0_{11}+\phi^c_1)-3\cot(\delta^1_{11}+\phi^c_1)+\cot(\delta^2_{11}+\phi^c_1) = 0^{(*)}$$

which is strongly violated by the data of Table I. The tensorial term is then needed, and these conclusions are supported by calculations per-

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(*) - The relation (17) is independent on potential strengths U_0, $U_{\sigma\sigma}$, U_{ls}, and only slightly dependent on the nuclear radius R, through the slow varying function $\phi^c_1(\rho)$. It is also independent on the approximation given for the coulomb potential.

formed by a Wood and Saxon potential (see forward).
The best fit with experience has been obtained by means of the following parameters:

(18) $R = 2.62f$; $U_0 = (35.62 - 1.08E)MeV$; $U_{\sigma\sigma} = 0$.

(19) $U_{ls} = 3MeV$; $U_T = 1$ MeV.

In Figs. 5 and 6 the comparison between calculations and experiments is shown. The P triplets obtained by pure spin-orbit interaction are drawn with dotted lines.

- In order to reproduce solution (b), the tensorial term is expected to be more and more important, because the phase shifts δ^1_{11} and δ^0_{11} must be inverted. A satisfactory agreement with experience (see Figs. 7 and 8), has been obtained with the central potential parameters given before, and with the following values of U_T and U_{ls}:

(20) $U_{ls} = 2MeV$; $U_T = -1MeV$.

It must be said for completeness that neither solution (a) nor (b) can be reproduced by a pure tensorial interaction.

As a final step of the analysis, solution (a) has been analyzed with a Wood and Saxon's potential, in order to ascertain that the general conclusions which have been drawn, are independent on the radial form of the interaction. The expression

(21) $$g(r) = \left[1 + \exp\left\{(r - R)/a\right\}\right]^{-1}$$

has been substituted into eqs. (10), and the Schroedinger equation has been numerically integrated by means of the electronic computer.

Starting from parameters (18) (19), and assuming initially a diffuseness $a = 0.5f$, the phase shift behaviour has been studied for variation of all potential parameters. It has been found that, as in the square well calculations, both spin-orbit and tensor interactions are needed in order to reproduce the P triplet. We may say therefore that the qualitative conclusions drawn in the square well approximation continue to hold even when a diffuseness is introduced into potential. The following values of the parameters

(22) $R = 2.37f$; $a = 0.5f$; $U_0 = (38.7-1.0E)MeV$;
$U_{\sigma\sigma} = 0$; $U_{ls} = 4MeV$; $U_T = 1.3$ MeV.

give a phase shift behaviour similar to that of Figs. 5 and 6, which re-

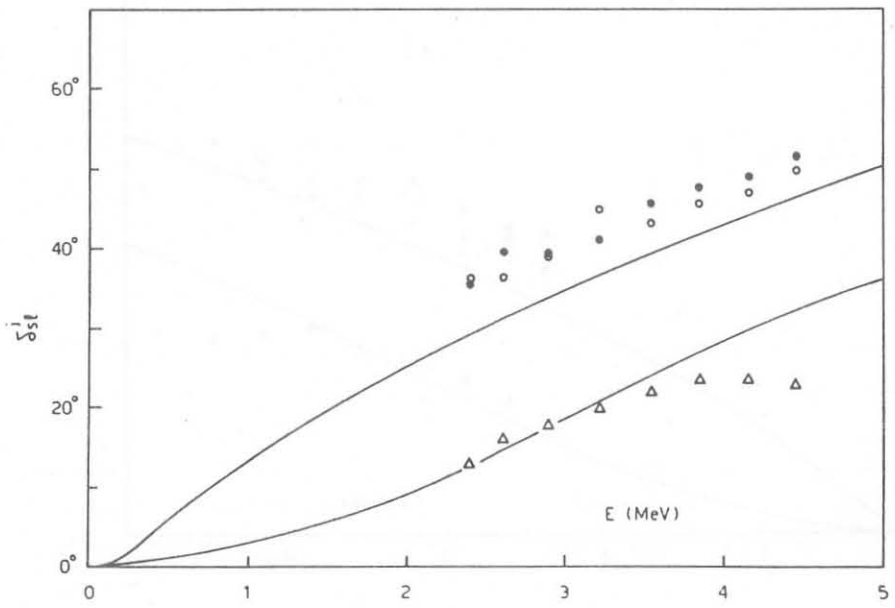

FIG. 5 - Solution a: The phase shifts $\delta^0_{00}(\circ)$, $\delta^1_{10}(\bullet)$ and $\delta^1_{01}(\Delta)$ are compared with theoretical curves, obtained from the potential parameters (18).

