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E. Gadioli, I. Iori and M. Sansoni : FINITE RANGE OF DATA ERRORS AFFECTING THE ANALYSIS OF A FLUCTUATING EXCITATION FUNCTION. -

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E. Gadioli, I. Iori, M. Sansoni: FINITE RANGE OF DATA ERRORS AFFECTING THE ANALYSIS OF A FLUCTUATING EXCITATION FUNCTION[x].

INTRODUCTION. -

The analysis of fluctuating excitation functions in the continuum energy region gives some interesting information on the reaction mechanism and the properties of nuclei at high excitation energy. A quantitative analysis of the experimental data requires some caution: in many cases it is necessary to take into account the variations of the average cross-section around which the fluctuations occur and the influence of the experimental conditions, i. e. the energy resolution and the energy step separating two neighbouring points of the excitation function[1, 2, 3]. Also when all these facts are taken into account, the results of the analysis are affected by bias and errors due to the finite sample of data which are analized. The evaluation of such errors has been given in different works. A recent paper by Dallimore and Hall[4] gives the expressions for evaluating some of such errors. As some of these estimations do not agree in a completely satisfactory way with the results of numerical calculations obtained with Monte Carlo methods[5], we have attempted to obtain more correct evaluations: our results are reported in this paper.

The errors due to the finite sample of data that have been estimated so far, have always been calculated in the hypothesis that the average cross-section be a constant function of the energy. We will show that, in such a case, the calculations can be very accurate. In practice, however, many experimental results show cross-sections fluctuating around an avera

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ge value that varies with the energy. One can try to extend to the new situation the results obtained in the case of a constant average. The evaluations of the errors, in these cases, are only first approximations: we believe, however, that, in many cases, these evaluations could be quite realistic.

EVALUATION OF THE ERRORS IN THE CASE OF PURELY STATISTICAL REACTIONS WITH A CONSTANT VALUE OF THE AVERAGE CROSS-SECTION. -

Let us first consider the case of purely statistical reactions and fluctuations around a constant average value. All the results reported may be obtained under the following assumptions:

1) - Only N incoherent channels contribute to the given statistical cross-section. Then the cross-section σ is a stochastic variable with a χ^2 distribution with 2N degrees of freedom[6,7]. In the case of integrated excitation functions, N is given by the expression[8]:

$$N = \frac{\left\{ \sum_{Jlsl's'} T_l(\alpha) T_{l'}(\alpha') \, e^{J(J+1)/2\sigma^2} / \Gamma_J \right\}^2}{\sum_{Jlsl's'} \left\{ T_l(\alpha) T_{l'}(\alpha') \, e^{J(J+1)/2\sigma^2} / \Gamma_J \right\}^2} \tag{1}$$

α and α' stand respectively for the initial and final channel, $T_l(\alpha)$, $T_{l'}(\alpha')$ are the transmission functions for the same channels, J and Γ_J are the spin and the width of the levels of the compound nucleus interested in the reaction, σ^2 is the spin cut-off factor of the compound nucleus.

In the case of differential excitation functions N is given approximately by eq. (2.49) of ref. (7), or by the inverse of the quantity $C(\theta, \theta)$ given by eq. (27) of ref. (9).

In expression (1) the energy dependence of N has not been considered: it is supposed to be constant with E.

2) Into an energy interval Δ of the excitation function, because of the correlations between neighbouring points, there are only $[(\Delta/\pi\Gamma) + 1]$ indipendent values of the cross-section. This result is due to Böhning and Gibbs as reported in ref. (5).

The following property of the χ^2 distributions is used: if σ_i are independent values following a χ^2 distribution with 2N degrees of freedom, the quantity $\langle\sigma\rangle = \frac{1}{n}\sum_1^n {}_i\, \sigma_i$ has a χ^2 distribution with 2Nn degrees of freedom[10].

Now we may calculate the biased average values of F(0) and C(0), that is the average values of the absolute and the relative autocorrelation functions for $\varepsilon = 0$, when they are calculated by taking into account only n independent values of the cross-sections.

In what follows the brackets will indicate a sample average, while the bar an ensemble average.

a) Biased average value of F(0).

By definition one has

$$F(0) = \langle(\sigma - \langle\sigma\rangle)^2\rangle = \langle\sigma^2 - 2\sigma\langle\sigma\rangle + \langle\sigma\rangle^2\rangle = \langle\sigma^2\rangle - \langle\sigma\rangle^2 \tag{2}$$

$$\overline{F(0)} = \overline{\langle\sigma^2\rangle} - \overline{\langle\sigma\rangle^2} \tag{3}$$

It is easy to see that

$$\overline{\langle\sigma^2\rangle} = \overline{\sigma^2}; \qquad \overline{\langle\sigma\rangle^2} = \frac{Nn+1}{Nn}\,\overline{\langle\sigma\rangle}^2 = \frac{Nn+1}{Nn}\,\bar{\sigma}^2$$

the last result being due to the cited property of the χ^2 distributions. With such results in mind one gets :

$$\overline{F(0)} = \overline{\sigma^2} - \frac{Nn+1}{Nn}\bar{\sigma}^2 = \overline{\sigma^2} - \bar{\sigma}^2 - \frac{1}{Nn}\bar{\sigma}^2$$

For the properties of the χ^2 distributions it is :

$$\overline{\sigma^2} = \frac{N+1}{N}\bar{\sigma}^2 ,$$

therefore $\overline{\sigma^2} - \bar{\sigma}^2 = \dfrac{\bar{\sigma}^2}{N}$ and

$$\overline{F(0)} = \frac{\bar{\sigma}^2}{N} - \frac{1}{Nn}\bar{\sigma}^2 = \left(1 - \frac{1}{n}\right)\frac{\bar{\sigma}^2}{N} = \left(\frac{n-1}{n}\right)\frac{\bar{\sigma}^2}{N} . \tag{4}$$

$\bar{\sigma}^2/N$ would be the expected value for F(0) if an ensemble of values of the cross-sections had been used in calculating it.

The result shows that the ensemble average of the values F(0) obtained taking into account a sample of n values of the cross-section, is smaller than the value of F(0) one would obtain taking into account the ensemble of the values of the cross-section.

In fig. 1 it is shown the comparison between the theoretical estimation of $\overline{F(0)}$ (full curve) as a function of n in the case N = 1, and the values obtained analyzing a fictitious excitation function constructed by means of a method first outlined by Brink and Stephen[11] and used also in ref. (3).

