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INTRODUCTION OF A QUANTUM OF TIME ("CHRONON") AND ITS CONSEQUENCES FOR QUANTUM MECHANICS(†)

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Abstract — We discuss the consequences of the introduction of a quantum of time τ_0 in the formalism of non-relativistic quantum mechanics, by referring ourselves in particular to the theory of the *chronon* as proposed by P.Caldirola. Such an interesting "finite difference" theory, forwards —at the classical level— a solution for the motion of a particle endowed with a non-negligible charge in an external electromagnetic field, overcoming all the known difficulties met by Abraham–Lorentz's and Dirac's approaches (and even allowing a clear answer to the question whether a free falling charged particle does or does not emit radiation), and —at the quantum level— yields a remarkable mass spectrum for leptons.

After having briefly reviewed Caldirola's approach, our first aim is to work out, discuss, and compare one another the *new* representations of Quantum Mechanics (QM) resulting from it, in the Schrödinger, Heisenberg and density–operator (Liouville–von Neumann) pictures, respectively.

Moreover, for each representation, three (*retarded, symmetric* and *advanced*) *formulations* are possible, which refer either to times t and $t - \tau_0$, or to times $t - \tau_0/2$ and $t + \tau_0/2$, or to times t and $t + \tau_0$, respectively. It is interesting to notice that, when the chronon tends to zero, the ordinary QM is obtained as the limiting case of the "symmetric" formulation only; while the "retarded" one does naturally appear to describe QM with friction, i.e., to describe *dissipative* quantum systems (like a particle

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moving in an absorbing medium). In this sense, *discretized* QM is much richer than the ordinary one.

We also obtain the (retarded) finite–difference Schrödinger equation within the Feynman path integral approach, and study some of its relevant solutions. We then derive the time–evolution operators of this discrete theory, and use them to get the finite–difference Heisenberg equations.

When discussing the mutual compatibility of the various pictures listed above, we find that they *can* be written down in a form such that they result to be equivalent (so as it happens in the "continuous" case of ordinary QM), even if the Heisenberg picture cannot be derived by "discretizing" directly the ordinary Heisenberg representation.

Afterwards, some typical applications and examples are studied, as the free particle, the harmonic oscillator and the hydrogen atom; and various cases are pointed out, for which the predictions of discrete QM differ from those expected from "continuous" QM.

At last, the density matrix formalism is applied to the solution of the *measurement problem* in QM, with very interesting results, so as a natural explication of "decoherence", which reveal the power of dicretized (in particular, *retarded*) QM.

1 Introduction

The idea of a discrete temporal evolution is not a new one and, as almost all the physical ideas, has from time to time been recovered from oblivion.* For instance, in classical Greece this idea came to light as part of the atomistic thought. In Middle Age, belief in the discontinuous character of time was at the basis of the "theistic atomism" held by the Arabic thinkers of the Kalām[2]. In Europe, discussions about the discreteness of space and time can be found for example in the writings of Isidore of Sevilla, Nicolaus Boneti and Henry of Harclay, who discussed the nature of continuum. In more recent times, the idea of the existence of a fundamental interval of time was rejected by Leibniz, since it was incompatible with his rationalistic philosophy. Within modern physics, however, Planck's famous work on black body radiation inspired a new view of the subject. In fact, the introduction of the quanta opened a wide range of new scientific possibilities regarding the way the physical world can be conceived. Including considerations, like those in the present paper, on the "discretization" of time within the framework of Quantum Mechanics (QM).

In the early years of our century, Mach regarded the concept of continuum to be a consequence of our physiological limitations: <<...le temps et l'espace ne représentent, au point de vue physiologique, qu'un continue apparent, qu'ils se composent très vraisemblablement d'elements discontinus, mais qu'on ne peut distinguer nettement les uns des autres.>>[3] Also Poincaré took into consideration the posible existence of what he called an "atom of time": the minimum amount of time which allows to distinguish between two states of a system.[4] Finally, in the twenties, J.J.Thomson[5] suggested the electric force to act in a discontinuous way, producing finite increments of momentum separated by finite intervals of time. Such a seminal work has ever since inspired a series of papers on the existence of a fundamental interval of time, named *chronon*; even if the repercussions of all that work was small, at that time. A further seminal article was the one by Ambarzumian and Ivanenko[6], appeared in 1930, which assumed space–time as being discrete and also stimulated a large number of subsequent papers.

It is important to stress that, in principle, time discretization can be introduced in two distinct (and completely different) ways:

*Historical aspects related to the introduction of a fundamental interval of time in Physics can be found in F.Casagrande, ref.[1].

(1) by attributing to time a discrete structure, i.e., by regarding time not as a continuum, but as a one-dimensional "lattice";

(2) by considering time as a continuum, in which events can take place (discontinuously) only at discrete instants of time.

Almost all the attempts to introduce a discretization of time followed the first way, generally as part of a more extended procedure in which the space-time as a whole is considered as intrinsically discrete (four-dimensional lattice). Recently, also T.D.Lee[7] introduced a time discretization on the basis of the *finite* number of experimental measurements performable in any finite interval of time. For an early approach in this direction, see e.g. T.Tati[8] and references therein.

The second approach was first adopted in the twenties (e.g., by Levi[61] and by Pokrowski[61]), after Thomson's work, and resulted in the first real example of a theory based on the existence of a fundamental interval of time: the one set forth by Caldirola, in the fifties[10]. Namely, Caldirola formulated a theory for the *classical* electron, with the aim of providing a consistent (classical) theory for its motion in an electromagnetic field. In the late seventies, Caldirola extended its procedure to non-relativistic quantum mechanics.

It is known that the classical theory of the electron in an electromagnetic field (despite the efforts by Abraham,[12] Lorentz,[13] Poincaré[14] and Dirac,[15] among others) actually presents many serious problems; *except —of course— when the field of the particle is neglected.*† By replacing Dirac's differential equation by two *finite–difference equations*, Caldirola developed a theory in which the main difficulties of Dirac's theory were overcome. As we shall see, in his relativistically invariant formalism the chronon characterizes the changes suffered by the dynamical state of the electron when it is submitted to external forces. So that the electron will be regarded as an (extended-like) object, which is pointlike only at discrete positions x_n (along its trajectory) such that the electron takes a quantum of proper time to travel from one position to the following one [or, rather, two chronons: see the following]. It is tempting to examine

†It is interesting to note that all those problems have been —necessarily— tackled by Yaghjian in its recent book,[16] when he faced the question of the relativistic motion of a charged, macroscopic sphere in an external electromagnetic field.

extensively the generalization of such a theory to the quantum domain; and this will be performed in the present work. Let us recall that one of the most interesting aspects of the *discretized* Schrödinger equations is that the mass of the *muon* and of the *tau* lepton followed as corresponding to the two levels of the first (degenerate) excited state of the electron.

In conventional QM there is a perfect equivalence among its various pictures: Schrödinger's, Heisenberg's, density matrix's . When discretizing the evolution equations, we shall succeed in writing down those pictures in a form such that they result to be still equivalent. However, in order to be compatible with the Schrödinger representation, our Heisenberg equations *cannot* in general be obtained by a *direct* discretization of the continuous Heisenberg equation.

The plan of this work is the following. In Chapt.2 we present a brief review of the main classical theories of the electron, including Caldirola's. In Chapt.3 we introduce the three discretized *forms* ("retarded", "advanced", "symmetrical") of the Schrödinger equation, analyze the main characteristics of such formulations, and derive the retarded one from Feynman's path integral approach. In Chapt.4, our discrete theory is applied to some simple quantum systems, such as the harmonic oscillator, the free particle and the hydrogen atom. The possible experimental deviations from the predictions of ordinary QM are investigated. In Chapt.5, a new derivation is presented of the discretized Liouville–von Neumann equation, starting from the *coarse grained* hypothesis. Such a representation is then adopted for tackling the *measurement problem* in QM, with rather interesting results. Finally, a discussion on the possible interpretation of our discretized equations is forwarded in Chapt.6.

2 The Introduction of the Chronon in the Classical Theory of the Electron

Almost a century after its discovery, the electron continues to be an object waiting for a convincing description (cf., e.g., [17]), both in classical and quantum electrodynamics. As Schroedinger put it, the electron is —still— a stranger in electrodynamics. Maxwell's electromagnetism is a field theoretical approach in which no reference is made to the existence of material corpuscles. Thus, one may say that one of the most controversial questions of the twentieth century physics, the wave–particle paradox, is not characteristic of QM only. In the electron classical theory, matching the description

of the electromagnetic fields (obeying Maxwell equations) with the existence of charge carriers like the electron is still a challenging task.

The hypothesis that electric currents could be associated with charge carriers was already present in the early "particle electrodynamics" formulated in 1846 by Fechner and Weber. But such an idea was taken into considerations again only a few decades later, in 1881, by Helmholtz. Up to that time, electrodynamics had been developed on the hypothesis of an electromagnetic *continuum*[18] and of an *ether*. In that same year, J.J.Thomson wrote his seminal paper in which the electron mass was regarded as purely electromagnetic in nature. Namely, the energy and momentum associated with the (electromagnetic) fields produced by an electron were held entirely responsible for the energy and momentum of the electron itself.[19]

Lorentz's electrodynamics, which described the particle–particle interaction via electromagnetic fields by the famous force law

$$\mathbf{f} = \rho \left(\mathbf{E} + \frac{\mathbf{1}}{\mathbf{c}} \mathbf{v} \wedge \mathbf{B} \right) , \tag{1}$$

ρ being the charge density of the particle on which the fields act, dates back to the beginning of the 1890 decade. The electron was finally discovered by Thomson in 1897, and in the following years various theories appeared. The famous (pre-relativistic) theories by Abraham, Lorentz and Poincaré regarded it as an extended–type object, endowed again with a purely electromagnetic mass. As wellknown, in 1903 Abraham proposed the simple-minded (and questionable) model of a rigid sphere, with a uniform electric charge density on its surface. The theory of Lorentz (1904) was quite similar, trying to improve the situation with the mere introduction of the effects resulting from the Lorentz–FitzGerald contraction.

2.1 The theory of the electron by Abraham–Lorentz

A main difficulty for an accurate description of the electron motion was the inclusion of the radiation reaction, i.e., of the effect produced on such a motion by the fields radiated by the particle itself. In the model proposed by Abraham–Lorentz the assumption of a purely electromagnetic structure for the electron implied that

$$\mathbf{F}_{\mathrm{p}} + \mathbf{F}_{\mathrm{ext}} = \mathbf{0} , \tag{2}$$

where $\mathbf{F}_{\mathrm{p}}$ is the self-force due to the self-fields of the particle, and $\mathbf{F}_{\mathrm{ext}}$ is the external force. According to Lorentz's law, the self-force was given by

$$\mathbf{F}_{\mathrm{p}} = \int \rho \left(\mathbf{E}_{\mathrm{p}} + \frac{1}{\mathbf{c}} \mathbf{v} \wedge \mathbf{B}_{\mathrm{p}} \right) \mathrm{d}^3\mathbf{r} \ ,$$

where $\mathbf{E}_{\mathrm{p}}$ and $\mathbf{B}_{\mathrm{p}}$ are the fields produced by the charge density ρ itself, according to the Maxwell–Lorentz equations. For the radiation reaction force, Lorentz obtained the following expression:

$$\mathbf{F}_{\mathrm{p}} = -\frac{4}{3c^2} W_{\mathrm{el}} \mathbf{a} + \frac{2}{3} \frac{ke^2}{c^3} \dot{\mathbf{a}} - \frac{2e^2}{3c^3} \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \frac{1}{c^n} \frac{\mathrm{d}^n \mathbf{a}}{\mathrm{d}t^n} \mathrm{O}(R^{n-1}) \ , \tag{3}$$

where $k \equiv (4\pi\epsilon_0)^{-1}$ [in the following, whenever convenient, we shall assume units such that numerically $k = 1$], and where

$$W_{\mathrm{el}} \equiv \frac{1}{2} \int \int \frac{\rho(\mathbf{r})\, \rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}^3\mathbf{r}\, \mathrm{d}^3\mathbf{r}'$$

is the electrostatic self-energy of the considered charge distribution, and R is the radius of the electron. All the terms in the sum are structure dependent; i.e., they depend on R and on the charge distribution. By identifying the electromagnetic mass of the particle with its electrostatic self-energy

$$m_{\mathrm{el}} = \frac{W_{\mathrm{el}}}{c^2} \ ,$$

it was possible to write eq.(2) as

$$\frac{4}{3} m_{\mathrm{el}} \dot{\mathbf{v}} - \boldsymbol{\Gamma} = \mathbf{F}_{\mathrm{ext}} \ , \tag{4}$$

so that one got:

$$\boldsymbol{\Gamma} = \frac{2}{3} \frac{e^2}{c^3} \dot{\mathbf{a}} \left(1 + \mathrm{O}(R)\right) \ , \tag{5}$$

which was the equation of motion in the Abraham–Lorentz model. Quantity $\boldsymbol{\Gamma}$ is the radiation reaction force, namely, the reaction force acting on the electron. A problem with equation (4) was constituted by the factor $\frac{4}{3}$. In fact, if the mass is supposed to be of electromagnetic origin only, then the total momentum of the electron would be given by

$$\mathbf{p} = \frac{4}{3} \frac{W_{\mathrm{el}}}{c^2} \mathbf{v} \ , \tag{6}$$

which is *not* invariant under Lorentz transformations. That model, therefore, was non-relativistic. Finally, we can observe from equation (3) that the structure dependent terms are functions of higher derivatives of the acceleration; even more, the resulting differential equation is *of the third order*, so that initial position and initial velocity are not enough to single out a solution. In order to suppress the structure terms, one ought to reduce the electron to a point, $(R \to 0)$, but in this case the self-energy $W_{\rm el}$ and mass $m_{\rm el}$ would diverge!

After the emergence of the special theory of relativity or, rather, after the publication by Lorentz in 1904 of his famous transformations, some attempts were made to adapt the model to the new requirements.[20, 21, 22] Abraham himself (1905) succeeded in deriving the following generalization of the radiation reaction term (5):

$$\Gamma_\mu = \frac{2}{3}\frac{e^2}{c}\left(\frac{\mathrm{d}^2 u_\mu}{\mathrm{d}s^2} + \frac{u_\mu u^\nu}{c^2}\frac{\mathrm{d}^2 u_\nu}{\mathrm{d}s^2}\right). \tag{7}$$

A solution for the problem of the electron momentum non-covariance was proposed by Poincaré in 1905, by the addition of cohesive forces of non-electromagnetic character, which —however— made the electron no longer purely electromagnetic in nature.