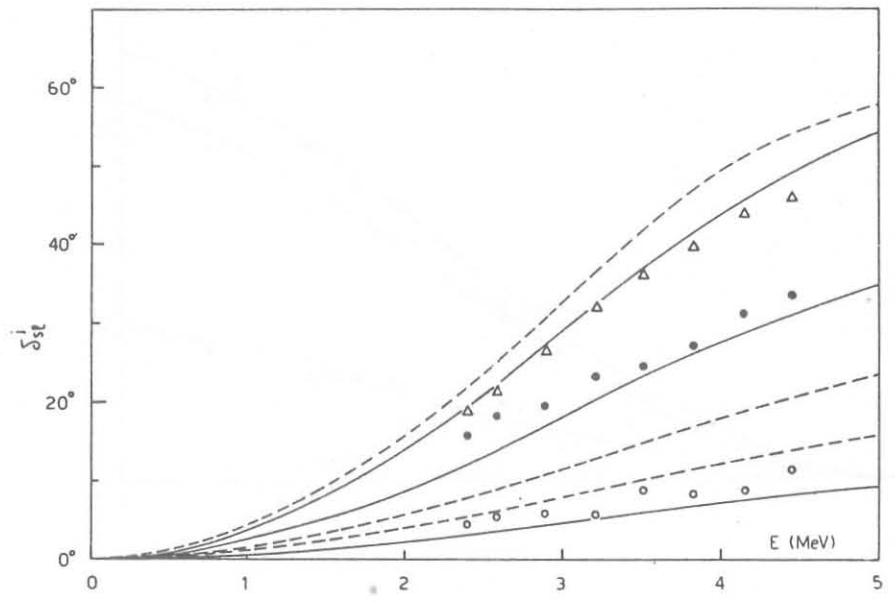

FIG. 6 - Solution a: comparison between phase shifts $\delta^0_{11}(\circ)$, $\delta^1_{11}(\bullet)$, $\delta^2_{11}(\Delta)$, and theoretical curves (parameters (18), (19)).

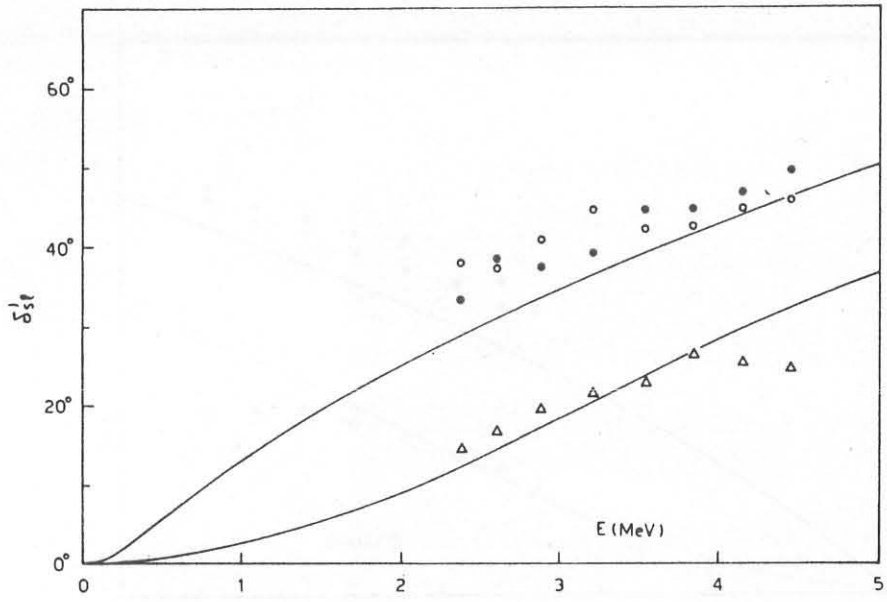

FIG. 7 - Solution b: comparison between phase shifts $\delta^{0}_{00}(\circ)$, $\delta^{1}_{10}(\bullet)$, $\delta^{1}_{01}(\Delta)$, and theoretical curves (parameters (18)).