The values of σ are obtained as the square modulus of the quantity[12, 13] :

$$\hat{S} = -i \sum_j \frac{a_j}{E - E_j + \frac{1}{2} i \Gamma} .$$

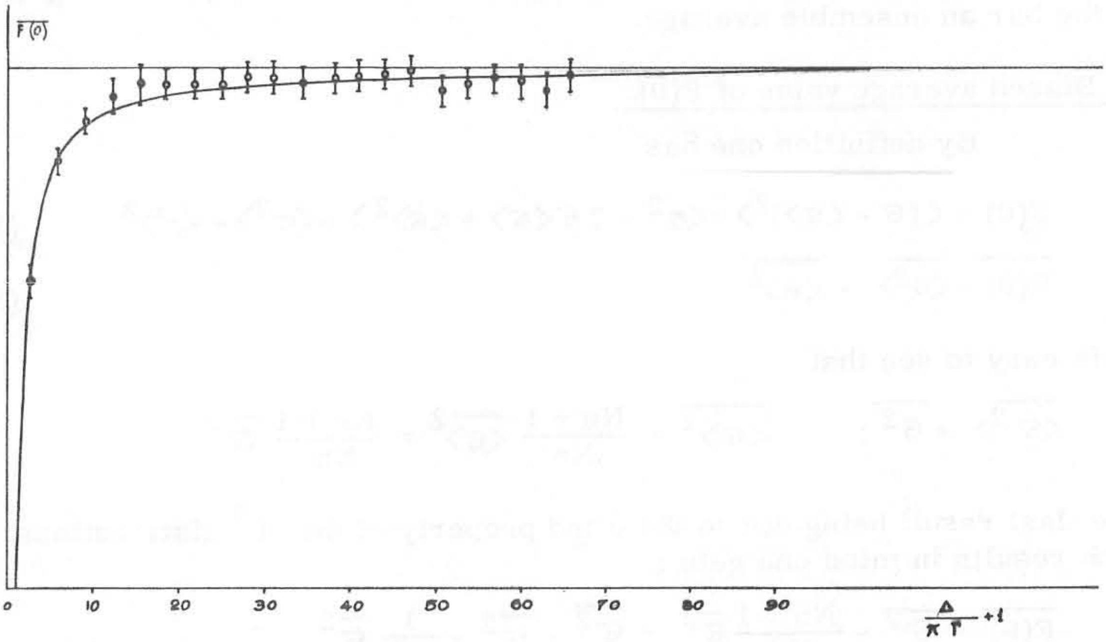


Fig. 1 - Comparison between the theoretical evaluation of the average biased value of F(0) as given by (4), in the case N = 1, and the estimations obtained with a Monte Carlo method as described in the text.

a_j are random complex variables with zero average value: two values a_i and a_j are uncorrelated if $i \neq j$; E_j is the energy of the levels of the compound nucleus involved in the fictitious statistical reaction, Γ is their average width. An equal spacing distribution of the energy levels has been assumed; the ratio between the average width and the average spacing has been taken equal to 20; the energy step between two neighbouring values of σ in the excitation function has been taken equal to $\Gamma/5$.

The biased values are the average of values obtained analyzing different intervals of the excitation function. Fig. 1 shows the errors one should attribute to these values.

The analysis is also a check of the estimation of n given by Böhning and Gibbs.

b) Biased average value of C(0).

C(0) is defined as follows :

$$C(0) = \frac{F(0)}{\langle\sigma\rangle^2} = \frac{\langle\sigma^2\rangle}{\langle\sigma\rangle^2} - 1 = \langle \frac{\sigma^2}{\langle\sigma\rangle^2} \rangle - 1$$

$$\overline{C(0)} = \overline{\langle \frac{\sigma^2}{\langle\sigma\rangle^2} \rangle} - 1 = \overline{(\frac{\sigma^2}{\langle\sigma\rangle^2})} - 1 .$$

The first term is calculated making the following approximation: if n is not too small, $\overline{\sigma^2}$ and $\langle\sigma\rangle^2$ are independent variables; the square root of the variance of $\langle\sigma\rangle^2$ is much smaller than $\overline{\langle\sigma\rangle^2}$ and we have approximately :

$$\overline{\left(\frac{\sigma^2}{\langle\sigma\rangle^2}\right)} \cong \frac{\overline{\sigma^2}}{\overline{\langle\sigma\rangle^2}} = \frac{\overline{\sigma^2}}{\frac{Nn+1}{Nn}\bar{\sigma}^2} = \frac{Nn}{Nn+1}\,\frac{\overline{\sigma^2}}{\bar{\sigma}^2} .$$

In this approximation it is

$$\overline{C(0)} \cong \frac{Nn}{Nn+1}\,\frac{\overline{\sigma^2}}{\bar{\sigma}^2} - 1 = \left[1 - \frac{1}{Nn+1}\right]\frac{\overline{\sigma^2}}{\bar{\sigma}^2} - 1 = \frac{\overline{\sigma^2}}{\bar{\sigma}^2} - 1 - \frac{1}{Nn+1}\,\frac{\overline{\sigma^2}}{\bar{\sigma}^2} .$$

As $\dfrac{\overline{\sigma^2}}{\bar{\sigma}^2} - 1 = \dfrac{1}{N}$

one gets

$$\overline{C(0)} \cong \frac{1}{N} - \left(\frac{1}{N}+1\right)\frac{1}{Nn+1} = \frac{1}{N}\left(\frac{nN}{Nn+1}\right) - \frac{1}{Nn+1} = \frac{nN - N}{N(Nn+1)} \qquad (5)$$

1/N is the value we would expect for C(0) if it had been calculated utilizing an ensemble of values of the cross-section. Actually $\overline{C(0)}$ is the ensemble average of the values C(0) obtained with a sample of n values of the cross-section.

In fig. 2 the theoretical evaluations of $\overline{C(0)}$ (full curve) as a function of n, are compared with the results of the Monte Carlo calculations of Gibbs in the cases N = 1, 2, 3[5].

One can see that our approximation is very good also for very low values of n.

c) Error affecting the biased value F(0).

The variance of F(0) is so defined :

$$\begin{aligned} \text{var } F(0) &= \overline{\left(\langle\sigma^2\rangle - \langle\sigma\rangle^2 - \overline{\langle\sigma^2\rangle} + \overline{\langle\sigma\rangle^2}\right)^2} = \\ &= \overline{\langle\sigma^2\rangle^2} + \overline{\langle\sigma\rangle^4} - 2\,\overline{\langle\sigma\rangle^2\langle\sigma^2\rangle} - \overline{F(0)}^2 \end{aligned} \qquad (6)$$

It is easy to show that :

i) $\text{var}\langle\sigma^2\rangle = \overline{\langle\sigma^2\rangle^2} - \overline{\sigma^2}^2$ then: $\overline{\langle\sigma^2\rangle^2} = \text{var}\langle\sigma^2\rangle + \overline{\sigma^2}^2$

But

$$\text{var}\langle\sigma^2\rangle = \text{var}\left(\frac{1}{n}\sum_{1}^{n}{}_i\,\sigma_i^2\right) = \frac{1}{n}\,\text{var}(\sigma^2) = \frac{4N+6}{N(N+1)}\,\frac{1}{n}\,\overline{\sigma^2}^2$$