On the other hand, electrons could not be considered point-like, due to the obvious divergence of their energy when $R \to 0$; thus, a description of the electron motion could not forget about the structure terms. Only Fermi succeeded to show, later, that the correct relation for the momentum of a purely electromagnetic electron could be obtained without Poincaré's cohesive forces.[23]

2.2 Dirac's theory of the classical electron

Notwithstanding its inconsistencies, the theory by Abraham–Lorentz was the most accepted theory of the electron, until the publication of Dirac's theory in 1938. During the long period in between, as well as afterwards, however, various further attempts to solve the problem were set forth, either by means of extended-type models (Mie, Page, Schott, etc.[24]), or by trying again to treat the electron as a point-like particle (Fokker, Wentzel, etc.[25]).

Dirac's approach[15] is the best known attempt to describe the classical electron. It by-passed the critical problem of the previous theories of Abraham and Lorentz by working out a for the point-like electron a trick which *avoided* divergences! By using the conservation laws of energy and momentum, and Maxwell equations, Dirac calculated the flux of the energy–momentum 4-vector through a tube of radius $\epsilon \ll R$ (quantity

R being the radius of the electron at rest) surrounding the world line of the particle, and obtained:

$$m\frac{\mathrm{d}u_\mu}{\mathrm{d}s} = F_\mu + \Gamma_\mu \ , \tag{8}$$

where Γ_μ is the Abraham 4-vector (7), that is, the reaction force acting on the electron itself; and F_μ is the 4-vector that represents the external field acting on the particle:

$$F_\mu = \frac{e}{c}F_{\mu\nu}u^\nu \ . \tag{9}$$

According to such a model, the rest mass m_0 of the electron is the limiting, finite value got as the difference of two quantities tending to infinity when $R \to 0$:

$$m_0 = \lim_{\epsilon\to 0}\left(\frac{1}{2}\frac{e^2}{c^2\epsilon} - k(\epsilon)\right) ,$$

the procedure followed by Dirac being an early example of elimination of divergences by means of a subtractive method.
At the non-relativistic limit, Dirac's equation goes into the one previously obtained by Abraham–Lorentz:

$$m_0\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} - \frac{2}{3}\frac{e^2}{c^3}\frac{\mathrm{d}^2\mathbf{v}}{\mathrm{d}t^2} = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\wedge\mathbf{B}\right) \ , \tag{10}$$

except for the fact that in Abraham–Lorentz's approach m_0 diverged. The last equation shows that the reaction force equals $\frac{2}{3}/;\frac{e^2}{c^3}\ \frac{\mathrm{d}^2\mathbf{v}}{rmdt^2}$.

Dirac's dynamical equation (8) was later reobtained also from different, improved models.[26] Wheeler and Feynman, for example, rederived eq.(8) by basing electromagnetism on an action principle applied to particles only, via their absorber hypothesis.[27] However, also eq.(8) does present many troubles, related to the infinite many non-physical solutions it admits. Actually, as already mentioned, it is a third–order differential equation, requiring three initial conditions for singling out a solution of its. In the description of a *free* electron, e.g., it yields also "self-accelerating" solutions (*runaway solutions*),[28] for which velocity and acceleration increase spontaneously and indefinitely.[25] Moreover, for an electron submitted to an electromagnetic pulse, further non-physical solutions appear, related this time to *pre-accelerations*:[29] If the electron comes from infinity with a uniform velocity v_0 and, at a certain instant of time t_0, is submitted to an electromagnetic pulse, then it starts accelerating *before* t_0.

Drawbacks like these do motivate further attempts to find out a coherent model for the classical electron.

2.3 Caldirola's theory for the classical electron

Among the various attempts to formulate a more satisfactory theory, we want to focus our attention on the one proposed by P.Caldirola. So as Dirac's, Caldirola's theory is also Lorentz invariant. Continuity, in fact, is not an assumption required by Lorentz invariance.[30] The theory postulates the existence of a universal interval τ_0 of *proper* time, even if time flows continuously as in the ordinary theory. When an external force acts on the electron, however, the reaction of the particle to the applied force is not continuous: The value of the electron velocity u_μ is supposed to jump from $u_\mu(\tau - \tau_0)$ to $u_\mu(\tau)$ only at certain positions s_n along its world line; these discrete positions being such that the electron takes a time τ_0 for travelling from one position s_{n-1} to the next one s_n.

In this theory[10] the electron in principle is still considered as pointlike, but the Dirac relativistic equations for the classical radiating electron are replaced: (i) by a corresponding *finite-difference* (retarded) equation in the velocity $u^\mu(\tau)$

$$\frac{m_0}{\tau_0}\left\{u_\mu(\tau) - u_\mu(\tau-\tau_0) + \frac{u_\mu(\tau)\,u_\nu(\tau)}{c^2}\left[u_\nu(\tau) - u_\nu(\tau-\tau_0)\right]\right\}^{\cdot} = \\ = \frac{e}{c}F_{\mu\nu}(\tau)\,u_\nu(\tau)\,, \tag{11}$$

which reduces to the Dirac equation (8) when $\tau_0 \to 0$, but cannot be derived from it (in the sense that it cannot be obtained by a simple discretization of the time derivatives appearing in Dirac's original equation); and (ii) by a second equation, connecting this time the "discrete positions" $x^\mu(\tau)$ along the world line of the particle. In fact, the dynamical law above is by itself unable to specify univocally the variables $u_\mu(\tau)$ and $x_\mu(\tau)$ which describe the motion of the particle. Caldirola named it the *transmission law*:

$$x_\mu(n\tau_0) - x_\mu[(n-1)\tau_0] = \frac{\tau_0}{2}\left\{u_\mu(n\tau_0) - u_\mu[(n-1)\tau_0]\right\}, \tag{12}$$

which is valid inside each discrete interval τ_0, and describes the *internal* or *microscopic* motion of the electron.

In these equations, $u^\mu(\tau)$ is the ordinary four-vector velocity satisfying the condition

$$u_\mu(\tau)u^\mu(\tau) = -c^2 \qquad \text{for } \tau = n\tau_0$$

where $n = 0, 1, 2, ...$ and $\mu, \nu = 0, 1, 2, 3$; $F^{\mu\nu}$ is the external (retarded) electromagnetic field tensor, and the quantity

$$\frac{\tau_0}{2} \equiv \theta_0 = \frac{2}{3}\frac{ke^2}{m_0c^3} \simeq 6.266 \times 10^{-24} \text{ s} \tag{13}$$

is defined as the chronon associated with the electron (as it will be justified below). The chronon $\theta_0 = \tau_0/2$ depends on the particle (internal) properties: namely, on its charge e and rest mass m_0.

As a result, the electron happens to appear eventually as an extended–like[31] particle, with internal structure, rather than as a point-like object (as initially assumed). For instance, one may imagine that the particle does not react instantaneously to the action of an external force because of its finite extension (the numerical value of the chronon is of the same order as the time spent by light to travel an electron classical diameter). As already said, eq.(11) describes the motion of an object that happens to be pointlike only at discrete positions s_n along its trajectory;[10] even if both position and velocity are still continuous and well-behaved functions of the parameter τ, they being differentiable functions of τ.

It is essential to notice that a discrete character is given to the electron merely by the introduction of the fundamental quantum of time, without any need of a "model" for the electron. Actually it is wellknown that many difficulties are met not only by the strictly pointlike models, but also by the extended-type particle models ("spheres", "tops", "gyroscopes", etc.). In A.O.Barut's words, <<If a spinning particle is not quite a point particle, nor a solid three dimensional top, what can it be?>>. We deem the answer stays with a third type of models, the "extended-like" ones, as the present theory; or as the (related) theoretical approach[31, 17] in which the center of the pointlike charge is spatially distinct from the particle center-of-mass. Anyway, it is not necessary to recall that the worst troubles met in quantum field theory (e.g., in quantumelectrodynamics), like the presence of divergencies, are due to the pointlike character still attributed to (spinning) particles; since the problem of a suitable model for elementary particles was transported, *unsolved*, from classical to quantum physics. One could say that problem to be the most important in modern particle physics.

Equations (11) and (12) together provide a full description of the motion of the electron. Notice that the global, "macroscopic" motion can be the same for different

solutions of the transmission law. The behaviour of the electron under the action of external electromagnetic fields is completely described by its macroscopic motion.

As in Dirac's case, the equations above are invariant under Lorentz transformations; but, as we are going to see, are *free* from pre-accelerations, self-accelerating solution, and problems (that had raised great debates in the first half of the century) with the hyperlbolic motion.

In the *non-relativistic limit* the previous (retarded) equations reduced to the form

$$\frac{m_0}{\tau_0}\left[\mathbf{v}(t)-\mathbf{v}(t-\tau_0)\right]=e\left[\mathbf{E}(t)+\frac{1}{c}\mathbf{v}(t)\wedge\mathbf{B}(t)\right], \tag{14}$$

$$\mathbf{r}(\mathbf{t})-\mathbf{r}(\mathbf{t}-\tau_0)=\frac{\tau_0}{2}\left[\mathbf{v}(\mathbf{t})-\mathbf{v}(\mathbf{t}-\tau_0)\right], \tag{15}$$

which *can* be obtained —this time– from eq.(10), by directly replacing the time derivatives by the corresponding finite–difference expressions. The macroscopic equation (14) had already been obtained also by other authors[21, 22, 33, 34] for the dynamics of extended–type electrons.

The important point is that eqs.(11),(12), or eqs.(14),(15), allow overcoming the difficulties met with the Dirac classical equation (8). In fact, the electron *macroscopic* motion is completely determined once velocity and initial position are given. Solutions of the relativistic equations (11),(12) for the radiating electron —or of the corresponding non-relativistic equations (14),(15)— were obtained for several problems, the resulting motions never presenting unphysical behaviour; so that the following questions can be regarded as having been solved:[10] A) *Exact relativistic solutions*: 1) Free electron motion; 2) Electron under the action of an electromagnetic pulse;[35] 3) Hyperbolic motion;[32] B) *Non-relativistic approximate solutions*: 1) Electron under the action of time-dependent forces; 2) Electron in a constant, uniform magnetic field;[36] 3) Electron moving along a straight line under the action of an elastic restoring force.[37]

Before going on, let us briefly study the electron radiation properties as deduced from the finite-difference relativistic equations (11), (12), with the aim of showing the advantages of the present formalism with respect to the Abraham–Lorentz–Dirac one. Such equations can be written[32, 10]

$$\frac{\Delta Q_\mu(\tau)}{\tau_0}+R_\mu(\tau)+S_\mu(\tau)\;=\;\frac{e}{c}F_{\mu\nu}(\tau)u^\nu(\tau)\;, \tag{16}$$

where:

$$\Delta Q_\mu \equiv m_0 \left[u_\mu(\tau) - u_\mu(\tau - \tau_0)\right] ; \tag{17}$$

$$R_\mu(\tau) \equiv -\frac{m_0}{2\tau_0} \left\{ \frac{u_\mu(\tau)u_\nu(\tau)}{c^2} \left[u^\nu(\tau+\tau_0) + u^\nu(\tau-\tau_0) - 2u^\nu(\tau)\right] \right\} ; \tag{18}$$

$$S_\mu(\tau) = -\frac{m_0}{2\tau_0} \left\{ \frac{u_\mu(\tau)u_\nu(\tau)}{c^2} \left[u^\nu(\tau+\tau_0) - u^\nu(\tau-\tau_0)\right] \right\} . \tag{19}$$

In eq.(16), the first term $\Delta Q_0/\tau_0$ represents the variation per unit of proper time (in the interval $\tau - \tau_0$ to τ) of the particle energy–momentum vector. The second one, $R_\mu(\tau)$, is a dissipative term, since it contains only even derivatives of the velocity as one can prove by expanding $u^\nu(\tau+\tau_0)$ and $u^\nu(\tau-\tau_0)$ in terms of τ_0; furthermore, it is never negative,[10, 32] and can therefore represent the energy-momentum *radiated* by the electron in the unit of proper time. The third term, $S_\mu(\tau)$, is conservative and represents the rate of change in proper time of the electron *reaction* energy-momentum.

The time component ($\mu = 0$) of eq.(16) writes:

$$\frac{T(\tau) - T(\tau-\tau_0)}{\tau_0} + R_0(\tau) + S_0(\tau) = P^{\text{ext}}(\tau) , \tag{20}$$

quantity $T(\tau)$ being the kinetic energy

$$T(\tau) = m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right) ; \tag{21}$$

so that in eq.(20) the first term replaces the proper-time derivative of the kinetic energy, the second one is the energy radiated by the electron in the unit of proper time, $S_0(\tau)$ is the variation rate in proper time of the electron reaction energy (radiative correction), and $P^{\text{ext}}(\tau)$ is the work done by the external forces in the unit of proper time.

We are now ready to show, e.g., that eq.(20) yields a clear explanation for the origin of the so-called "acceleration energy" (Schott energy), appearing in the energy–conservation relation for the Dirac equation. In fact, by expanding in power series with respect to τ_0 the l.h. sides of eqs.(16),(17), (18),(19) for $\mu = 0$, and keeping only the first-order terms, one gets

$$\frac{T(\tau) - T(\tau-\tau_0)}{\tau_0} \simeq \frac{\mathrm{d}T}{\mathrm{d}\tau} - \frac{2}{3}\frac{e^2}{c^2}\frac{\mathrm{d}a_0}{\mathrm{d}\tau} ; \tag{22}$$

$$R_0(\tau) \simeq \frac{1}{\sqrt{1-\beta^2}} \frac{2}{3} \frac{e^2}{c^3} a_\mu a^\mu \; ; \tag{23}$$

$$S_0(\tau) \simeq 0 \; ; \tag{24}$$

where a^μ is the four-acceleration

$$a^\mu \equiv \frac{drmu^\mu}{drm\tau} = \gamma \frac{drmu^\mu}{drmt}$$

quantity γ being the Lorentz factor. Therefore eq.(20), to the first order in τ_0, becomes:

$$\frac{\mathrm{d}T}{\mathrm{d}\tau} - \frac{2}{3} \frac{e^2}{c^2} \frac{\mathrm{d}a_0}{\mathrm{d}\tau} + \frac{2}{3} \frac{e^2}{c^3} \frac{a_\mu a^\mu}{\sqrt{1-\beta^2}} \simeq P^{\mathrm{ext}}(\tau) \tag{25}$$

or, passing from the proper time τ to the observer's time t:

$$\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{2}{3} \frac{e^2}{c^2} \frac{\mathrm{d}a_0}{\mathrm{d}t} + \frac{2}{3} \frac{e^2}{c} a_\mu a^\mu \simeq P^{\mathrm{ext}}(\tau) \frac{\mathrm{d}\tau}{\mathrm{d}t} \; . \tag{26}$$

Last relation is identical with the energy conservation law found by Fulton and Rohrlich[40] for the Dirac equation. In it, the derivative of $(2e^2/3c^2)a_0$ appears, which is just the *acceleration energy.* Our approach does clearly shows that *it arises only by expanding in a power series of* τ_0 the kinetic energy increment suffered by the electron during the fundamental proper-time interval τ_0; whilst such a Schott energy (as well as higher-order energy terms) does not show up when adopting the full formalism of finite–difference equations. We shall come back to this important point in Subsection 2.4.