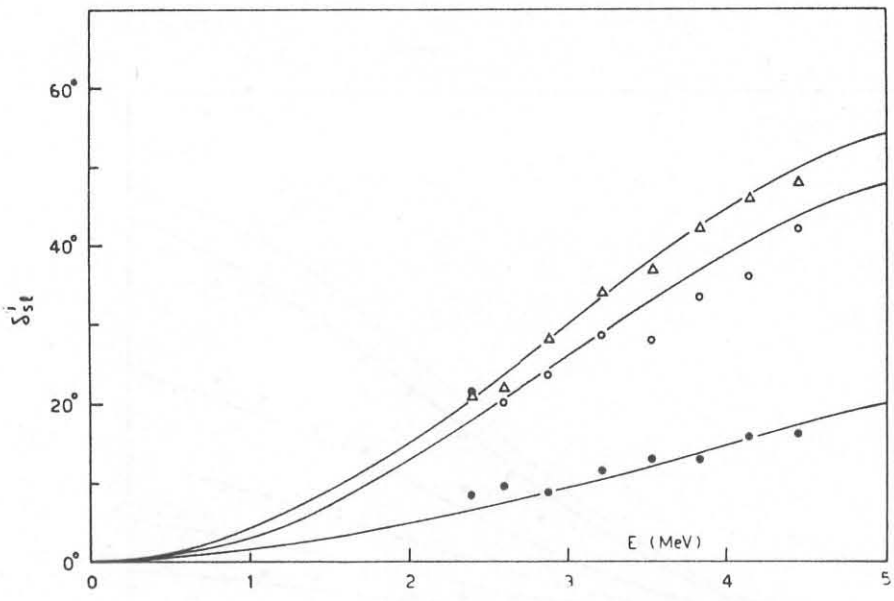

FIG. 8 - Solution b: comparison between phase shifts $\delta^{0}_{11}(\circ)$, $\delta^{1}_{11}(\bullet)$, $\delta^{2}_{11}(\Delta)$, and theoretical curves (parameters (18), (20)).

present the best agreement with the experiment obtained by us. It is nevertheless believed that a more systematic analysis (which unfortunately requires long calculation times) should probably improve slightly the fit.

Finally the potential model can be used in order to calculate the D phase shifts, whose experimental determination is very uncertain.

The problem of extrapolating the potential parameters is cumbersome, because the linear dependence of U_0 and the approximated treatment of the tensor interaction are expected to fail as soon as the energy is raised. It is nevertheless reasonable to expect, from these calculations, realistic indications at least on the sign of the D waves. We have found, at the energy of 4-5 MeV, small ($\leq 1^o \simeq 2^o$) and positive values for the 4 D phase shifts, in contrast with the calculations performed by us (see Table II), and with the results of Ref.[3]. These conclusions are stable against sensitive variations of the potential parameters, and independent on the employment of the square well or Wood and Saxon radial form.

CONCLUSIONS -

Four phase shift sets have been obtained, able to reproduce cross section and polarization curves between zero and ~ 7 MeV. Only solution (a) has for the moment the support of higher energies calculations.

On the ground of the interaction form here considered, only solutions (a) and (b) can be accounted for by a potential model. Both spin-orbit and tensor forces seem to be necessary and also sufficient (within the limits of experimental errors), in order to reproduce these solutions. On the contrary spin-spin forces are negligible in all cases.

It is believed that a definitive word on the discrimination will be said by experiments with polarized protons scattered by polarized targets. We are not able, for example, to exclude completely solutions (c) and (d), but we only say that these solutions should require completely different potential forms: for example potentials not spherically symmetric in spatial coordinates.

Of course one should not be surprised if further experiments would disclose that more realistic potential forms are needed, in order to account for possible "anomalous" phase shift behaviour, arising from the well known anomalous properties of the He^3 system.