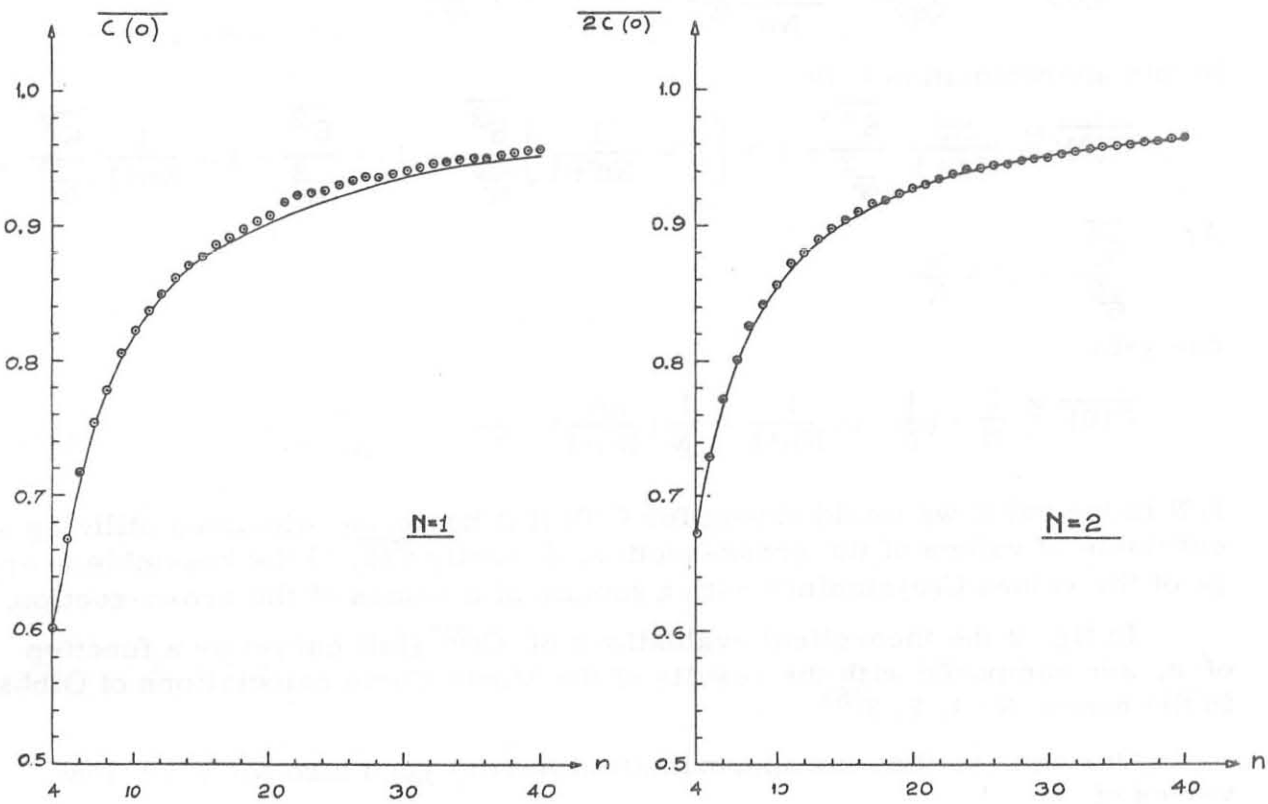
and $\overline{\sigma^2}^2 = \left(\frac{N+1}{N}\bar{\sigma}^2\right)^2 = \left(\frac{N+1}{N}\right)^2\bar{\sigma}^4$

so that

6.

$$\overline{\langle \sigma^2 \rangle^2} = \left\{ \frac{4N+6}{N(N+1)} \; \frac{1}{n} + 1 \right\} \left(\frac{N+1}{N} \right)^2 \overline{\sigma}^4 = \tag{7}$$

$$= \frac{4N^2 + 10N + nN^3 + 2nN^2 + nN + 6}{nN^3} \; \overline{\sigma}^4 \; .$$



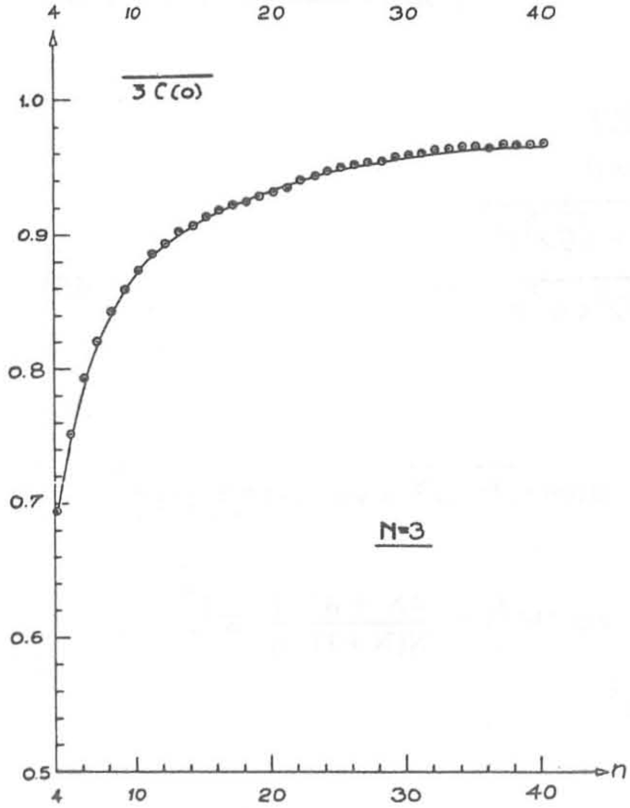


Fig. 2 - Comparison between the theoretical evaluation of the average biased value of C(0) as given by (5) and the results of a Monte Carlo calculation given by Gibbs in the case N = 1, 2, 3.

ii) Taking into account the probability distribution of $\langle\sigma\rangle$ we obtain

$$\overline{\langle\sigma\rangle^4} = \frac{n^3N^3 + 6n^2N^2 + 11nN + 6}{n^3N^3}\,\bar{\sigma}^4 \tag{8}$$

iii)

$$2\overline{\langle\sigma\rangle^2\langle\sigma^2\rangle} = \overline{\frac{2}{n^3}\left\{\sum_1^n {}_i\sigma_i\right\}^2\left\{\sum_1^n {}_i\sigma_i^2\right\}} =$$

$$= 2\left(\frac{\overline{\sigma^4}}{n^2} + \frac{n^2-n}{n^3}\overline{\sigma^2}^2 + 2\,\frac{n^2-n}{n^3}\,\overline{\sigma^3}\,\overline{\sigma} + \frac{n^3-3n^2+2n}{n^3}\,\bar{\sigma}^2\,\overline{\sigma^2}\right)$$

$$\overline{\sigma^2} = \frac{N+1}{N}\,\bar{\sigma}^2 \tag{9a}$$

$$\overline{\sigma^3} = \frac{(N+1)(N+2)}{N^2}\,\bar{\sigma}^3 \tag{9b}$$

$$\overline{\sigma^4} = \frac{(N+1)(N+2)(N+3)}{N^3}\,\bar{\sigma}^4 \tag{9c}$$

so that

$$2\overline{\langle\sigma\rangle^2\langle\sigma^2\rangle} = \frac{12nN + 12n + 10n^2N^2 + 10n^2N + 2n^3N^3 + 2n^3N^2}{n^3N^3}\,\bar{\sigma}^4 \tag{10}$$

iiii)

$$\overline{F(0)}^2 = \left(\frac{n-1}{n}\right)^2 \frac{\bar{\sigma}^4}{N^2} \tag{11}$$

Substituting (7), (8), (10), (11) into (6), we obtain

$$\mathrm{var}\,F(0) = \frac{\bar{\sigma}^4}{N^2}\left\{\frac{2N+6}{nN} - \frac{2N+12}{n^2N} + \frac{6}{n^3N}\right\} \tag{12}$$

In fig. 3 the theoretical evaluation of $(\mathrm{var}\,F(0))/\overline{F(0)}^2$ (full curve) is compared with the results of a Monte Carlo calculation in the case N = 1. The Monte Carlo results are obtained by computing var F(0) and $\overline{F(0)}$ on samples of numbers having a χ^2 distribution with 2 degrees of freedom.

d) Error affecting the biased value C(0).