Let us finally observe[10] that, when setting

$$\frac{m_0}{ec\tau_0} \left[u_\mu(\tau)u_\nu(\tau-\tau_0) - u_\mu(\tau-\tau_0)u_\nu(\tau)\right] \equiv F^{\mathrm{self}}_{\mu\nu} \; , \tag{27}$$

the relativistic equation of motion (11) writes:

$$e \; overc \left(F^{\mathrm{self}}_{\mu\nu} + F^{\mathrm{ext}}_{\mu\nu}\right) u^\nu = 0 \; , \tag{28}$$

confirming that $F^{\mathrm{self}}_{\mu\nu}$ represents the (retarded) self-field associated with the moving electron.

2.4 The three alternative formulations of Caldirola theory

Two more (alternative) formulations are possible of Caldirola's equations, based on different discretization procedures. In fact, equations (11) and (12) describe an intrinsically radiating particle. And, by expanding the equation (11) in terms of τ_0, a radiation reaction term appears. Caldirola called those equations the *retarded* form of the electron equations of motion.

By rewriting the finite–difference equations, on the contrary, in the form:

$$\frac{m_0}{\tau_0}\left\{u_\mu(\tau+\tau_0)-u_\mu(\tau)+\frac{u_\mu(\tau)\,u_\nu(\tau)}{c^2}\left[u_\nu(\tau+\tau_0)-u_\nu(\tau)\right]\right\} = \\ = \frac{e}{c}F_{\mu\nu}(\tau)\,u_\nu(\tau)\,, \tag{29}$$

$$x_\mu\left[(n+1)\,\tau_0\right]-x_\mu(n\tau_0)=\tau_0 u_\mu(n\tau_0)\,, \tag{30}$$

one gets the *advanced* formulation of the electron theory, since the motion —according to eqs.(29),(30)— is now determined by advanced actions. In contrast with the retarded formulation, the advanced one describes an electron which is absorbing energy from the external world.

Finally, by adding together retarded and advanced actions, Caldirola wrote down the *symmetric* formulation of the electron theory:

$$\frac{m_0}{2\tau_0}\left\{u_\mu(\tau+\tau_0)-u_\mu(\tau-\tau_0)+\frac{u_\mu(\tau)\,u_\nu(\tau)}{c^2}\left[u_\nu(\tau+\tau_0)-u_\nu(\tau-\tau_0)\right]\right\} = \\ = \frac{e}{c}F_{\mu\nu}(\tau)u_\nu(\tau), \tag{31}$$

$$x_\mu\left[(n+1)\,\tau_0\right]-x_\mu\left((n-1)\,\tau_0\right)=2\tau_0 u_\mu(n\tau_0)\,, \tag{32}$$

which does not include any radiation reactions, and describes a non radiant electron.

Before closing this brief introduction to Caldirola's theory, it is worthwhile to present two more relevant results derived from it, one of them following below, in the next Subsection. If we consider a free particle and look for the "internal solutions" of the equation (15), we get —for a periodical solution of the type

$$\dot{x} = -\beta_0\, c\, \sin\left(\frac{2\pi\tau}{\tau_0}\right)$$
$$\dot{y} = -\beta_0\, c\, \cos\left(\frac{2\pi\tau}{\tau_0}\right)$$
$$\dot{z} = 0$$

(which describes a uniform circular motion) and by imposing the kinetic energy of the internal rotational motion to equal the intrinsic energy m_0c^2 of the particle— that the amplitude of the oscillations is given by $\beta_0^2 = \frac{3}{4}$. Thus, the magnetic moment corresponding to this motion is exactly the *anomalous magnetic moment* of the electron,[39] obtained here in a purely classical context:

$$\mu_a = \frac{1}{4\pi}\,\frac{e^3}{m_0c^2}\,.$$

This shows that the anomalous magnetic moment is an intrinsically classical, and not quantum, result; and the absence of $\hbar$ in the last expression is a remarkable confirmation of this fact.

2.5 Hyperbolic motions

In a review paper on the theories of electron including radiation–reaction effects, Erber[38] criticized Caldirola's theory for its results in the case of hyperbolic motion.

Let us recall that the opinion of Pauli and von Laue (among others) was that a charge performing uniformly accelerated motions —e.g., an electron in free fall— could not emit radiation. That opinion was strengthened by the invariance of Maxwell equations under the group of conformal transformations,[67] which in particular includes transformations from rest to uniformly accelerated motions. Since the first decades of the twentieth century, this had been —however— an open question, as the works by Born and Schott had on the contrary suggested a radiation emission in such a case. In 1960, Fulton and Rohrlich[40] *demonstrated* that from Dirac's equation for the classical electron it does follow the emission of radiation during the hyperbolic motion.

A solution of this paradox is possible within Caldirola's theory, and it was worked out in ref.[32] By analysing the energy conservation law for an electron submitted to an external force and following a procedure similar to that of Fulton and Rohrlich, Lanz obtained the expression (20) above. By expanding it in terms of τ and keeping only the the first order terms, he arrived to expression (25), identical to the one obtained

by Fulton and Rohrlich, in which —let us repeat— the Schott energy appears: a term that Fulton and Rohrlich (having obtained it from Dirac's expression for the radiation reaction) interpreted as a part of the internal energy of the charged particle.

For the particular case of *hyperbolic motion*, it is

$$a_\mu a^\mu = \frac{\mathrm{d}a_0}{\mathrm{d}\tau}$$

so that there is *no* radiation reaction [cf. eqs.(25) or (26)]. However, neither the acceleration energy neither the energy radiated by the charge per unit of proper time, $\frac{2}{3}e^2 a_\mu a^\mu$, are zero!

The difference is that in the discrete case this acceleration energy does not exist as such, being a term proceeding from the discretized expression for the charged particle kinetic energy variation. As we have seen in eq.(22), the Schott term appears when the variation of the kinetic energy during the fundamental interval of proper time is expanded in powers of τ_0:

$$\frac{T(\tau) - T(\tau - \tau_0)}{\tau_0} \cong \frac{\mathrm{d}}{\mathrm{d}\tau}T - \frac{2}{3}\frac{e^2}{c^2}\frac{\mathrm{d}}{\mathrm{d}\tau}a_0.$$

This is an interesting result, since it was not easy to understand the physical meaning of the Schott acceleration energy. With the introduction of the fundamental interval of time, as we know, the changes in the kinetic energy are no longer continuous and the Schott's term merely expresses, to first order, the variation of the kinetic energy when passing from one discrete instant of time to the subsequent one.

In eqs.(22) and (25), the derivative $\mathrm{d}T/\mathrm{d}\tau$ is a point function, forwarding the kinetic energy slope *at* the instant τ. And the dissipative term $\frac{2}{3}e^2 a_\mu a^\mu$ is just a relativistic generalization of the Larmor radiation law: then, if there is acceleration, there is also radiation emission.

For the hyperbolic motion, however, the energy dissipated because of the acceleration has just the same magnitude as the energy gain due to the kinetic energy increase. We are not forced to resort to 'pre-accelerations' in order to justify the origin of such energies...[41] Thus, the present theory provides a clear picture of the physical processes involved in the uniformly accelerated motion of a charged particle.

3 The Hypothesis of the Chronon in Quantum Mechanics

Let us pass to the main topic of the present paper: the chronon in Quantum Mechanics. The speculations about the discreteness of time (on the basis of of possible physical evidences) in QM go back to the first decades of this century, and various theories have been proposed developing QM on a space-time lattice (cf., e.g., refs.[42]). This is not the case with the hypothesis of the chronon, where we do **not** actually have a discretization of the time coordinate. In the twenties, for example, Pokrowski[61] suggested the introduction of a fundamental interval of time, starting from an analysis of the shortest wavelengths detected at that time in cosmic radiation. More recently, for instance, Ehrlich[43] proposed a quantization of the elementary particle lifetimes, suggesting the value 4.4×10^{-24} s for the *quantum* of time. But a time discretization is suggested by the very foundations of QM. There are physical limits that prevent the distinction of arbitrarily close successive states in the time evolution of a quantum system. Basically, such limitations result from the Heisenberg relations: in such a way that, if a discretization is to be introduced in the description of a quantum system, it cannot possess a universal value, since those limitations depend on the characteristics of the particular system under consideration. In other words, the value of the fundamental interval of time has to change a priori from system to system. All these points make justifiable the extension of Caldirola's procedure to QM.

In the seventies, Caldirola extended to QM the introduction of the chronon, following the same guidelines that led him to his theory of the electron. So, time is still a continuous variable, but the evolution of the system along its world line is discontinuous. As for the electron theory in the non-relativistic limit, one has to substitute the corresponding finite–difference expression for the time derivatives; e.g.:

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} \rightarrow \frac{f(t) - f(t - \Delta t)}{\Delta t} \tag{33}$$

where proper time is now replaced by the local time t. Such a procedure was then applied to obtain the finite–difference form of the Schrödinger equation. As for the electron case, there are three different ways to perform the discretization, and three "Schrödinger equations" can be obtained[44]:

$$i\frac{\hbar}{\tau}\left[\Psi\left(\mathbf{x},\mathbf{t}\right)-\Psi\left(\mathbf{x},\mathbf{t}-\tau\right)\right]=\hat{H}\Psi\left(\mathbf{x},\mathbf{t}\right), \tag{34}$$

$$i\frac{\hbar}{2\tau}\left[\Psi\left(\mathbf{x},\mathbf{t}+\tau\right)-\Psi\left(\mathbf{x},\mathbf{t}-\tau\right)\right]=\hat{H}\Psi\left(\mathbf{x},\mathbf{t}\right), \tag{35}$$

$$i\frac{\hbar}{\tau}\left[\Psi\left(\mathbf{x},\mathbf{t}+\tau\right)-\Psi\left(\mathbf{x},\mathbf{t}\right)\right]=\hat{H}\Psi\left(\mathbf{x},\mathbf{t}\right), \tag{36}$$

which are, respectively, the *retarded*, *symmetric* and *advanced* Schrödinger equations, all of them transforming into the (same) continuous equation when the fundamental interval of time (that now can be called just τ) goes to zero. It can be immediately observed that the symmetric equation is of the second order, while the other two are first order equations. As in the continuous case, for a finite–difference equation of order n a single and complete solution requires n initial conditions to be specified.

As the equations are different, the solutions they provide are also fundamentally different. In order to study the properties of such equations there are two basic procedures. For some special cases, they can be solved by one of the various existing methods of solving finite-difference equations, or by means of an attempt solution, an *ansatz*. The other way is to find a new hamiltonian $\tilde{H}$ such that a new continuous Schrödinger equation

$$i\hbar\frac{\partial\Psi\left(\mathbf{x},\mathbf{t}\right)}{\partial t}=\tilde{H}\Psi\left(\mathbf{x},\mathbf{t}\right) \tag{37}$$

reproduces, at the points $t=n\tau$, the same results obtained from the discretized equations. As has been shown by Casagrande and Montaldi[45], it is always possible to find a continuous generating function which makes possible to obtain a differential equation equivalent to the original finite-difference one, such that in every point of interest their solutions are identical. This procedure is useful since it is generally very difficult to work with the finite–difference equations on a qualitative basis. Except for some very special cases, they can be only numerically solved. This equivalent hamiltonian $\tilde{H}$ is, however, non-hermitian and it is frequently very difficult to be obtained. For the special case where the hamiltonian is time independent, the equivalent hamiltonian is quite easy to be calculated. For the symmetric equation, e.g., it is given by:

$$\tilde{H}=\frac{\hbar}{\tau}\sin^{-1}(\frac{\tau}{\hbar}\hat{H}). \tag{38}$$

As expected, $\tilde{H} \rightarrow \hat{H}$ when $\tau \rightarrow 0$. Caldirola[46] used the *symmetric* equation to describe the non-radiating electron (bound electron) since for hamiltonians explicitly independent of time its solutions are always of oscillating character:

$$\Psi(\mathbf{x},\mathbf{t}) = \exp\left(-i\frac{t}{\tau}\sin^{-1}\left(\frac{\tau}{t}\hat{H}\right)\right) f(\mathbf{x}).$$

In the classical theory of electron, the symmetric equation also represents a non-radiating motion. It provides only an approximate description of the motion without taking into account the effects due to the self fields of the electron. However, in the quantum theory it plays a fundamental role. In the discrete formalism, it is the only way to describe a bound non-radiating particle.

The solutions of the *advanced* and *retarded* equations show completely different behaviour. For a hamiltonian explicitly independent of time the solutions have a general form given by

$$\Psi(\mathbf{x},\mathbf{t}) = \left[1 + i\frac{\tau}{\hbar}\hat{H}\right]^{-t/\tau} f(\mathbf{x})$$

and, expanding $f(x)$ in terms of the eigenfunctions of $\hat{H}$,

$$\hat{H}u_n(\mathbf{x}) = W_n u_n(\mathbf{x})$$

$$f(\mathbf{x}) = \sum_n c_n u_n(\mathbf{x})$$

with

$$\sum_n |c_n|^2 = 1,$$

it can be obtained that

$$\Psi(\mathbf{x},\mathbf{t}) = \sum_n c_n \left[1 + i\frac{\tau}{\hbar}W_n\right]^{-t/\tau} u_n(\mathbf{x}).$$

In particular, the norm of this solution is given by

$$|\Psi(\mathbf{x},\mathbf{t})|^2 = \sum_n |c_n|^2 \exp(-\gamma_n t)$$

with

$$\gamma_n = \frac{1}{\tau} \ln \left(1 + \frac{\tau^2}{\hbar^2} W_n^2 \right) = \frac{W_n^2}{\hbar^2} \tau + O\left(\tau^3\right).$$

The presence of a damping factor depending critically on the value τ of the chronon must be remarked.