APPENDIX -

In the approximation $l_{max} = 1$, and neglecting mixing parameter, the differential cross section (1) reads:

$$k^2 \sigma(\theta) = |f_c(\theta)|^2 = \sum_{n=1}^{6} A_n a_n(\theta), \tag{A1}$$

where the angular functions $a_n(\theta)$ and the distribution coefficients A_n can be written as follows:

$$a_1(\theta) = 1-3\cos^2\theta, \tag{A2a}$$

$$a_2(\theta) = \cos\theta, \tag{A2b}$$

$$a_3(\theta) = \mathrm{Re}\left[f_c(\theta)\right] + \frac{3}{2}\cos^2\theta, \tag{A2c}$$

$$a_4(\theta) = \mathrm{Im}\left[f_c(\theta)\right], \tag{A2d}$$

$$a_5(\theta) = \mathrm{Re}\left[f_c(\theta)\right]\cos\theta + \frac{3}{2}\cos 2\omega_1 \cos^2\theta, \tag{A2e}$$

$$a_6(\theta) = \mathrm{Im}\left[f_c(\theta)\right]\cos\theta + \frac{3}{2}\sin 2\omega_1 \cos^2\theta, \tag{A2f}$$

$$\cos 2\delta^0_{00} + 3\cos 2\delta^1_{10} = 4(1-A_3), \tag{A3a}$$

$$\sin 2\delta^0_{00} + 3\sin 2\delta^1_{10} = -4A_4, \tag{A3b}$$

$$\begin{aligned} &3\cos 2\delta^1_{01}(\cos 2(\omega_1 - \delta^0_{00}) - \cos 2(\omega_1 - \delta^1_{10})) - \\ &-3\sin 2\delta^1_{01}(\sin 2(\omega_1 - \delta^0_{00}) - \sin 2(\omega_1 - \delta^1_{10})) = \\ &= 8A_2 - 4A_5(1-\cos 2\delta^1_{10}) + 4A_6\sin 2\delta^1_{10} + 3\cos 2(\omega_1 - \delta^0_{00}) - \\ &-3\cos 2(\omega_1 - \delta^1_{10}), \end{aligned} \tag{A3c}$$

$$\begin{aligned} &\cos 2(\omega_1 + \delta^0_{11}) + 3\cos 2(\omega_1 + \delta^1_{11}) + 5\cos 2(\omega_1 + \delta^2_{11}) = \\ &= 12\cos 2\omega_1 - 3\cos 2(\omega_1 + \delta^1_{01}) - 4A_5, \end{aligned} \tag{A3d}$$

$$\begin{aligned} &\sin 2(\omega_1 + \delta^0_{11}) + 3\sin 2(\omega_1 + \delta^1_{11}) + 5\sin 2(\omega_1 + \delta^2_{11}) = \\ &= 12\sin 2\omega_1 - 3\sin 2(\omega_1 + \delta^1_{01}) - 4A_6, \end{aligned} \tag{A3e}$$

$$4\cos 2(\delta^0_{11} - \delta^2_{11}) + 9\cos 2(\delta^1_{11} - \delta^2_{11}) = 13 + 16A_3 - 32A_1. \tag{A3f}$$

If the coefficients A_n are known from experience, the solution of the trigonometric system (A3) leads to determination of the phase shifts, with certain mathematical ambiguities. The solution of the system and the mathematical ambiguities encountered will be now briefly discussed.

The system of equations (A3 a, b) is very well known from the analysis of the scattering between 0 spin and 1/2 spin particles[13]. We write down, for easy reference, the system in a more general form, and the resolutive formulas:

(A4a) $$x\cos\alpha + y\cos\beta = u,$$

(A4b) $$x\sin\alpha + y\sin\beta = v.$$

(A5a) $$\cos\alpha = F_+(\Omega, u, v)/x,$$

(A5b) $$\sin\alpha = F_+(\Omega, v, -u)/x,$$

(A5c) $$\cos\beta = F_-(\Omega, u, v)/y,$$

(A5d) $$\sin\beta = F_-(\Omega, v, -u)/y,$$

where

(A6) $$F_{\pm}(\Omega, p, q) = \left\{ p\left[(p^2 + q^2) \pm (x^2 - y^2)\right] \pm \right.$$
$$\left. \pm \Omega q\left[(2xy)^2 - (p^2+q^2-x^2-y^2)^2\right] 1/2\right\} \left\{2(p^2+q^2)\right\}^{-1}.$$

Assuming in eqs (A5), (A6) either $\Omega = +1$ or $\Omega = -1$, we obtain two solution sets, which satisfy equation

(A7) $$\alpha(\Omega=1) - \beta(\Omega=1) = \beta(\Omega=-1) - \alpha(\Omega=-1),$$

and represent an ambiguity known as the Fermi-Yang ambiguity in the case of the π^+-p process.