The variance of C(0) is so defined :

$$\mathrm{var}\,C(0) = \overline{\left\{\left(\frac{\langle\sigma^2\rangle}{\langle\sigma\rangle^2} - 1\right) - \overline{C(0)}\right\}^2} = \overline{\left\{\frac{\langle\sigma^2\rangle}{\langle\sigma\rangle^2}\right\}^2} - \left\{\overline{C(0)} + 1\right\}^2 \tag{13}$$

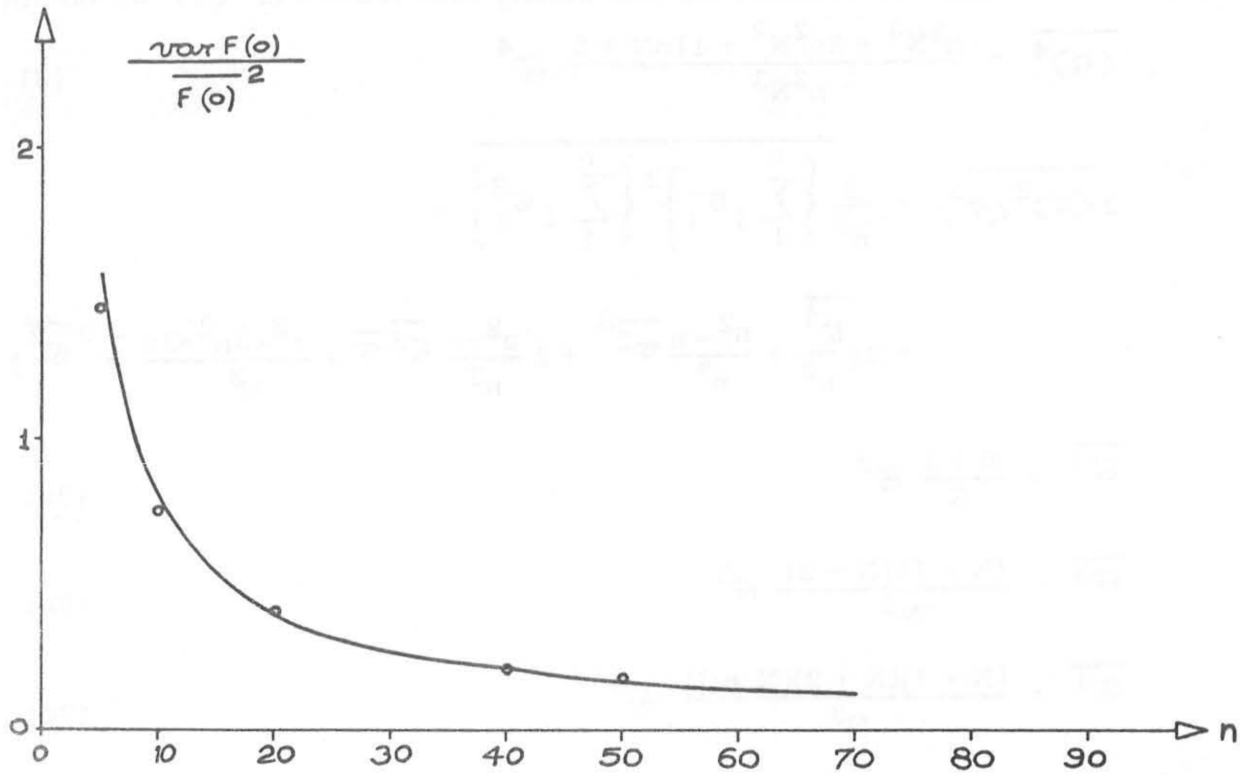


Fig. 3 - Comparison of the theoretical calculation of $\frac{\mathrm{var}\,F(0)}{\overline{F(0)}^2}$, full curve, with the results of a Monte Carlo calculation, open points, in the case N = 1.

It is easy to show the following relations :

i) $$\overline{\left\{\frac{\langle\sigma^2\rangle}{\langle\sigma\rangle^2}\right\}^2} = \overline{\left\{\left\langle\frac{\sigma^2}{\langle\sigma\rangle^2}\right\rangle\right\}^2} = \frac{1}{n^2}\overline{\left\{\sum_1^n{}_i\,\frac{\sigma_i^2}{\langle\sigma\rangle^2}\right\}^2} \cong \frac{1}{n}\,\frac{\overline{\sigma^4}}{\overline{\langle\sigma\rangle^4}} + \frac{n^2-n}{n^2}\,\frac{\overline{\sigma^2}^2}{\overline{\langle\sigma\rangle^4}}$$

$\overline{\langle\sigma\rangle^4}$ is given by (8), $\overline{\sigma^4}$ nad $\overline{\sigma^2}$ by (9c) and (9a), so that we have

$$\overline{\left\{\frac{\langle\sigma^2\rangle}{\langle\sigma\rangle^2}\right\}^2} \cong \frac{n^3N^3 + 2n^3N^2 + n^3N + 4n^2N^2 + 10n^2N + 6n^2}{(nN+1)(nN+2)(nN+3)} \tag{14}$$

ii)

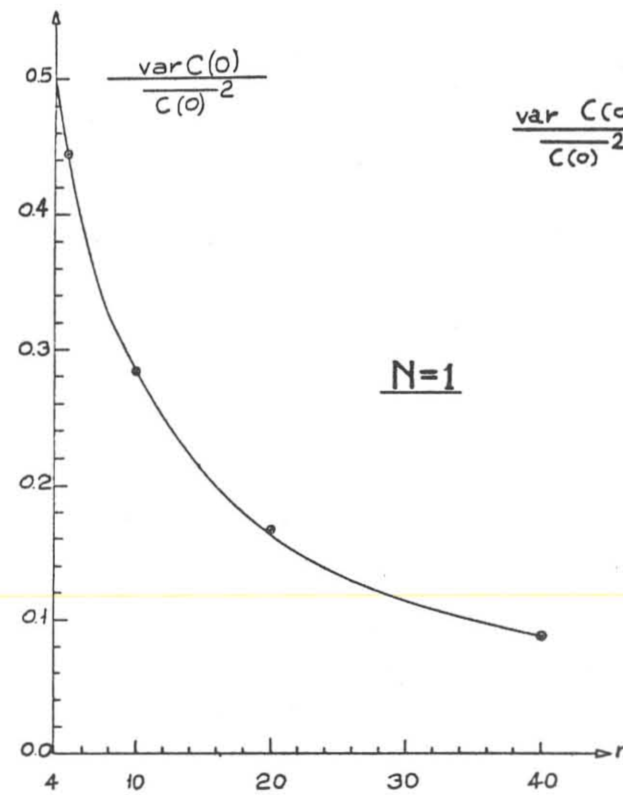
$$\overline{C(0)} + 1 = \frac{nN - N}{N(nN+1)} + 1 = \frac{nN + nN^2}{N(nN+1)} \tag{15}$$