This *dissipative* behaviour originates from the *retarded* character of the equation. The analogy with the electron theory also holds and the retarded equation shows intrinsically dissipative solutions representing a radiant system. The hamiltonian has the same status as in the continuous case: it is an observable since it is a hermitian operator and its eigenvectors form a basis of the state space. However, due to the damping term, the norm of the state vector is not constant anymore. An opposite behaviour is observed for the solutions of the advanced equation, in the sense that they increase exponentially.

3.1 The mass of the muon

The most impressive achievement due to the introduction of the chronon hypothesis in the realm of QM is obtained in the description of a bound electron using the new formalism. Bound states are described by the symmetric Schrödinger equation and a hamiltonian that does not depend explicitly on time. A general solution can be obtained by using a convenient *ansatz*:

$$\Psi(\mathbf{x}, \mathbf{t}) = \sum_n u_n(\mathbf{x}) \exp\left(-i\alpha_n t\right),$$

where $\hat{H} u_n(\mathbf{x}) = E_n u_n(\mathbf{x})$ gives the spectrum of eigenvalues of the hamiltonian. If the fundamental interval of time τ corresponds to the chronon θ_0 associated with the classical electron, it can be straightforwardly obtained that

$$\alpha_n = \frac{1}{\theta_0} \sin^{-1}\left(\frac{E_n \theta_0}{\hbar}\right).$$

This solution gives rise to an upper limit for the eigenvalues of the hamiltonian due to the condition

$$\left|\frac{E_n \theta_0}{\hbar}\right| \leq 1.$$

Since θ_0 is finite, there is a maximum value for the energy of the electron given by

$$E_{\max} = \frac{\hbar}{\theta_0} = \frac{2}{3}\frac{\hbar m_0 c^3}{e^2} \approx 105.04 \ \text{MeV}.$$

Now, including the rest energy of the electron, we finally get

$$E = E_{\max} + E_0^{\text{electron}} \approx 105.55 \ \text{MeV}.$$

which is very close (an error of 0.1%) to the measured value of the rest mass of the muon. Using the equivalent hamiltonian method it is possible to extend the basis of eigenstates out of the critical limit. However, for the eigenvalues above the critical limit the corresponding eigenstates are unstable and decay in time:

$$\Psi(\mathbf{x}, \mathbf{t}) = \sum_n c_n u_n(\mathbf{x}) \exp(-i\gamma_n t) \exp(-k_n t),$$

As for the retarded equation, the norm of the state vector is not constant and decays exponentially with time for those eigenstates out of the stability range. Caldirola[46] interpreted this norm as giving the probability of the existence of the particle in its original Hilbert space and associated a mean lifetime with these states.

The considerations regarding the muon as an excited state of the electron can be traced back to the very days of its discovery. Particularly, it has already been observed that the ratio between the masses of the two particles is almost exactly $3/(2\alpha)$, where α is the fine structure constant.[48] It has already been remarked that $\frac{2}{3}\alpha$ is just the coefficient of the radiative reaction term in Dirac's equation for the classical electron.[49] Bohm and Weinstein[33] put forward the hypothesis that various kinds of "mesons" could be excited states of the electron. Dirac[50] has even proposed a specific model for an extended electron so as to interpret the muon as an excited state of the electron.

Caldirola[47] observed that by means of the Heisenberg uncertainty relations it is possible to associate the existence of the muon as an excited state of the electron with the introduction of the chronon in the theory of electron. The relation

$$\Delta\tau \, \Delta E \geq \hbar/2$$

imposes limitations in the determination, at a certain instant τ, of the energy E associated with the internal motion of the electron. If excited states of the particle corresponding to larger values of mass exist, thus it is only possible to speak of an "electron with rest mass m_0" when $\Delta E \leq (\mu_0 - m_0)c^2$, where μ_0 is the rest mass of

the internal excited state. Such internal states could only be excited in the presence of sufficiently strong interactions. From the uncertainty relation we have that

$$\Delta\tau \geq \frac{\hbar}{2\left(\mu_0 - m_0\right)c^2},$$

and, supposing the muon as an excited state, we get

$$\left(\mu_0 - m_0\right)c^2 \cong \frac{3}{2}\frac{\hbar c}{e^2} m_0 c^2.$$

Thus, it can be finally obtained that

$$\Delta\tau \geq \frac{1}{3}\frac{e^2}{m_0 c^2} = \frac{\tau_0}{2},$$

i.e., that the value of the rest mass of an interacting electron can be taken only inside an interval of the proper time larger than half a chronon. So, when we take into account two successive states, each one endowed with the same uncertainty $\Delta\tau$, then they must be separated by a time interval of at least $2\,\Delta\tau$, which corresponds exactly to the chronon τ_0.

3.2 The mass spectrum of leptons

In order to obtain the mass of another particle a possibility to be considered is to take the symmetric equation as describing the muon. According to this *naïve* argumentation, the equation also foresees a maximum limit for the energy of the eigenstates of the muon. By taking the equation as successively describing the particles corresponding to these maxima, an expression can be set up for the various limit values, given by

$$E_0^{(n)} = m_0 c^2 \left[\frac{3}{2}\frac{\hbar c}{e^2} + 1\right]^n = m_0 c^2 \left[\frac{3}{2}\frac{1}{\alpha} + 1\right]^n \tag{39}$$

such that, for

$$\begin{array}{lcll} n = 0 & \rightarrow & E^{(0)} = 0.511 \ \text{MeV} & \text{(electron)} \\ n = 1 & \rightarrow & E^{(1)} = 105.55 \ \text{MeV} & \text{(muon)} \\ n = 2 & \rightarrow & E^{(2)} = 21801.54 \ \text{MeV} & \text{(heavy lepton?)} \end{array}$$

the masses for the first excited states can be obtained, including a possible heavy lepton which, according to the experimental results up to now, probably does not exist.

Following a suggestion of Barut[52], according to which it should be possible to obtain the excited states of the electron from the coupling of its intrinsic magnetic moment with its self field, Caldirola[53][54], considering a model of the extended electron as a micro-universe, succeeded in evaluating also the mass of the lepton τ .

Caldirola took into account, for the electron, a model of a point-object moving around in a 4-dimensional *de Sitter* micro-universe characterized by

$$c^2t^2 - x^2 - y^2 - z^2 = c^2\tau_0^2,$$

where τ_0 is the chronon associated with the electron and the radius of the micro-universe is given by $a = c\tau_0$. Considering the spectrum of excited states obtained from the naïve argumentation above, we find that each excited state determines a characteristic radius for the micro-universe. Thus, for each particle, the trajectory of the point-object is confined to a spherical shell defined by its characteristic radius and by the characteristic radius of its excited state. For the electron, e.g., the point-object moves around inside the spherical shell defined by its correspondent radius and that one associated with its excited state: the muon. Such radii are given by

$$a^{(n)} = \tau_0 c \left[\frac{3}{2}\frac{1}{\alpha} + 1\right]^{-n}. \tag{40}$$

According to the model —supposing that the intrinsic energy of the lepton $e^{(n)}$ is given by $m^{(n)}c^2$— the lepton moves in its associated micro-universe along a circular trajectory with a velocity $\beta = \frac{\sqrt{3}}{2}$, to which it corresponds an intrinsic magnetic moment

$$\mu_a^{(n)} = \frac{1}{4\pi}\frac{e^2}{m^{(n)}c^2}. \tag{41}$$

Starting from Barut's suggestion, Caldirola obtained for the lepton $e^{(n)}$ an extra self-energy given by

$$E^{(n,p)} = (2p)^4 m^{(n)}c^2.$$

The condition set down on the trajectory of the point-object so that it remains confined to its corresponding spherical shell is given by

$$E^{(n,p)} \leq \left[\frac{3}{2}\frac{1}{\alpha} + 1\right] m_0 c^2,$$

and the values attainable by p are: $p = 0$ for $n = 0$, and $p = 0, 1$ for $n \neq 0$. The spectrum of mass is then finally given by

$$m^{(n,p)} = \left[1 + (2p)^4\right] m^{(n)} = m_0 \left[1 + (2p)^4\right] \left[\frac{3}{2}\frac{1}{\alpha} + 1\right]^n . \tag{42}$$

Thus, for different values of n and p we have:

n	p	$m^{(n)}$	
0	0	0.511 MeV	electron
1	0	105.55 MeV	muon
	1	1794.33 MeV	tau

It must be remarked that the *tau* appears as an internal excited state of the muon and its mass is in fair agreement[55] with the experimental values: $m_\tau \approx$ 1784 MeV. The difference between these values is less than 1%. This is quite amazing if we consider the simplicity of the model. The model foresees the existence of other excited states which seems not to exist. This is to some extent justifiable once the muon is obtained as an excited electron and the description of the electron does not anticipate the existence of any other state. In order to obtain the lepton *tau* it was necessary to introduce in the formalism the coupling of the intrinsic magnetic moment with the self-field of the electron.

3.3 Feynman path integrals

The discretized Schrödinger equations can be easily obtained using Feynman's path integral approach. This is particularly interesting since it allows a clearer idea of the meaning of these equations. According to the hypothesis of a chronon, time is still a continuous variable and the introduction of the fundamental interval of time is connected —let us recall– only with the reaction of the system to the action of a force. It is convenient to restrict the derivation to the one-dimensional case, considering a particle under the action of a potential $V(x,t)$. We take the time coordinate as continuous and take into account a discretization at instants separated by steps given by τ.

The transition amplitude for a particle going from an initial point (x_1, t_1) of the space-time to a final point (x_n, t_n) is given by the propagator

$$K\left(x_n, t_n; x_1, t_1\right) = \langle x_n, t_n \mid x_1, t_1 \rangle . \tag{43}$$

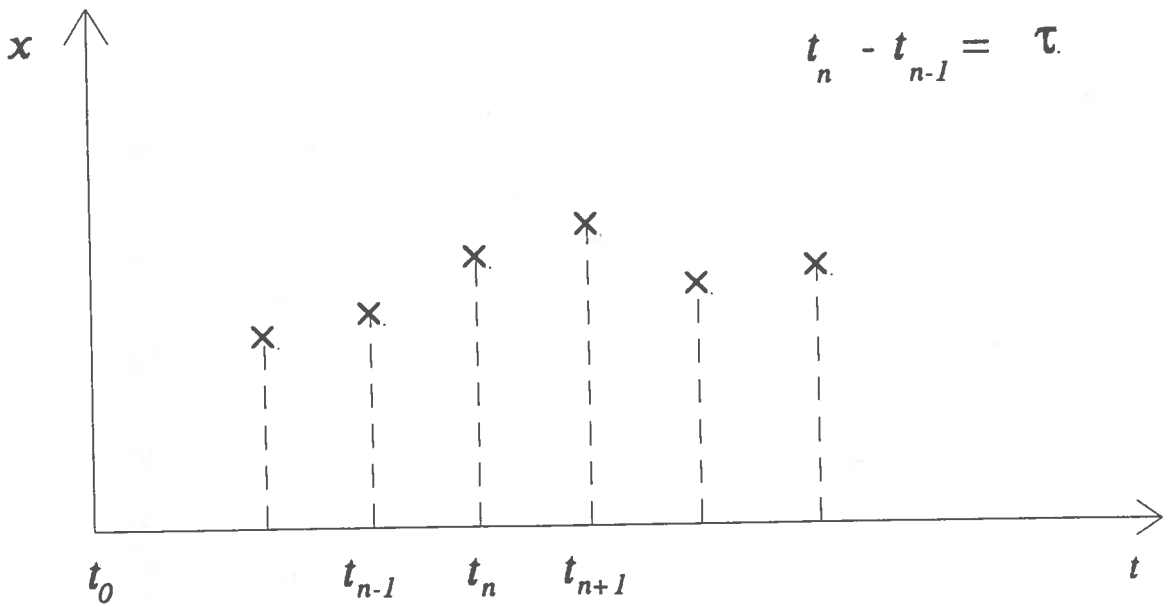


Figure 1: Discrete steps in the time evolution

In Feynman's approach this transition amplitude is associated with a path integral, where the classical *action* plays a fundamental role. It is convenient to introduce the notation

$$S(n, n-1) \equiv \int_{t_{n-1}}^{t_n} \mathrm{d}t\, \mathcal{L}(x, \dot{x}) \tag{44}$$

such that $\mathcal{L}(x, \dot{x})$ is the classical lagrangian and $S(n, n_1)$ is the classical action. Thus, for two consecutive instants of time, the propagator is given by

$$K(x_n, t_n; x_{n-1}, t_{n-1}) = \frac{1}{A} \exp\left(\frac{i}{\hbar} S(x_n, t_n; x_{n-1}, t_{n-1})\right). \tag{45}$$

The path integral has the meaning of a sum over all the possible paths traversed by the particle and can be written as

$$\langle x_n, t_n \mid x_1, t_1 \rangle = \lim_{N \to \infty} A^{-N} \int \mathrm{d}x_{N-1} \int \mathrm{d}x_{N-2} \cdots \int \mathrm{d}x_2 \prod_{n-2}^{N} \exp\left(\frac{i}{\hbar} S(n, n-1)\right), \tag{46}$$

where A is a normalization factor. In order to obtain the discretized Schrödinger equations we have to consider the evolution of a quantum state between two consecutive configurations (x_{n-1}, t_{n-1}) e (x_n, t_n). The state of the system at t_n is:

$$\Psi(x_n, t_n) = \int_{-\infty}^{+\infty} K(x_n, t_n; x_{n-1}, t_{n-1}) \Psi(x_{n-1}, t_{n-1})\, \mathrm{d}x_{n-1}. \tag{47}$$

By the other side, it follows from the definition of the classical action (eq. 44) that

$$S(x_n, t_n; x_{n-1}, t_{n-1}) = \frac{m}{2\tau}(x_n - x_{n-1})^2 - \tau V\left(\frac{x_n + x_{n-1}}{2}, t_{n-1}\right). \tag{48}$$