By particularization of the formulas (A4), (A5), (A6) to the case (A3a, b), two S wave doublets are obtained.

The eq. (A3c), considered now in the unknown δ^1_{01} has the form

(A8) $$x\cos\alpha + y\sin\alpha = z,$$

and can be solved by the following expressions:

(A9a) $$\cos\alpha = \left[xz + \Omega y\sqrt{x^2 + y^2 - z^2}\right]\left[x^2 + y^2\right]^{-1},$$

(A9b) $$\sin\alpha = \left[yz - \Omega x \sqrt{x^2+y^2-z^2} \right] \left[x^2+y^2 \right]^{-1}.$$

By means of eqs (A9), we obtain two values of the singlet P phase shift, for each one of the S doublets found above.

The system (A3d, e, f) can be solved for example by parametrization in respect of the phase shift δ^2_{11}. By this procedure the system (A3d, e) assumes the form (A4) (in the unknown δ^0_{11}, δ^1_{11}), and is immediately solved through eqs (A5). The eq. (A3f) is then employed as a "freezing" condition. One gets this way four sets of P triplets for each one of the sets found before. The four P triplets (corresponding to the same S and singlet P phase shifts) satisfy the following equation[14]:

(A10) $$(\delta^j_{11})^I + (\delta^j_{11})^{II} + (\delta^j_{11})^{III} + (\delta^j_{11})^{IV} = \text{constant} \qquad (j=0, 1, 2)$$

We have therefore in general $2^4 = 16$ solution sets, which give the same differential cross section. In practical cases the total number of mathematical solutions can be reduced drastically, because several of the sets now discussed happen to be imaginary.

In the same approximation, the polarization can be put into the form

(A11) $$P(\theta)k^2\sigma(\theta)/\sin\theta = \sum_{n=1}^{4} B_n b_n(\theta)$$

where the angular functions $b_n(\theta)$ are:

(A12a) $$b_1(\theta) = 1,$$

(A12b) $$b_2(\theta) = \cos\theta,$$

(A12c) $$b_3(\theta) = \mathrm{Re}\left[f_c(\theta) \right],$$

(A12d) $$b_4(\theta) = \mathrm{Im}\left[f_c(\theta) \right],$$

and where the coefficients B_n satisfy the following equations:

(A13a) $$B_2\cos 2(\delta^1_{10} + \omega_1) + B_3 \sin 2(\delta^1_{10} + \omega_1) = -2B_0 + B_2,$$

(A13b) $$B_2\cos 2(\delta^2_{11} + \omega_1) + B_3 \sin 2(\delta^2_{11} + \omega_1) =$$
$$= -(2/3)B_1 + B_2 \cos 2\omega_1 + B_3 \sin 2\omega_1,$$

(A13c) $$2\cos 2(\delta^{0}_{11} + \omega_1) + 3\cos 2(\delta^{1}_{11} + \omega_1) =$$

$$= 8B_3 + 5\cos 2(\delta^{2}_{11} + \omega_1),$$

(A13d) $$2\sin 2(\delta^{0}_{11} + \omega_1) + 3\sin 2(\delta^{1}_{11} + \omega_1) =$$

$$= -8B_2 + 5\sin 2(\delta^{1}_{11} + \omega_1).$$

The eqs. (A13a, b) have the structure (A8), and can be solved with formulas (A9). We obtain therefore two values of the phase shift δ^{1}_{10} and (independently) two values of δ^{2}_{11}. Finally, the system (A13c, d) (in the unknowns δ^{0}_{11} and δ^{1}_{11}) is equivalent to the system (A4) and gives two doublets $\delta^{0}_{11}\,\delta^{1}_{11}$ (for each one of the two δ^{2}_{11} phase shifts) which are connected by the Fermi-Yang ambiguity. Since these P triplet states can be combined indifferently with both S triplet solutions, we have altogether eight phase shift sets, which give rise to the same polarization curves.

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