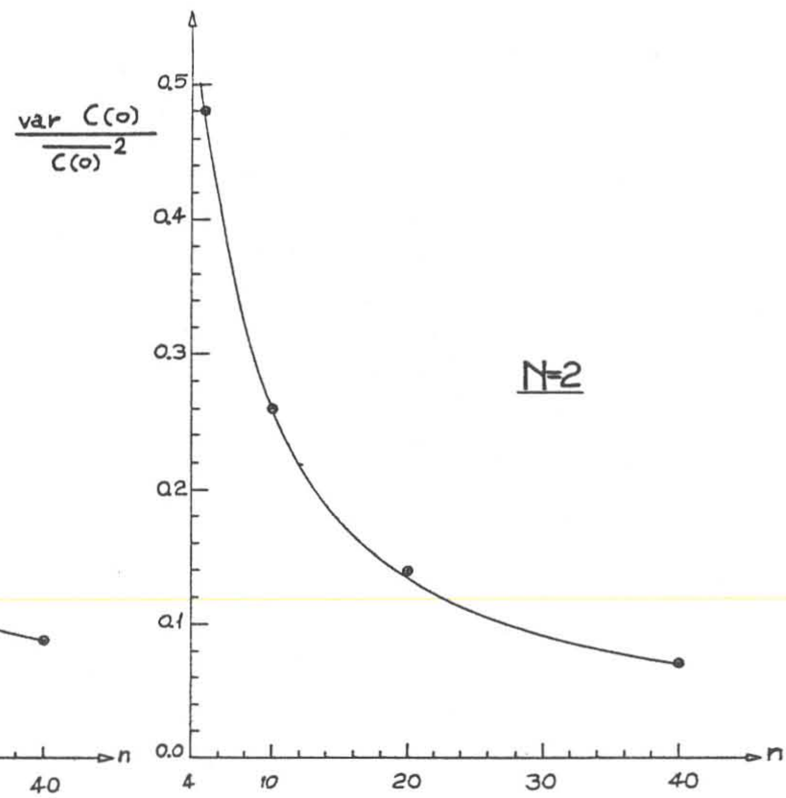
Putting the expressions (14) and (15) into (13) one has :

$$\mathrm{var}\,C(0) = \frac{2n^3N^2 + 2n^3N - 2n^2N^2 - 2n^2N}{(nN+1)^2(nN+2)(nN+3)} \tag{16a}$$

$$\frac{\mathrm{var}\,C(0)}{\overline{C(0)}^2} = \frac{2n^2N^3(N+1)(n-1)}{(nN+2)(nN+3)(nN-N)^2} \tag{16b}$$

In Fig. 4 the theoretical calculations of $\{\mathrm{var}\ C(0)/\overline{C(0)}^2\}$ (full curve) as a function of n, are compared with the results of the Monte Carlo estimations of Gibbs for N = 1, 2, 3[5].





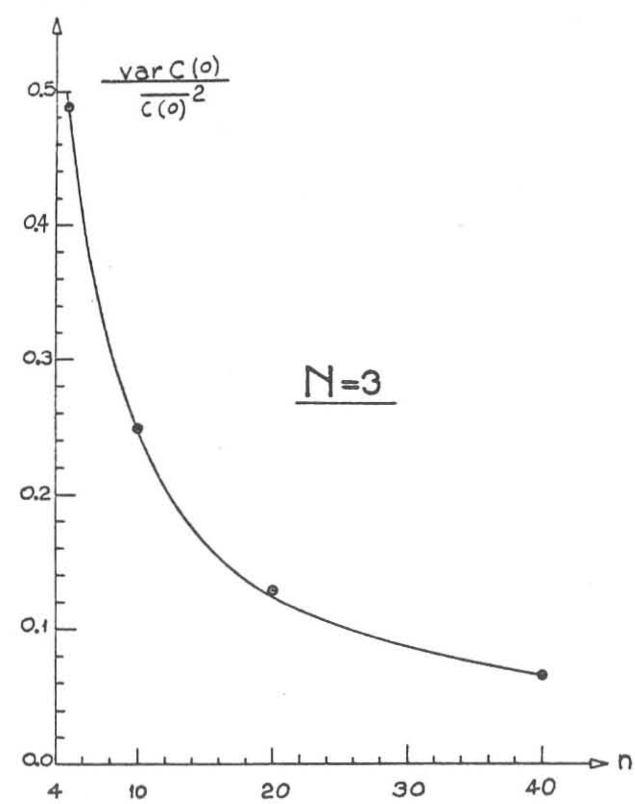


Fig. 4 - Comparison of the theoretical calculation of $\mathrm{var}\,C(0)/C(0)^2$ as given by (16b), full curve, with the results of Monte Carlo calculations, black points, given by Gibbs in the cases N = 1, 2, 3.

A quantity of principal interest, which characterizes the fluctuations, is the coherence energy Γ. It is usually defined as the value of ε for which $F(\varepsilon)$ $(C(\varepsilon))$ becomes $1/2\ F(0)$ $(1/2\ C(0))$. The bias and errors on the values of Γ are due to the bias and errors on the values of $F(\varepsilon)$.

It has to be noticed that the errors and the bias on the experimental values of Γ depend on the method one uses to extract it from the autocorrelation function.

It is easy to show that, in the case we examine, if Γ is defined as the value of ε corresponding to which the ordinate of the curve connecting the experimental values of $F(\varepsilon)$ $(C(\varepsilon))$ becomes $1/2\ F(0)$ $(1/2\ C(0))$, and we assume a complete correlation between values of the cross-section less than Γ apart, we have approximately

$$\Gamma \sim \Gamma_t \left(\frac{n-1}{n}\right) \left[1 \pm \frac{\sqrt{\mathrm{var}\ F(0)}}{\overline{F(0)}}\right] \tag{17}$$

where Γ_t is the true value of Γ.

The proof goes as follows. Let us consider the bias on Γ. By definition it is

$$F(\varepsilon) = \langle(\sigma(E) - \langle\sigma\rangle)(\sigma(E+\varepsilon) - \langle\sigma\rangle)\rangle \simeq \langle\sigma(E)\sigma(E+\varepsilon)\rangle - \langle\sigma\rangle^2 \tag{18}$$

$$\begin{aligned}\overline{F(\varepsilon)} &= \overline{\langle\sigma(E)\sigma(E+\varepsilon)\rangle} - \overline{\langle\sigma\rangle^2} = \overline{\sigma(E)\sigma(E+\varepsilon)} - \frac{Nn+1}{Nn}\bar{\sigma}^2 = \\ &= \overline{\sigma(E)\sigma(E+\varepsilon)} - \bar{\sigma}^2 - \frac{1}{Nn}\bar{\sigma}^2 = \overline{\sigma(E)\sigma(E+\varepsilon)} - \bar{\sigma}^2 - \frac{1}{n}F_{th}(0)\end{aligned} \tag{19a}$$

where $F_{th}(0)$ is the value we would obtain for the absolute autocorrelation function for $\varepsilon = 0$, if it had been calculated taking into account an ensemble of values of the cross-section.