Thus, the state at t_n is given by

$$\Psi(x_n, t_n) = \int_{-\infty}^{+\infty} \exp\left\{\frac{im}{2\hbar\tau}(x_n - x_{n-1})^2 - i\frac{\tau}{\hbar}V\left(\frac{x_n + x_{n-1}}{2}, t_{n-1}\right)\right\} \Psi(x_{n-1}, t_{n-1})\mathrm{d}x_{n-1}. \tag{49}$$

When $\tau \approx 0$, for x_n slightly different from x_{n-1}, the integral due to the quadratic term is rather small. The contributions are considerable only for $x_n \approx x_{n-1}$. Thus, we can make the following approximation

$$x_{n-1} = x_n + \eta \quad \rightarrow \quad \mathrm{d}x_{n-1} \equiv \mathrm{d}\eta,$$

such that

$$\Psi(x_{n-1}, t_{n-1}) \cong \Psi(x_n, t_{n-1}) + \left(\frac{\partial \Psi(x_n, t_{n-1})}{\partial x}\right)\eta + \left(\frac{\partial^2 \Psi}{\partial x^2}\right)\eta^2.$$

By inserting this expression into equation (49), supposing that[‡]

$$V\left(x + \frac{\eta}{2}\right) \approx V(x),$$

and taking into account only the terms to the first order in τ, we get

$$\Psi(x_n, t_n) = \frac{1}{A}\exp\left(-\frac{i}{\hbar}\tau V(x_n, t_{n-1})\right)\left(\frac{2i\hbar\pi\tau}{m}\right)^{1/2}\left(\Psi(x_n, t_{n-1}) + \frac{i\hbar\tau}{2m}\frac{\partial^2\Psi}{\partial x^2}\right)$$

Notwithstanding the fact that $\exp(-i\tau V(x_n, t_n)/\hbar)$ is a function defined only for certain well-determined values, it can be expanded in powers of τ, around an arbitrary

[‡]The potential is supposed to vary slowly with x.

position (x_n, t_n). Choosing $A = (2i\hbar\pi\tau/m)^{-1/2}$, such that $\tau \to 0$ in the continuous limit, we obtain

$$\Psi(x_n, t_{n-1} + \tau) - \Psi(x_n, t_{n-1}) = -\frac{i}{\hbar}\tau V(x_n, t_{n-1})\Psi(x_n, t_{n-1}) + \\ + \frac{i\hbar\tau}{2m}\frac{\partial^2\Psi}{\partial x^2} + O(\tau^2) \tag{50}$$

By a simple reordering of terms we finally get

$$i\frac{\Psi(x_n, t_{n-1} + \tau) - \Psi(x_n, t_{n-1})}{\tau} = \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x_n, t_{n-1})\right\}\Psi(x_n, t_{n-1})$$

Following this procedure we obtain the **advanced** finite–difference Schrödinger equation which describes a particle performing an one-dimensional motion under the potential $V(x, t)$.

The solutions of the advanced equation show an amplification factor which may suggest that the particle absorbs energy from the field described by the hamiltonian in order to evolve in time. In the classical domain the *advanced* equation is interpreted as describing a *positron*. However, in the realm of the non-relativistic QM, it is more naturally interpreted as representing a system which takes in energy from the environment.

In order to obtain the discrete Schrödinger equation only the terms to the first order in τ have been taken into account. Since the limit $\tau \to 0$ has not been accomplished, the equation thus obtained is only an approximation. The meaning of this fact may be related to another one we are going to face ahead in this paper when considering the measurement problem in QM.

It is rather interesting to remark that in order to obtain the *retarded* equation we have to regard the propagator as acting backward in time. The conventional procedure used in the continuous case always provides us with the *advanced* equation. Therefore, the potential describes a mechanism for transferring energy from a field to the system. The *retarded* equation can obtained only by assuming an inversion of the time order, considering the expression

$$\Psi(x_{n-1}, t_{n-1}) = \int_{-\infty}^{+\infty} \frac{1}{A}\exp\left\{\frac{i}{\hbar}\int_{t_n}^{t_{n-1}} \mathcal{L}dt\right\}\Psi(x_n, t_n)\,dx_n, \tag{51}$$

which can be rigorously obtained by merely using the closure relation for the eigenstates of the position operator and then redefining the propagator in the inverse time order. With this expression, it is possible to obtain the *retarded* Schrödinger equation. The symmetric equation can be easily obtained by a similar procedure.

An interesting characteristic related to these apparently opposed equations is the impossibility of obtaining one from the other by a simple time inversion. The time order in the propagators must be related to the inclusion, in these propagators, of something like the advanced and retarded potentials. Thus, in order to obtain the retarded equation we have to take into account effects that act backward in time. Considerations like these that led to the derivation of the three discretized equations can supply useful guidelines for the comprehension of their meaning.

3.4 The Schrödinger and Heisenberg pictures

In discrete QM, as well as in the "continuous" one, the utilization of discretized Heisenberg equations is expected to be preferable for certain types of problem. As it happens for the continuous case, the discretized versions of the Schrödinger and Heisenberg pictures are also equivalent. However, we show below that the Heisenberg equations **cannot** in general be obtained by a direct discretization of the continuous equations.

First of all, it is convenient to introduce the discrete time evolution operator for the symmetric

$$\hat{U}(t, t_0) = \exp\left[-\frac{i(t-t_0)}{\tau}\sin^{-1}\left(\frac{\tau\hat{H}}{\hbar}\right)\right] \tag{52}$$

and for the retarded equation,

$$\hat{U}(t, t_0) = \left[1 + \frac{i}{\hbar}\tau\hat{H}\right]^{-(t-t_0)/\tau} \tag{53}$$

In order to simplify the equations, the following notation will be used in all along this Section

$$\Delta f(t) \longleftrightarrow \frac{f(t+\tau) - f(t-\tau)}{2\tau} \tag{54}$$

$$\Delta_R f(t) \longleftrightarrow \frac{f(t) - f(t-\tau)}{\tau} \tag{55}$$

For both the operators above it can be easily demonstrated that, if the hamiltonian $\hat{H}$ is a hermitian operator, the following equations are valid:

$$\Delta\hat{U}(t,t_0) = \frac{1}{i\hbar}\hat{U}(t,t_0)\hat{H}, \tag{56}$$

$$\Delta\hat{U}^\dagger(t,t_0) = -\frac{1}{i\hbar}\hat{U}^\dagger(t,t_0)\hat{H}. \tag{57}$$

In the Heisenberg picture the time evolution is transferred from the state vector to the operator representing the observable according to the definition

$$\hat{A}^{\mathrm{H}} \equiv \hat{U}^\dagger(t,t_0=0)\hat{A}^{\mathrm{S}}\hat{U}(t,t_0=0) \tag{58}$$

For the *symmetric* case, for a given operator $\hat{A}^{\mathrm{S}}$, the time evolution of the operator $\hat{A}^{\mathrm{H}}(t)$ is given by

$$\Delta\hat{A}^{\mathrm{H}}(t) = \Delta\left[\hat{U}^\dagger(t,t_0=0)\hat{A}^{\mathrm{S}}\hat{U}(t,t_0=0)\right]$$

$$\Delta\hat{A}^{\mathrm{H}}(t) = \frac{1}{i\hbar}\left[\hat{A}^{\mathrm{H}},\hat{H}\right] \tag{59}$$

which has exactly the same form as the equivalent equation for the continuous case. The important feature of the time evolution operator which is used to derive the expression above is that it is an *unitary* operator. This is true for the symmetric case. For the retarded case, however, this property is not satisfied anymore. Differently from the symmetric and continuous cases, the state of the system is also time dependent in the retarded Heisenberg picture:

$$\left|\Psi^{\mathrm{H}}(t)\right\rangle = \left[1+\frac{\tau^2\hat{H}^2}{\hbar^2}\right]^{-(t-t_0)/\tau}\left|\Psi^{\mathrm{S}}(t_0)\right\rangle \tag{60}$$

By using the property $\left[\hat{A},f\left(\hat{A}\right)\right]=0$, it is possible to show that the evolution law for the operators in the retarded case is given by:

$$\Delta\hat{A}^{\mathrm{H}}(t) = \left\{\frac{1}{i\hbar}\left[\hat{A}^{\mathrm{S}}(t),\hat{H}^{\mathrm{S}}(t)\right]+\Delta\hat{A}^{\mathrm{S}}(t)\right\}^{\mathrm{H}} \tag{61}$$

In short, we can conclude that the discrete symmetric case and the continuous case are formally very similar and the Heisenberg equation can be obtained by a direct discretization of the continuous equation. For the retarded and advanced cases, however,

this does not hold. In the Appendix A we analyse the compatibility between the Heisenberg and Schrödinger pictures.

3.5 Time dependent hamiltonians

When we restricted the analysis of the discretized equations to the time independent hamiltonians this was made aiming at simplicity. When the hamiltonian is explicitly time dependent the situation is very similar to the continuous case. It is always difficult to work with such hamiltonians but, as in the continuous case, the theory of small perturbations can also be applied. For the symmetric equation, when the hamiltonian is of the form

$$\hat{H} = \hat{H}_0 + \hat{V}(t), \tag{62}$$

such that $\hat{V}$ is a small perturbation related to $\hat{H}_0$, the resolution method turns out to be very similar to the usual one. The solutions are equivalent to the continuous solutions followed by an exponentially varying term. It is always possible to solve this kind of problem using an appropriate ansatz.

However, another factor must be considered, related to the existence of a limit beyond which $\hat{H}$ does not have stable eigenstates. For the symmetric equation, the equivalent hamiltonian is given by

$$\tilde{H} = \frac{\hbar}{\tau} \sin^{-1}\left(\frac{\tau}{\hbar}\hat{H}\right). \tag{63}$$

Thus, as previously stressed, beyond the critical value the eigenvalues are not real and the operator $\tilde{H}$ is not hermitian anymore. Below that limit, $\tilde{H}$ is a densely defined and self-adjoin operator in the $\mathcal{L} \subset L^2$ subspace defined by the eigenfunctions of $\tilde{H}$. When the limit value is exceeded the system changes to an excited state and the previous state loses physical meaning. In this way, it is convenient to restrict the observables to self-adjoin operators that keep invariant the subspace $\mathcal{L}$. The perturbation $\hat{V}$ is assumed to satisfy this requirement.

In the usual QM it is convenient, in order to deal with time dependent perturbations, to work with the interaction representation (Dirac's picture). In this representation, the evolution of the state is determined by the time dependent potential $\hat{V}(t)$, while the evolution of the observable is determined by the stationary part of the the hamiltonian $\hat{H}_0$. In the discrete formalism, the time evolution operator defined for $\hat{H}_0$, in the symmetric case, is given by

$$\hat{U}_0(t,t_0) = \exp\left[-\frac{i(t-t_0)}{\tau}\sin^{-1}\left(\frac{\tau\hat{H}_0}{\hbar}\right)\right] \tag{64}$$

In the interaction picture the vector state is defined, from the state in the Schrödinger picture, as

$$\left|\Psi^{\mathrm{I}}(t)\right\rangle = \hat{U}_0^{\dagger}(t)\left|\Psi^{\mathrm{S}}(t_0)\right\rangle, \tag{65}$$

where $\hat{U}_0^{\dagger}(t) \equiv \hat{U}_0^{\dagger}(t, t_0 = 0)$. On the other hand, the operators are defined as

$$\hat{A}^{\mathrm{I}} = \hat{U}_0^{\dagger}(t)\,\hat{A}^{\mathrm{S}}\hat{U}_0(t) \tag{66}$$

So, it is possible to show that, in the interaction picture, the evolution of the vector state is determined by the equation

$$i\hbar\Delta\Psi^{\mathrm{I}}(\mathrm{x},\mathrm{t}) = \frac{i\hbar}{2\tau}\left[\Psi^{\mathrm{I}}(\mathrm{x},\mathrm{t}+\tau) - \Psi^{\mathrm{I}}(\mathrm{x},\mathrm{t}-\tau)\right] = \hat{V}^{\mathrm{I}}\Psi^{\mathrm{I}}(\mathrm{x},\mathrm{t}), \tag{67}$$

which is equivalent to a direct discretization of the continuous equation. For the operators we get that

$$\Delta\hat{A}^{\mathrm{I}}(t) = \frac{\hat{A}^{\mathrm{I}}(t+\tau) - \hat{A}^{\mathrm{I}}(t-\tau)}{2\tau} = \frac{1}{i\hbar}\left[\hat{A}^{\mathrm{I}}, \hat{H}_0\right]_{\tau} \quad . \tag{68}$$

which is also equivalent to the continuous equation.

Thus, for the symmetric case, the discrete interaction picture keeps the same characteristics of the continuous one for the evolution of the operators and state vectors once, obviously, the eigenstates of $\hat{H}$ remain below the stability limit. We can adopt, for the discrete case, a procedure similar to that one commonly used in QM to deal with small time dependent perturbations.

We consider, in the interaction picture, the same basis of eigenstates associated with the stationary hamiltonian $\hat{H}_0$, given by $|n\rangle$. Then,

$$|\Psi(t)\rangle^{\mathrm{I}} = \sum_n \Psi(t)\,\langle n \mid \Psi(t)\rangle^{\mathrm{I}}\,|n\rangle = \sum_n c_n(t)\,|n\rangle$$

is the expansion, in this basis, of the state of the system in a certain instant t. It must be observed that the evolution of the state of the system is determined once the coefficients $c_n(t)$ are known. Using the evolution equation (67) it can be obtained that

$$i\hbar\Delta \langle n \mid \Psi(t)\rangle^{\mathrm{I}} = \sum_m \langle n| \hat{V}^{\mathrm{I}} |m\rangle \langle m \mid \Psi(t)\rangle^{\mathrm{I}}.$$

Using the evolution operator to rewrite the perturbation $\hat{V}$ in the Schrödinger picture we get that

$$i\hbar\Delta c_n(t) = \sum_m c_m(t) V_{nm}(t) \exp(i\omega_{nm} t), \tag{69}$$

such that

$$\omega_{nm} = \frac{1}{\tau}\left[\sin^{-1}\left(\frac{\tau E_n}{\hbar}\right) \sin^{-1}\left(\frac{\tau E_m}{\hbar}\right)\right],$$

and we obtain the evolution equation for the coefficients $c_n(t)$, the solution of which gives the time evolution of the system.