Comparing expressions (4) and (19a), we notice that $\overline{F(0)}$ and $\overline{F(\varepsilon)}$ differ from the corresponding non biased values for the same quantity independent from ε. If ΔF is this quantity, we have

$$\overline{F(0)} = F_{th}(0) - \Delta F \tag{19b}$$

$$\overline{F(\varepsilon)} = F_{th}(\varepsilon) - \Delta F \tag{19c}$$

$$F_{th}(\Gamma_t) = (1/2)F_{th}(0) \tag{19d}$$

and, for the definition given for Γ:

$$\overline{F(\Gamma)} = \frac{1}{2}\overline{F(0)} \tag{19e}$$

Taking into account (19c), (19e) becomes

$$F_{th}(\Gamma) - \Delta F = (1/2)\overline{F(0)}$$

lastly, taking into account (19b) we obtain :

$$F_{th}(\Gamma) = \frac{1}{2}\overline{F(0)} + \Delta F = \frac{1}{2} F_{th}(0) + \frac{\Delta F}{2} \tag{20}$$

We observe that

$$\Gamma = \Gamma_t - \Delta\Gamma \tag{21}$$

So that taking into account (19d), (20) becomes

$$F_{th}(\Gamma_t - \Delta\Gamma) = F_{th}(\Gamma_t) + (1/2)\Delta F$$

or

$$F_{th}(\Gamma_t - \Delta\Gamma) - F_{th}(\Gamma_t) = (1/2)\Delta F \tag{22}$$

but

$$F_{th}(\Gamma_t - \Delta\Gamma) - F_{th}(\Gamma_t) \backsimeq \frac{\Delta\Gamma}{2\Gamma_t} F_{th}(0) \tag{23}$$

so that (22) becomes

$$\frac{\Delta\Gamma}{2\Gamma_t} F_{th}(0) \backsimeq \frac{1}{2}\Delta F \qquad \text{or} \qquad \frac{\Delta\Gamma}{\Gamma_t} = \frac{\Delta F}{F_{th}(0)} \tag{24}$$

Remembering that $\frac{\Delta F}{F_{th}(0)} = \frac{1}{n}$ and the expression (21), we have :

$$\Gamma \backsimeq \Gamma_t - \frac{1}{n}\Gamma_t = \frac{n-1}{n}\Gamma_t \ .$$

When we consider the errors which affect Γ, the proof goes exactly in the same way, but we must notice that ΔF is the error of F(0), Γ_t, $\overline{F}$ and F_{th} become $\frac{(n-1)}{n}\Gamma_t$, F and $\overline{F}$; the analogous of expression(19c) is valid only assuming a complete correlation between points less than Γ apart. With this approximation we obtain the expression (17).

EVALUATION OF THE ERRORS WHEN A NON STATISTICAL EFFECT IS PRESENT AND THE AVERAGE CROSS SECTION IS CONSTANT. -

Let now consider the case in which a non statistical effect contributes to the considered reaction, but the average cross-section is constant.

Let σ_c be the statistical cross-section, σ_{DI} the cross-section of that part of non statistical effect proceeding through the N channels equally contributing in an incoherent way to the statistical reaction, σ^t_{DI} be the total non statistical cross-section. Both σ_{DI} and σ^t_{DI} are supposed to be energy independent.

Assuming that only N channels contribute to the statistical reaction, we have approximately

$$\sigma \cong \sum_{1}^{N}{}_i |S_i|^2 + \sigma^t_{DI} - \sigma_{DI} \cong$$
$$= \sum_{1}^{N}{}_i \left| ReS_{C_i} + iImS_{C_i} + ReS_{DI_i} + iImS_{DI_i} \right|^2 + \sigma^t_{DI} - \sigma_{DI} \tag{25}$$

It has been assumed that the scattering amplitude S_i may be separated in an average part S_{DI} and a fluctuating part S_C (11, 12).

We obtain the following relations :

$$\sigma = \sigma_C + \sigma_{DI} + \mathcal{J} + \sigma^t_{DI} - \sigma_{DI} = \sigma_C + \sigma^t_{DI} + \mathcal{J} \tag{26a}$$

$$\sigma_C = \sum_{1}^{N}{}_i (Re^2 S_{Ci} + Im^2 S_{Ci}) \tag{26b}$$

$$\sigma_{DI} = \sum_{1}^{N}{}_i (Re^2 S_{DI_i} + Im^2 S_{DI_i}) \tag{26c}$$

$$\mathcal{J} = 2\sum_{1}^{N}{}_i (ReS_{C_i}\, ReS_{DI_i} + ImS_{C_i}\, ImS_{DI_i}) \tag{26d}$$

$$\overline{Re^2 S_C} = \overline{Im^2 S_C} = \overline{\sigma_C}/2N \tag{26e}$$

$$\overline{\sigma} = \overline{\sigma}_C + \sigma^t_{DI} \tag{26f}$$

a) Biased average value of $F(0)_{CN+DI}$.

$F(0)_{CN+DI}$ is so defined

$$F(0)_{CN+DI} = \langle (\sigma - \langle \sigma \rangle)^2 \rangle$$

It is easy to show that

$$F(0)_{CN+DI} = \langle (\sigma_C - \langle \sigma_C \rangle)^2 \rangle + 2\langle (\sigma_C - \langle \sigma_C \rangle)(\mathcal{J} - \langle \mathcal{J} \rangle) \rangle + $$
$$+ \langle \mathcal{J}^2 \rangle - \langle \mathcal{J} \rangle^2 \tag{27a}$$

when the number of independent points is not too small, we have

$$\langle (\sigma_C - \langle \sigma_C \rangle)(\gamma - \langle \gamma \rangle) \rangle \cong 0$$

Remembering expression (2) we obtain

$$F(0)_{CN+DI} \cong F(0)_{CN} + \langle \gamma^2 \rangle - \langle \gamma \rangle^2 \tag{27b}$$

$$\overline{F(0)}_{CN+DI} \cong \overline{F(0)}_{CN} + \overline{\gamma^2} - \overline{\langle \gamma \rangle^2}$$

$\overline{F(0)}_{CN}$ is given by (4);

$$\overline{\gamma^2} = 2 \frac{\overline{\sigma}_C \, \sigma_{DI}}{N} \tag{28a}$$

$$\overline{\langle \gamma \rangle^2} = \overline{\left\{ 2 \sum_1^N {}_i (\langle ReS_{C_i} \rangle ReS_{DI_i} + \langle ImS_{C_i} \rangle ImS_{DI_i}) \right\}^2} =$$

$$= 4 \sum_1^N {}_i (\overline{\langle ReS_{C_i} \rangle^2} \, Re^2 S_{DI_i} + \overline{\langle ImS_{C_i} \rangle^2} \, Im^2 S_{DI_i})$$

but

$$\overline{\langle ReS_{C_i} \rangle^2} = var \langle ReS_{C_i} \rangle = \frac{1}{n} \overline{Re^2 S_{C_i}}$$

so that we obtain

$$\overline{\langle \gamma \rangle^2} = \frac{4}{n} \frac{\overline{\sigma}_C \, \sigma_{DI}}{2N} = \frac{2}{n} \frac{\overline{\sigma}_C \, \sigma_{DI}}{N} \tag{28b}$$

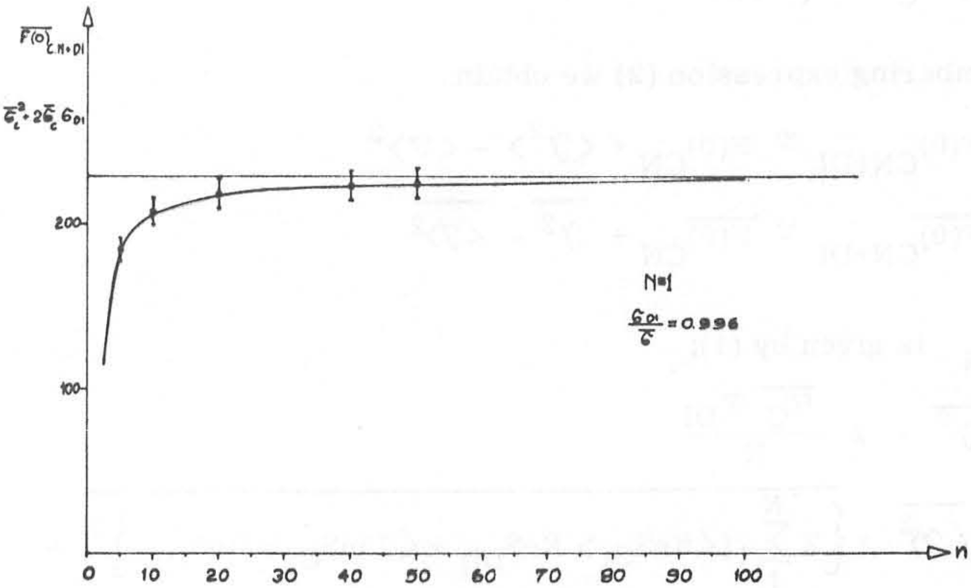
Taking into account (4), (28a), (28b), we get

$$\begin{aligned} \overline{F(0)}_{CN+DI} &= \frac{n-1}{n} \frac{\overline{\sigma}_C^2}{N} + \frac{n-1}{n} 2 \frac{\overline{\sigma}_C \, \sigma_{DI}}{N} = \\ &= \frac{n-1}{n} \left\{ \frac{\overline{\sigma}_C^2}{N} + 2 \frac{\overline{\sigma}_C \, \sigma_{DI}}{N} \right\} \end{aligned} \tag{29}$$

$\frac{\overline{\sigma}_C^2}{N} + 2 \frac{\overline{\sigma}_C \, \sigma_{DI}}{N}$ is the value we would obtain for $F(0)_{CN+DI}$ if an ensemble of values of the cross-section had been considered in deducing it. In fig. 5 the theoretical evaluation (29) (full curve) is compared with the results of a Monte Carlo calculation in the cases $\sigma_{DI}/\overline{\sigma} = 0.51$, and $\sigma_{DI}/\overline{\sigma} = 0.996$ when $N = 1$ and $\sigma^t_{DI} = \sigma_{DI}$.

b) Error affecting the biased value $F(0)_{CN+DI}$

As the term $\langle \gamma \rangle^2$ will give only a minor contribution to var $F(0)_{CN+DI}$, in order to semplify the calculation, it has been dropped.



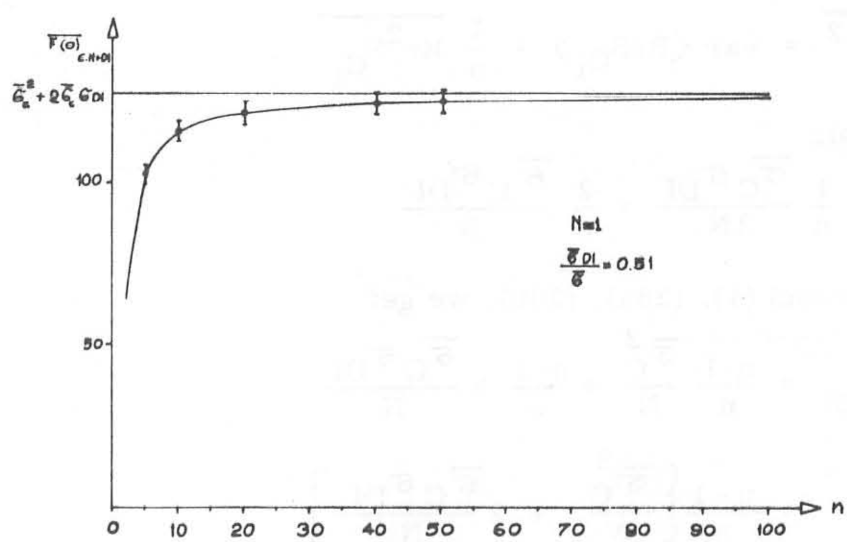


Fig. 5 - Comparison of the quantity $F(0)_{CN+DI}$ given by (29) with the results of a Monte Carlo calculation in the case $\sigma_{DI}/\bar{\sigma} = 0.51$, $\sigma_{DI}/\bar{\sigma} = 0.996$, when $N = 1$ and $\sigma^t_{DI} = \sigma_{DI}$.

We have

$$\mathrm{var}\, F(0)_{CN+DI} \simeq \mathrm{var}\, F_{CN}(0) + \mathrm{var}\langle \mathfrak{J}^2\rangle + \overline{+\, 2(\langle \mathfrak{J}^2\rangle - \overline{\langle \mathfrak{J}^2\rangle})(F(0)_{CN} - \overline{F(0)}_{CN})}$$

The term $\mathrm{var}\, F(0)_{CN}$ is given by (12),

$$\mathrm{var}\langle \mathfrak{J}^2\rangle = 16\,\sigma^2_{DI}\, \mathrm{var}\langle \mathrm{Re}^2 S_C\rangle = \frac{8\sigma^2_{DI}\,\bar{\sigma}^2_C}{nN^2} \tag{30}$$

$$\overline{2(\langle \mathfrak{J}^2\rangle - \overline{\langle \mathfrak{J}^2\rangle})(F(0)_{CN} - \overline{F(0)}_{CN})} =$$

$$= 2\,\overline{\left\{4\,\sigma_{DI}(\langle \mathrm{Re}^2 S_C\rangle - \overline{\mathrm{Re}^2 S_C})\left[2N(\langle \mathrm{Re}^4 S_C\rangle - \langle \mathrm{Re}^2 S_C\rangle^2) - \left(\frac{n-1}{n}\right)\frac{\bar{\sigma}^2_C}{N}\right]\right\}} = 16\,\frac{\sigma_{DI}\,\bar{\sigma}^3_C}{n^2N^2}\,(n-1) \tag{31}$$

Lastly, taking into account the expressions (12), (30) and (31), we have

$$\mathrm{var}\, F(0)_{CN+DI} \simeq \frac{\bar{\sigma}^4_C}{N^2}\left\{\frac{2N+6}{nN} - \frac{2N+12}{n^2 N} + \frac{6}{n^3 N}\right\} + 8\,\frac{\sigma^2_{DI}\,\bar{\sigma}^2_C}{n\,N^2} + 16\,\frac{\sigma_{DI}\,\bar{\sigma}^3_C}{n^2N^2}(n-1) \tag{32}$$

The results given by (29) and (32) have been obtained without assuming an equal contribution of non statistical effect to the different channels.