As in the usual QM, it is also possible to work with the interaction picture evolution operator, $\hat{U}^{\mathrm{I}}(t, t_0)$, which is defined as

$$|\Psi(t)\rangle^{\mathrm{I}} = \hat{U}^{\mathrm{I}}(t, t_0) |\Psi(t_0)\rangle^{\mathrm{I}},$$

such that (67) can be written as

$$i\hbar\Delta\hat{U}^{\mathrm{I}}(t, t_0) = \hat{V}^{\mathrm{I}}(t)\hat{U}^{\mathrm{I}}(t, t_0). \tag{70}$$

The operator $\hat{U}^{\mathrm{I}}(t, t_0)$ have to satisfy the initial condition $\hat{U}^{\mathrm{I}}(t, t_0) = 0$. Given this condition, we have for the finite–difference equation above the solution

$$\hat{U}^{\mathrm{I}}(t, t_0) = \exp\left[\frac{-i(t - t_0)}{\tau} \sin^{-1}\left(\frac{\tau \hat{V}^{\mathrm{I}}(t)}{\hbar}\right)\right].$$

Differently from the continuous case, where the approximate evolution operator turns out to be an infinite Dyson series, a well determined expression is obtained. The solution of the problem is obtained by correlating the elements of the matrix associated with such operator to the evolution coefficients $c_n(t)$.

In general, the finite–difference equations are harder to be analytically solved than the equivalent differential equations. In particular, such difficulty is far more stressed for the system of equations obtained from the formalism above.

An alternative approach is to use the equivalent hamiltonians.[47] Once the equivalent hamiltonian is found the procedure is exactly the same as for the continuous theory. If the perturbation term $\hat{V}$ is small the equivalent hamiltonian can be written as

$$\tilde{H} = \frac{\hbar}{\tau}\sin^{-1}\left(\frac{\tau}{\hbar}\hat{H}_0\right) + \hat{V}(t) = \hat{H}_0 + \hat{V}(t).$$

In the interaction picture, the state of the system is now defined as

$$\left|\Psi^{\mathrm{I}}(t)\right\rangle = \exp i\frac{\tilde{H}_0 t}{\hbar}\left|\Psi^{\mathrm{S}}(t)\right\rangle, \tag{71}$$

and the operators are given by

$$\hat{A}^{\mathrm{I}} = \exp\left(i\frac{\tilde{H}_0 t}{\hbar}\right)\hat{A}^{\mathrm{S}}\exp\left(-i\frac{\tilde{H}_0 t}{\hbar}\right). \tag{72}$$

The states (71) evolve according to the equation

$$i\hbar\frac{\partial}{\partial t}\left|\Psi^{\mathrm{I}}(t)\right\rangle = \hat{V}^{\mathrm{I}}\left|\Psi^{\mathrm{I}}(t)\right\rangle, \tag{73}$$

where $\hat{V}^{\mathrm{I}}$ is obtained according to definition (72).

Now, small time dependent perturbations can be dealt with by taking into account the time evolution operator defined by

$$\left|\Psi^{\mathrm{I}}(t)\right\rangle = \hat{U}^{\mathrm{I}}(t,t_0)\left|\Psi^{\mathrm{I}}(t_0)\right\rangle. \tag{74}$$

According to the evolution law (73) we have

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}^{\mathrm{I}}(t,t_0) = \hat{V}^{\mathrm{I}}(t)\hat{U}^{\mathrm{I}}(t,t_0). \tag{75}$$

Thus, once given that $\hat{U}^{\mathrm{I}}(t_0,t_0) = 1$ it turns out that the time evolution operator is given by

$$\hat{U}^{\mathrm{I}}(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t}\hat{V}^{\mathrm{I}}(t')\hat{U}^{\mathrm{I}}(t',t_0)\mathrm{d}t'$$

or

$$\hat{U}^{\mathrm{I}}(t,t_0) = 1 + \sum_{n=1}\left(-\frac{i}{\hbar}\right)^n\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2\cdots\int_{t_0}^{t_{n-1}}\mathrm{d}t_n\hat{V}^{\mathrm{I}}(t_1)\hat{V}^{\mathrm{I}}(t_2)\cdots\hat{V}^{\mathrm{I}}(t_n),$$

where the evolution operator is obtained in terms of a Dyson series.

In order to draw a parallel between the elements of the matrix of the evolution operator and the evolution coefficients $c_n(t)$ obtained from the continuous equation equivalent to (69), we have to use the basis of eigenstates of the stationary hamiltonian $\hat{H}_0$. If the initial state of the system is an eigenstate $|m\rangle$ of that operator then, at a subsequent time, we have

$$c_n(t) = \left\langle n|\hat{U}^{\mathrm{I}}(t,t_0)|m\right\rangle .$$

The method of the equivalent hamiltonian is simpler to be used, since it takes full advantage of the continuous formalism.

4 Some Applications of the Discretized Quantum Equations

Turning back to more general questions, it is interesting to analyse the physical consequences resulting from the introduction of the fundamental interval of time in QM. In this Section we apply the discretized equations to some typical problems.

4.1 The simple harmonic oscillator

The hamiltonian that describes a simple harmonic oscillator does not depend explicitly on time. The introduction of the discretization in the time coordinate does not affect the outputs obtained from the continuous equation for the spatial branch of the solution. This is always the case when the potential does not have an explicit time dependence. For potentials like this, the solutions of the discrete equations are always formally identical, with changes in the numerical values which depends on the eigenvalues of the hamiltonian considered and on the value of the chronon associated with the system described. We have the same spectrum of eigenvalues and the same basis of eigenstates but with the time evolution given by a different expression.

For the simple harmonic oscillator, the hamiltonian is given by

$$\hat{H} = \frac{1}{2m}\hat{\mathbf{P}}^2 + \frac{m\omega^2}{2}\hat{\mathbf{X}}^2, \tag{76}$$

to which corresponds the eigenvalue equation

$$\hat{H}\,|u_n\rangle = E_n\,|u_n\rangle\,, \tag{77}$$

so that E_n gives the energy eigenvalue spectrum of the oscillator.

As mentioned previously, since this hamiltonian does not depends explicitly on time, there is always an upper limit for the possible values of its energy eigenvalues. In the basis of eigenfunctions of $\hat{H}$ a general state of the oscillator can be written as

$$|\Psi(t)\rangle = \sum_n c_n(0)\,|u_n\rangle \exp\left[-i\frac{t}{\tau}\sin^{-1}\left(\frac{E_n\tau}{\hbar}\right)\right],$$

with $c_n(0) = \langle u_n|\Psi(t=0)\rangle$. Naturally, when $\tau \to 0$, the solution above recovers the continuous expression with its time dependency given by $\exp\left(\frac{-iE_nt}{\hbar}\right)$. Therefore, there is only a small phase difference between the two expressions. For the mean value of an arbitrary observable,

$$\left\langle \Psi(t)|\hat{A}|\Psi(t)\right\rangle = \sum_{m=0}\sum_{n=0} c_m^*(0)c_n(0)A_{mn}\exp\left[\frac{i}{\hbar}\left(E_m - E_n\right)t\right]\cdot$$

$$\cdot\ \exp\left[i\left(E_m^3 - E_n^3\right)\frac{t\tau^2}{3!\hbar^3}\right] \ +\ \mathrm{O}(\tau^4)\,,$$

with $A_{mn} = \left\langle u_m|\hat{A}|u_n\right\rangle$, we obtain an additional phase term which implies a small deviation of the resulting frequencies when compared to the Bohr frequencies of the harmonic oscillator. To first approximation, this deviation is given by the term depending on τ^2 in the expression above.

It must be emphasized that the restrictions imposed on the spectrum of eigenvalues of $\hat{H}$ break the basis of eigenvectors: the number of eigenvectors becomes *finite* and does not constitute a complete set and, therefore, does not form a basis anymore. For eigenstates beyond the upper limit the states are unstable and decay exponentially with time.

For a time independent hamiltonian, the *retarded* equation always furnishes damped solutions characteristic of radiating systems. In this case there is neither stationary solutions nor upper limit for the energy eigenvalues. The larger the eigenvalue the larger the damping factor and more quickly its contribution to the state of the system tends to zero. If we write the state of the oscillator as

$$|\Psi(t)\rangle = \sum_n c_n(0)\,|u_n\rangle\left[1 + \frac{i}{\hbar}\tau E_n\right]^{\frac{t}{\tau}},$$

which has a norm decaying according to

$$\langle\Psi(t)|\Psi(t)\rangle = \sum_n c_n(0)\,|u_n\rangle \left[1 + \frac{\tau^2 E_n^2}{\hbar^2}\right]^{\frac{t}{\tau}}, \tag{78}$$

we have for an arbitrary observable that [with $\langle A(t)\rangle \equiv \langle A\rangle(t)$]:

$$\langle A(t)\rangle = \sum_m \sum_n c_m^*(0)c_n(0)A_{mn} \exp\left[-\frac{t}{\tau}\ln\left[1 + \frac{\tau^2}{\hbar^2}E_n E_m - \frac{i\tau}{\hbar}(E_m - E_n)\right]\right]$$

or, to the first order in τ,

$$\langle A(t)\rangle = \sum_m \sum_n c_m^*(0)c_n(0)A_{mn} \exp\left[\frac{i}{\hbar}(E_m - E_n)\,t\right] \exp\left[-i\left(E_m^2 - E_n^2\right)\frac{\tau}{2\hbar^2}\right],$$

so that, beside the Bohr frequencies defining the emission and absorption frequencies of the oscillator, we obtain a damping term which causes the average value of the observable —which is explicitly independent of time— to tend to zero with time. A cursory analysis shows that even for very small eigenvalues, smaller then 1.0 eV, the damping factor is large, so that the decay of the average values is very fast. The damping factor of the norm in equation (78) can be evaluated, and its behaviour can be seen in picture 2.

4.2 Free particle

For a free particle, an electron for example, the general solution of the symmetric equation (35) can be obtained, in the coordinate representation, using for ansatz the solution for the continuous case. Thus, it is obtained a spectrum of eigenfunctions (plane waves) given by

$$\Psi_p(\mathbf{x}, t) = (2\pi\hbar)^{-3/2} \exp\left(-i\alpha(|\mathbf{p}|)t + i\frac{(\mathbf{p}\cdot\mathbf{x})}{\hbar}\right).$$

Inserting this expression into the symmetric equation, we obtain for the frequency $\alpha(|\mathbf{p}|)$ that

$$\alpha(|\mathbf{p}|) = \frac{1}{c}\sin^{-1}\left(\frac{\tau}{\hbar}\frac{\mathbf{p}^2}{2m_0}\right). \tag{79}$$

When $\tau \to 0$, $\alpha(|\mathbf{p}|)\hbar$ coincides with the energy of the particle. As has been observed for the bound particle, here we also have an upper limit for the spectrum of eigenvalues. Thus the upper limit for the possible values of momentum is given by

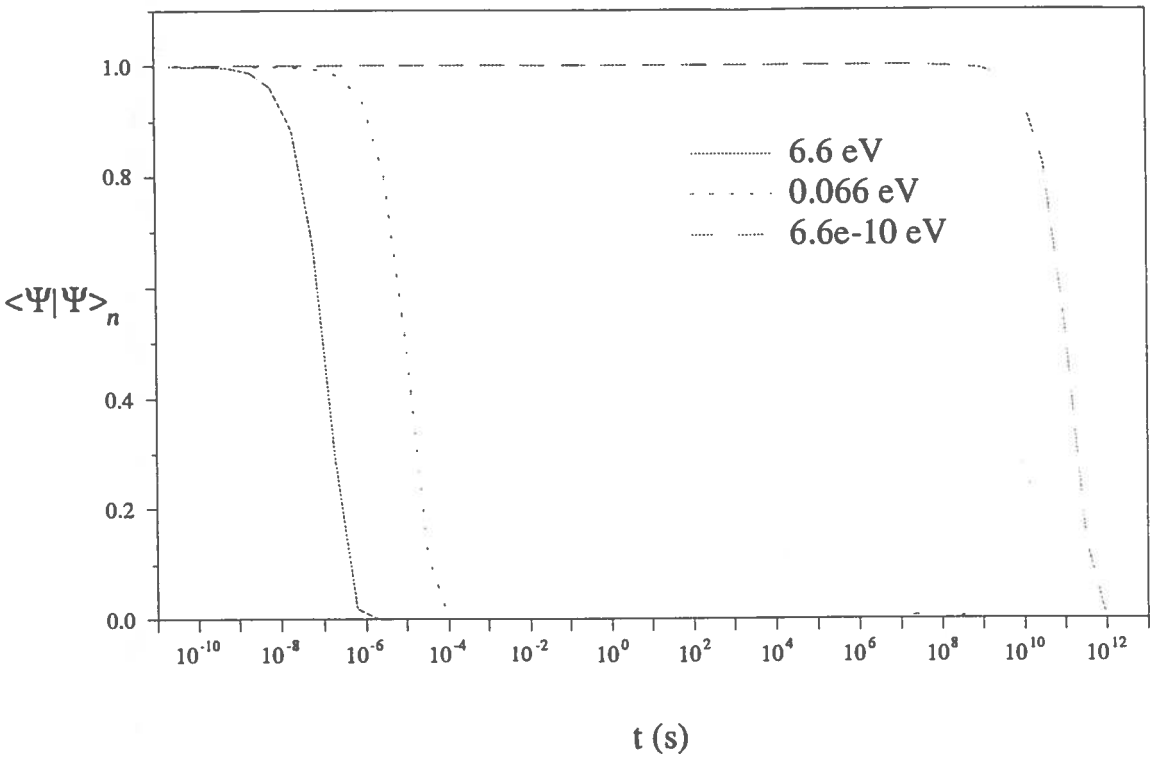


Figure 2: Typical behaviour of the damping factor associated with different eigenvalues of energy (retarded case)

$$p \leq p_{\text{Max}} \equiv \sqrt{\frac{2m_0\hbar}{\tau}} = 10 \text{ MeV}/c \tag{80}$$

for the electron. In other words, there is a limit beyond which the frequencies cease to be real.

As in the continuous case, the state of the particle is described by a superposition of the eigenstates and can be written as

$$\Psi(\mathbf{x}, t) = \frac{1}{(2\pi\hbar)^{3/2}} \int \mathrm{d}^3p c(\mathbf{p}) \exp\left(-i\alpha(|\mathbf{p}|)t + i\frac{(\mathbf{p}\cdot\mathbf{x})}{\hbar}\right).$$

The coefficients $c(\mathbf{p})$ are determined from the initial condition $\Psi(\mathbf{x}, 0) = \Psi_0(\mathbf{x})$. From the expression for α, it can be observed that beyond a certain value of p the expression loses meaning. When $p \geq \sqrt{\frac{2m_0}{\tau}}$, the complete solution will be defined only if $c(\mathbf{p})$. From the stationary phase condition we have that

$$\mathbf{x} = \frac{\mathbf{p}}{m_0} \frac{t}{\sqrt{1 - \left(\frac{\tau}{\hbar}\right)^2 \left(\frac{p^4}{4m_0^2}\right)}},$$

and, supposing that $c(\mathbf{p})$ corresponds to a distribution of probabilities with a peak at $\mathbf{p} = \mathbf{p}_0$, then the wave packet will move in the direction $\mathbf{p}_0$ with uniform velocity

$$v = \frac{p_0}{m_0}\left[1 - \left(\frac{\tau}{\hbar}\right)^2 \left(\frac{p^4}{4m_0^2}\right)\right]^{-1/2}$$

which coincides with the group velocity of the packet. It can be promptly observed that when p reaches its maximum value permitted, the velocity diverges: $v \to \infty$. Thus, the introduction of a fundamental interval of time does not bring in any restriction to the velocity of the particle, although results in a limit for the canonical momentum of the eigenfunctions. Starting from the condition of stationary phase it is possible to redefine the momentum associated with the particle, so that this new momentum does not suffer any restriction at all. So, it is possible to conclude that it is possible the existence of free electrons with any energy, differently from what happens for the bound electron.

For $p > p_{\text{Max}}$ the frequency $\alpha(|\mathbf{p}|)$ fails to be real and its dependence on p is shown in figure (3). An analysis of equation (79) shows that if $\alpha(|\mathbf{p}|)$ is complex then, for $p \leq p_{\text{Max}}$, the imaginary component is null and the real part is given by expression (79). When $p \geq p_{\text{Max}}$, then

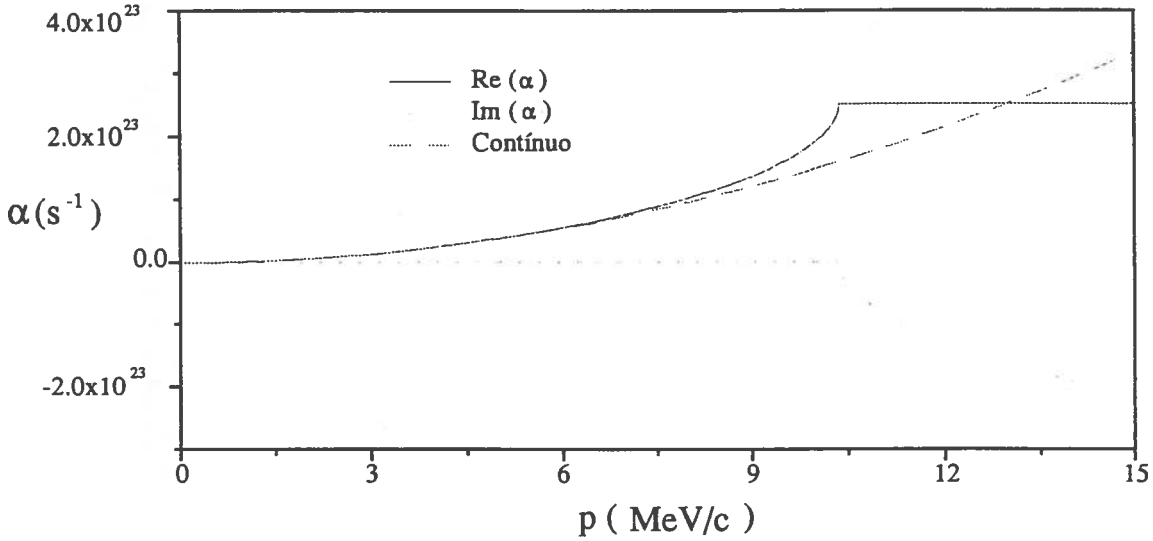


Figure 3: Real and imaginary components of the $\alpha(|\mathbf{p}|)$ obtained for the *symmetric* equation compared to continuous case.

$$\mathrm{Re}\,(\alpha(p)) = \frac{\pi}{2\tau},$$

$$\mathrm{Im}\,(\alpha(p)) = -\frac{1}{\tau}\ln\left[\left|\frac{\tau p^2}{2m_0\hbar}\right| + \sqrt{\left(\frac{\tau p^2}{2m_0\hbar}\right)^2 - 1}\right],$$

with the real part being a constant and the imaginary one tending logaritmically to $-\infty$. Using the expressions above we can observe that, for $p > p_{\mathrm{Max}}$, the eigenstates become unstable, with a time dependent decay term. When we look for an equivalent hamiltonian $\tilde{H}$ that, for the continuous Schrödinger equation, supplies equivalent outputs, we have that this is only possible if $\tilde{H}$ is a non-hermitian operator. It is straightforward to see that this is the case for $\tilde{H} = H_1 + iH_2$, with H_1 and H_2 hermitian and such that $H_1\,|p\rangle = \hbar\,\mathrm{Re}(\alpha(p))\,|p\rangle$ and $H_2\,|p\rangle = \hbar\,\mathrm{Im}\,(\alpha(p))\,|p\rangle$.

For the *retarded* equation, using the same *ansatz* of the symmetric case, the damping factor appears for every value of p. There is not any limitation on the values of p but,

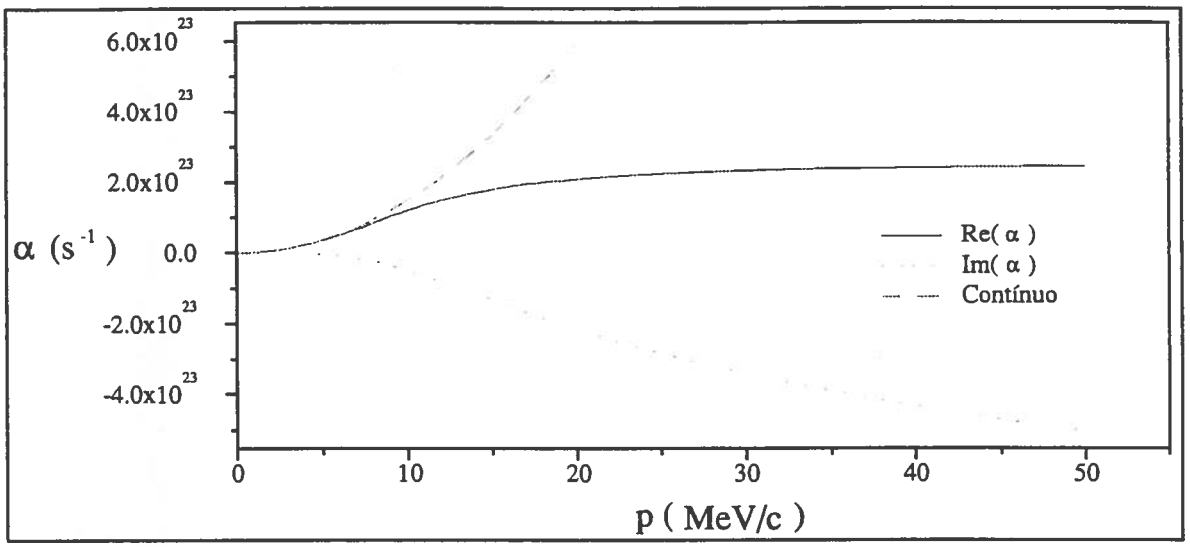


Figure 4: Real and imaginary components of the $\alpha(|\mathbf{p}|)$ obtained for the *retarded* equation compared to continuous case.

when $p \to \infty$, the real part of $\alpha(|\mathbf{p}|)$ tends to the same limit value observed for the symmetric case. Figure (4) illustrates the behaviour of the components of $\alpha(|\mathbf{p}|)$. The general expression for an eigenfunction is found to be

$$\Psi_p(x,t) \propto \exp\left[\frac{ipx}{\hbar} - \frac{it}{\tau}\tan^{-1}\frac{p^2\tau}{2m\hbar}\right]\exp\left[-\frac{t}{2\tau}\ln\left[1+\left(\frac{p^2\tau}{2m\hbar}\right)^2\right]\right].$$

Performing a Taylor expansion and keeping only the terms to the first order in τ we obtain the continuous solution multiplied by a damping factor:

$$\Psi_p(x,t) \propto \exp\left(\frac{ipx}{\hbar} - i\omega t\right)\exp\left(\frac{1}{2}\omega^2\tau t\right) \tag{81}$$

with $\omega = p^2/2m\hbar$ is the frequency obtained for the continuous case.

It must be remarked that the damping term depends only on the hamiltonian, through the frequency ω, and on the chronon associated with the particle. As the later is constant for a given particle, that term shows that for very high frequencies the

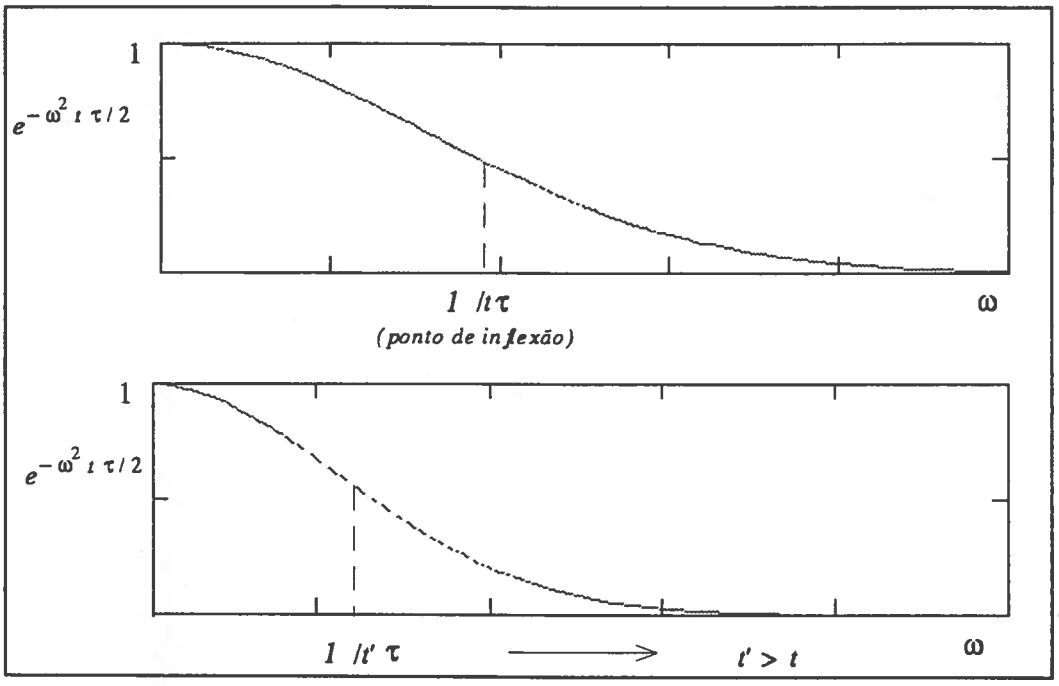


Figure 5: Displacement of the inflection point with time.

solutions decay quite fast and, as the system evolves, the decay for smaller frequencies also also comes true.

In figure (5) it can be observed that the inflection point, delimiting the region of the spectrum where the decay is faster, moves in the direction of smaller frequencies as time goes by. The consequence of this decay is the narrowing of the frequency bandwidth which are relevant for the wave packet describing the particle. This is an echo of its continuous decrease of the energy. As in the symmetric case, obtaining an equivalent hamiltonian is only possible if non-hermitian operators are considered.

At this point, it is worthwhile to reconsider the question of the physical meaning of the three discretized Schrödinger equations. Apparently, the choice of the equation to be used in a particular situation is determined by the boundary conditions, by the restrictions they impose on the system. The symmetric equation is used for special situations for which the system neither emits nor absorbs radiation, or does it in a

perfectly balanced way. This is the case for the electrons in their 'atomic orbits'. So, the particle is stable until a certain energy limit beyond which the behaviour of the states is similar to that of the retarded solutions. For energies far below that limit, the particle behaves almost identically to the continuous case, only that the new frequencies associated with each wave function differ from the continuous frequencies by a factor of order τ^2. The probability that a particle is found with energy larger than the limit value decreases exponentially with time. For the bound electron, the limit is that one equivalent to the rest mass of the muon. If a parallel with the classical approach is valid, the symmetric equation describes an isolated system, which does not exchange energy with the surrounding environment; or a situation of perfect thermodynamic equilibrium, in which a perfect balance between absorbed and dissipated energies is verified. For the classical theory of the electron the symmetric equation is only an approximation which ignores the radiation reaction effects. In QM, however, the existence of non-radiating states are related to the own essence of the theory. The symmetric equation shows that, below the critical limit, the states are physically identical to the outputs from the continuous theory: are *non-radiating* states.

The retarded equation represents a system which somehow loses energy for the environment. The mechanism of such energy dissipation is related to the hamiltonian of the system but also to some property of the environment —even the vacuum— as it can be inferred from the description of the free particle. From the solutions obtained it is now observed that time has a well defined direction of flux and that the frequency composition of the wave packet associated with the particle depends on the instant of time considered. It is clear that it is always possible to normalize the state in a certain instant and consider it as being an initial state. This is permitted by the formalism. However, in a strictly rigorous description, the frequency spectrum corresponds to a specific instant of time which took place after the emission. This is an aspect that can be interesting on the point of view of possible experimental verifications.

4.3 The discretized Klein–Gordon equation

Another interesting application is the description of a free scalar particle —a *scalar* or *zero spin* photon,— using a finite–difference form of the Klein–Gordon equation for massless particles.