For the coherence energy Γ we have an expression like (17)

$$\Gamma \simeq \frac{n-1}{n}\,\Gamma_t\left(1 \pm \frac{\sqrt{\mathrm{var}\, F(0)_{CN+DI}}}{\overline{F(0)}_{CN+DI}}\right) \tag{33}$$

We notice that, for a fixed value of N, as the percentage of non statistical contribution increases, the ratio

$$\frac{\sqrt{\mathrm{var}\, F(0)_{CN+DI}}}{\overline{F(0)}_{CN+DI}}$$

becomes smaller and smaller.

Expression (32) agrees very well with the evaluations given by Monte Carlo calculations.

EXTENSION OF THE PRECEEDING RESULTS TO THE CASE OF A VARYING AVERAGE CROSS-SECTION.

In the case of varying average cross-sections,the results we give are only an extrapolation of the results obtained so far, but we believe that, in most cases, they give a realistic estimation of the true values of the bias and the errors.

This is the case when N, $\overline{\sigma}_C$, σ_{DI}, σ^t_{DI}, have a much slower variation with the energy than the fluctuating cross-section.

In this case it is possible to reproduce the shape of the average cross-section using a moving averaging interval[1, 3].

Putting the varying average cross-section in the definitions of $F(\varepsilon)$ and $C(\varepsilon)$ it is possible to obtain some information on the same quantities which have been studied in the case of a constant average. In this case, the values of F(0) and C(0) are about the same one could obtain substituting to the true excitation function a new "equivalent" excitation function whose points are distributed around a step average, the width of the steps being equal to the moving averaging interval.

This is a consequence of the approximation that the mean value and the variance of the square of the difference between the points of the excitation function and the average cross-section, are about the same in the moving averaging interval.

Let Δ be the total energy interval of the excitation function with extreme values E_1 and E_2 , Δ_1 the moving averaging interval, $n_1 = (\Delta_1/\pi\Gamma) + 1$ and $n_2 = \Delta/\Delta_1$. In the new excitation function the energy interval Δ may be thought of as devided into n_2 sub-intervals Δ_1.

In the case of purely statistical reactions the approximate expressions of the biased values of F(0) and C(0) and of the errors are:

$$\overline{F(0)} \simeq \frac{1}{n_2}\left[\sum_{1}^{n_2}{}_k \frac{\overline{\sigma}^2_C(E_k)}{N(E_k)}\right]\frac{n_1 - 1}{n_1} \tag{34}$$

$$\overline{C(0)} \simeq \frac{1}{n_2}\left[\sum_{1}^{n_2}{}_k \frac{1}{N(E_k)}\,\frac{n_1 N(E_k) - N(E_k)}{n_1 N(E_k) + 1}\right] \simeq \frac{1}{N^*}\,\frac{n_1 N^* - N^*}{n_1 N^* + 1} \tag{35}$$

being

$$E_k = E_1 + (2k-1)\frac{\Delta_1}{2} \qquad \text{and} \qquad \frac{1}{N^*} = \frac{1}{n_2}\sum_{1}^{n_2}{}_k \frac{1}{N(E_k)}$$

$$\operatorname{var} F(0) \simeq \overline{F(0)}^2 \left(\frac{n_1}{n_1 - 1}\right)^2 \left\{ \frac{2N^* + 6}{n_1 n_2 N^*} - \frac{2N^* + 12}{n_1^2 n_2 N^*} + \frac{6}{n_1^3 n_2 N^*} \right\} \tag{36}$$

$$\mathrm{var}\,C(0) \cong \overline{C(0)}^2 \left\{ \frac{2n_1^2 N^{*3}(N^*+1)(n_1-1)}{n_2(n_1N^*+2)(n_1N^*+3)(n_1N^*-N^*)^2} \right\} \tag{37}$$

The value of Γ^2 obtained by the absolute autocorrelation function $F(\varepsilon)$ is given by a weighted average of the values Γ_k^2 corresponding to each sub-interval Δ_1 (see expression (5) of ref. (8)). Each biased value Γ_k is reduced of a factor $(n_1-1)/n_1$, also the biased value Γ will be reduced of the same factor.

The error $\Delta\Gamma$ of Γ is given approximately by $\Delta\Gamma/\Gamma \sim \sqrt{\mathrm{var}\,F(0)}/\overline{F(0)}$. When a non statistical effect contributes to the examined reaction, we have approximately

$$\overline{F(0)}_{CN+DI} \cong \frac{1}{n_2}\left[\sum_{1}^{n_2}{}_k \frac{\overline{\sigma}_C^2(E_k) + 2\,\overline{\sigma}_C(E_k)\,\sigma_{DI}(E_k)}{N(E_k)} \right] \frac{n_1 - 1}{n_1} \tag{38}$$

$$\mathrm{var}\,F(0)_{CN+DI} \cong \overline{F(0)}^2_{CN} \left(\frac{n_1}{n_1-1}\right)^2 \left\{ \frac{2N^*+6}{n_1 n_2 N^*} - \frac{2N^*+12}{n_1^2 n_2 N^*} + \frac{6}{n_1^3 n_2 N^*} \right\} +$$

$$+\ 8\,\frac{\sigma^2_{DI}(\overline{E})\,\overline{\sigma}^2_C(\overline{E})}{n_1 n_2\, N^{*2}} + 16\,\frac{\sigma_{DI}(\overline{E})\,\overline{\sigma}^3_C(\overline{E})}{n_1^2\, n_2\, N^{*2}}(n_1-1) \tag{39}$$

with $\overline{E} = 1/2\,(E_1+E_2)$.

The value of Γ obtained by means of $F(\varepsilon)$ is again reduced by the same factor $(n_1-1)/n_1$; the error on Γ is given approximately by

$$\Delta\Gamma/\Gamma \cong \sqrt{\mathrm{var}\,F(0)_{CN+DI}}\,/\,\overline{F(0)}_{CN+DI}\ .$$

All the expressions given for the case of a varying average cross-section, are only first approximations: caution must be used in applying them in view of the assumptions which have been made in their derivation.

REFERENCES. -

(1) - G. Pappalardo, Phys. Letters, 13, 320 (1964).
(2) - E. Gadioli, Phys. Letters 16, 288 (1965).
(3) - E. Gadioli, I. Iori, A. Marini, Nuovo Cimento 39, 996 (1965).
(4) - P. J. Dallimore, I. Hall, Phys. Letters 18, 138 (1965).
(5) - W. R. Gibbs, Los Alamos Sc. Lab., Report LA 3266 (1965).
(6) - T. Ericson, Phys. Letters 4, 258 (1963).
(7) - R. O. Stephen, Clarendon Lab. Report, Oxford 1963.
(8) - E. Gadioli, I. Iori, A. Marini, M. Sansoni, Report INFN/BE-66/3 (1966) and to be published on Il Nuovo Cimento.
(9) - M. G. Kendall, The advanced theory of statistics (C. Griffin e C. Ldt., London, 1946).
(10) - D. M. Brink, R. O. Stephen, Phys. Letters 8, 324 (1964).
(11) - T. Ericson, Ann. Phys. 23, 590 (1963).
(12) - D. M. Brink, R. O. Stephen, Phys. Letters 5, 77 (1963).