In the symmetric form the equation is written as

$$\Box^2 A_\mu = 0 \longrightarrow \frac{\Psi(t+2\tau) - 2\Psi(t) + \Psi(t-2\tau)}{4c^2\tau^2} - \nabla^2\Psi(t) = 0. \tag{82}$$

Using a convenient *ansatz* we obtain, for this equation, in the coordinate representation, that

$$\Psi_k(x,t) = A\exp\left(-i\frac{t}{2\tau}\cos^{-1}(1-2c^2\tau^2k^2)\right)\exp(ikx),$$

which can be written as

$$\Psi_p(x,t) = A\exp\left(-i\frac{t}{2\tau}\cos^{-1}(1-2c^2\tau^2E^2/\hbar^2)\right)\exp(ipx/\hbar),$$

since $E = p^2c^2$ and $p = \hbar k$. Expanding the time exponential in powers of τ, we find that, to the second order in τ, a solution which is very similar to the continuous expression:

$$\Psi_{\mathrm{p}}(x,t) = A\exp\left(-\frac{i}{\hbar}(E't - px)\right),$$

with

$$E' \approx E\left(1 + \frac{E^2\tau^2}{6\hbar^2}\right).$$

A difference of the order of τ^2 is observed between the energy values of the photons in the continuous and discrete approaches. The general solution is given by a linear combination of the eigenfunctions found. *A priori*, the value of the chronon for the particle is not known. The time dependent exponential term in the expressions above lead to a upper limit for the energy allowed, which is given by $E \leq \hbar/\tau$. We could suppose that the value of the chronon for this 'photon' is of about the fundamental time interval of the the electromagnetic interactions, around 10^{-9} s , resulting in a critical value of approximately 6.6 keV, which is a very low limit. A smaller chronon should increase this limit but, if there is any generality in the classical expression obtained for the electron, we should expect a larger value for this massless particle.

Whether, instead of a 'photon' we consider a scalar 'neutrino' , taking for the value of the chronon $\tau \sim 10^{-13}$ s —typical time for the weak decay—, the limit for the energy associated with the eigenfunctions is now approximately 0.007 eV. This means that in the composition of the wave packet describing this particle the only contribution comes from eigenfunctions the energy of which is below that limit.

The eigenfunctions obtained for the hamiltonian considered are "plane waves" solutions. The dependence of these solutions on energy and time is displayed on figures (6) and (7). For smaller values of τ the decay of the modes with energy above the maximum is faster.

Apparently, it seems to be possible to determine a limiting value for the chronon starting from the uncertainty relations. This could be obtained, when describing particles, using the expression

$$\tau < \frac{\hbar}{2m_o c^2}$$

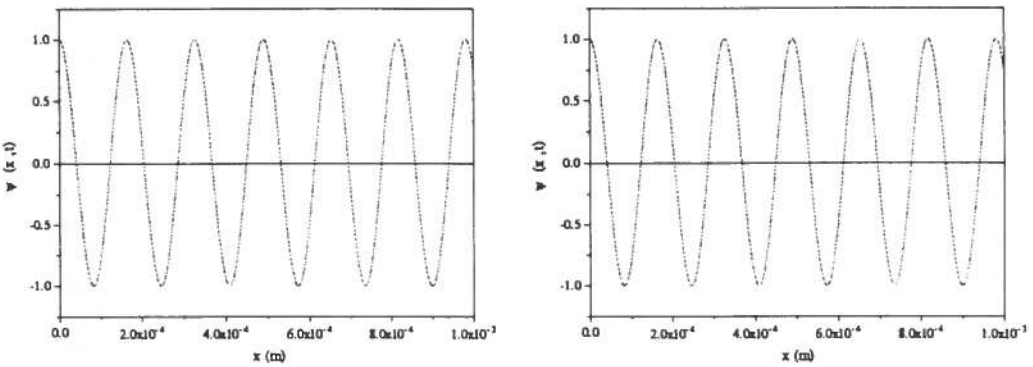
that provides for the electron a maximum limit given by 6.4×10^{-22} s. However, this value is two degrees of magnitude larger than the classical value of the chronon for the electron, which is a considerable difference. For a complex system it is also possible to make use of this relation, as we are going to mention ahead.

We have also to consider the conditions a 'photon' must be supplied with in order to be described by the symmetric equation. For the electron it seems clear that do not irradiate in a bound state —which is imposed by QM— implies using the symmetrical equation. For the photon, as for a free particle, but using the retarded form of the Klein–Gordon equation, it is also obtained a solution wherein the highest frequencies decay faster than the lowest. There is always a tendency in the sense that the lowest frequencies prevail. If we are allowed to assign physical meaning to such discretized Klein–Gordon equation we are also allowed to think that, the farther the light source more the spectrum of the emitted light will be shifted for the larger wavelengths, even whether the source is at rest related to the observer. Thus, we could obtain a *red shift* effect as consequence of the introduction of the chronon which could be used in the construction of a 'tired-light' theory.

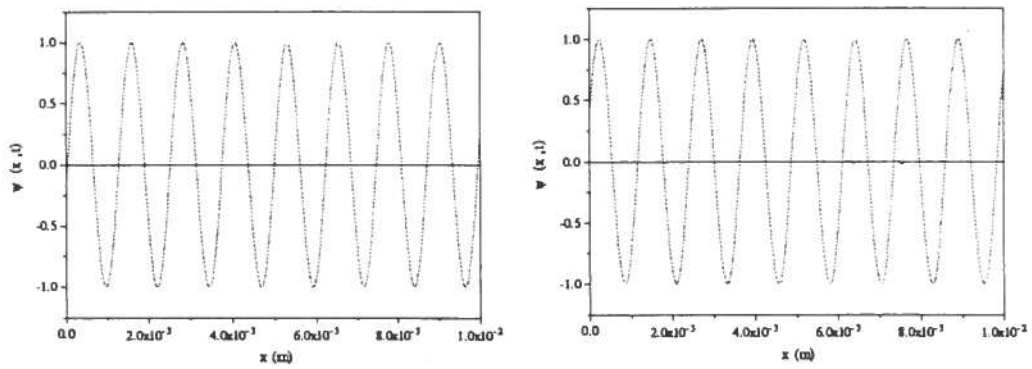
Finally, we have to keep in mind that the discretization considered for the Klein–Gordon equation does not follow exactly the same procedure which led to the discretized Schrödinger equation, since it is a relativistic invariant equation. We did not change the proper time but the time coordinate itself into the discretized form. We considered a discretized version of the hamiltonian operator by applying the transformations:

$$p \longrightarrow \frac{\hbar}{i}\nabla,$$

$$H \longrightarrow i\hbar\Delta,$$

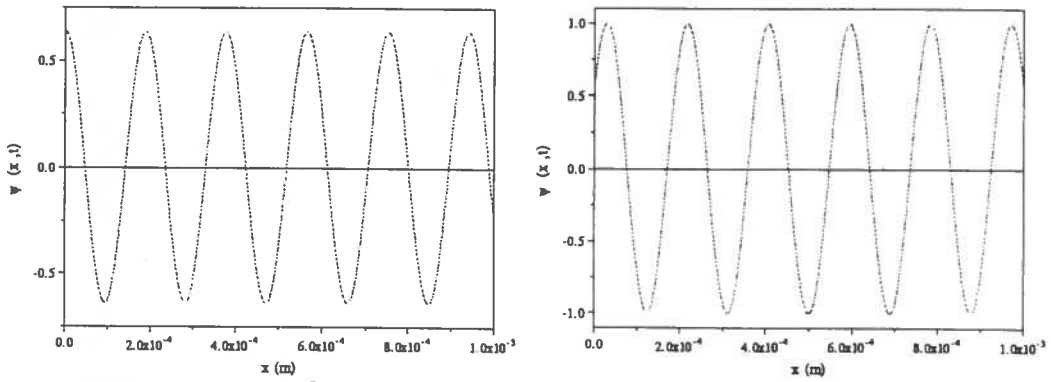


(a) $E < E_M$: $E = 0.001\text{eV}$; $t = 0$. Discrete and continuous solutions are identical.

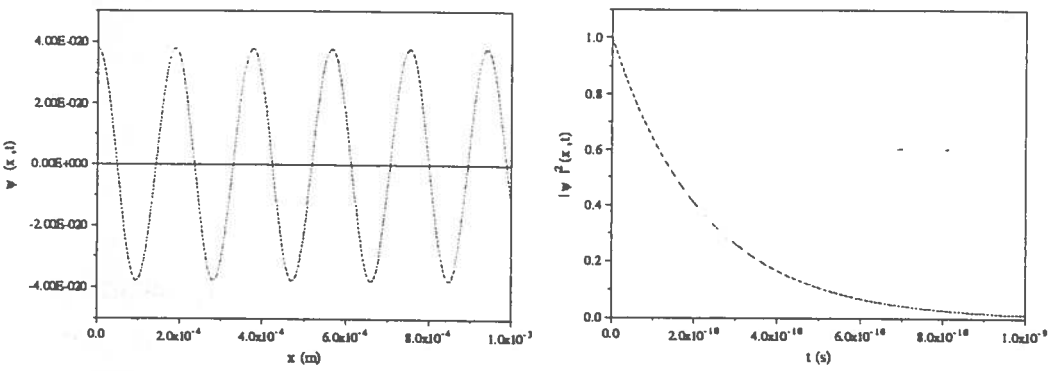


(b) $E < E_M$: $E = 0.001\text{eV}$; $t = 1\times10^{-10}$s. Discrete and continuous solutions differ in phase.

Figure 6: Solution of the discretized Klein–Gordon equation when the energy is smaller than the critical limit, depicted for different values of energy and time.



(a) $E > E_M$: $E = E_M(1+1\times10^{-7})$ eV; $t = 1\times10^{-10}$s. Discrete and continuous solutions differ in phase and amplitude.



(b) $E > E_M$: $E = E_M(1+1\times10^{-7})$ eV. Left: $t = 1\times10^{-8}$s. Right: Damping of the amplitude with time.

Figure 7: Solution of the discretized Klein–Gordon equation when the energy is larger than the critical limit, depicted for different values of energy and time. In this case the amplitude decays quite fast.

with Δ as defined in Subsection (3.4), on the hamiltonian of a relativistic free particle,

$$H = \sqrt{p^2c^2 + m^2c^4}$$

as usual in the continuous case.

4.4 Time evolution of the position and momentum Operators: The harmonic oscillator

It is possible to apply the discretized equations to determine the time evolution of the position and momentum operators, which is rather interesting for the description of the simple harmonic oscillator. In order to do that, we use the discretized form of the Heisenberg equations which, in the symmetric case, can be obtained by a direct discretization of the continuous equation. Starting from this equation we determine the coupled Heisenberg equations for the two operators:

$$\frac{\hat{\mathrm{p}}(t+\tau) - \hat{\mathrm{p}}(t-\tau)}{2\tau} = -m\omega^2\hat{\mathrm{x}}(t), \tag{83}$$

$$\frac{\hat{\mathrm{x}}(t+\tau) - \hat{\mathrm{x}}(t-\tau)}{2\tau} = \frac{1}{m}\hat{\mathrm{p}}(t). \tag{84}$$

Such coupled equations yield two finite–difference equations of second order the general solutions of which are easily obtained. The most immediate way to determine the evolution of these operators is to use the creation and annihilation operators. Keeping the Heisenberg equation and remembering that for the harmonic oscillator we have $\hat{H} = \omega\left(\hat{A}^\dagger\hat{A} + \frac{1}{2}\right)$, we get for the symmetric case:

$$\frac{\hat{A}(t+\tau) - \hat{A}(t-\tau)}{2\tau} = -i\omega\hat{A}(t), \tag{85}$$

$$\frac{\hat{A}^\dagger(t+\tau) - \hat{A}^\dagger(t-\tau)}{2\tau} = i\omega\hat{A}^\dagger(t), \tag{86}$$

such that

$$\hat{A}(t) = \hat{A}(0)\exp\left(-i\frac{t}{\tau}\sin^{-1}(\omega\tau)\right), \tag{87}$$

$$\hat{A}^\dagger(t) = \hat{A}^\dagger(0)\exp\left(i\frac{t}{\tau}\sin^{-1}(\omega\tau)\right), \tag{88}$$

where we used the fact that, for $t = 0$, the Heisenberg and Schrödinger pictures are equivalent: $\hat{A}(t = 0) = \hat{A} = \hat{A}(0)$ and $\hat{A}^\dagger(t = 0) = \hat{A}^\dagger = \hat{A}^\dagger(0)$, with $\hat{A}$ and $\hat{A}^\dagger$ independent of time. In order to obtain these equations we considered that, for the non-relativistic case, there is neither creation nor annihilation of particles, such that we can impose restrictions on the frequencies in the phase term of the operators. For the creation operators, e.g., the terms with negative frequencies —associated with antiparticles— are discarded.

We can observe that the **Number**, as well as the hamiltonian operator are not altered:

$$\hat{N} = \hat{A}^\dagger(t)\hat{A}(t) = \hat{A}^\dagger(0)\hat{A}(0),$$

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right) = \hbar\omega\left(\hat{A}^\dagger(0)\hat{A}(0) + \frac{1}{2}\right).$$

Thus, starting from these operators, we obtain for the symmetric case:

$$\hat{\mathrm{x}}(t) = \hat{\mathrm{x}}(0)\cos\left[\frac{t}{\tau}\sin^{-1}(\omega\tau)\right] + \frac{\hat{\mathrm{p}}(0)}{m\omega}\sin\left[\frac{t}{\tau}\sin^{-1}(\omega\tau)\right]$$

$$\hat{\mathrm{p}}(t) = \hat{\mathrm{p}}(0)\cos\left[\frac{t}{\tau}\sin^{-1}(\omega\tau)\right] - m\omega\hat{\mathrm{x}}(0)\sin\left[\frac{t}{\tau}\sin^{-1}(\omega\tau)\right]$$

which differ from the continuous case since the frequency ω is here replaced by a new frequency $\frac{1}{\tau}\sin^{-1}(\omega\tau)$ that, for $\tau \to 0$, tends to the continuous one. Also, there is now a upper limit for the possible oscillation frequencies given by $\omega \leq 1/\tau$. Above this frequency the motion becomes unstable, as it an be observed in figure (8).

The existence of a maximum limit for the frequency is equivalent to a upper limit for the energy eigenvalues given by $E_n = (n + \frac{1}{2})\hbar\omega \leq \hbar/\tau$, which is equal to the upper limit obtained using Schrödinger' s picture. Since,

$$\frac{t}{\tau}\sin^{-1}(\omega\tau) \cong \omega + \frac{1}{3!}\omega^3\tau^2 + \mathrm{O}(\tau^4),$$

the difference expected in the behaviour of the oscillator faced to the continuous solution is quite small. If we take, for example, the vibration frequency of the hydrogen molecule (H_2), we have that $\omega \sim 10^{14}$ Hz, while the term of the second order in τ is smaller than 10^{-3} Hz (if the analogy with the classical theory is valid, the chronon is expected to be smaller for more massive systems). In terms of average values we have that, for the position operator

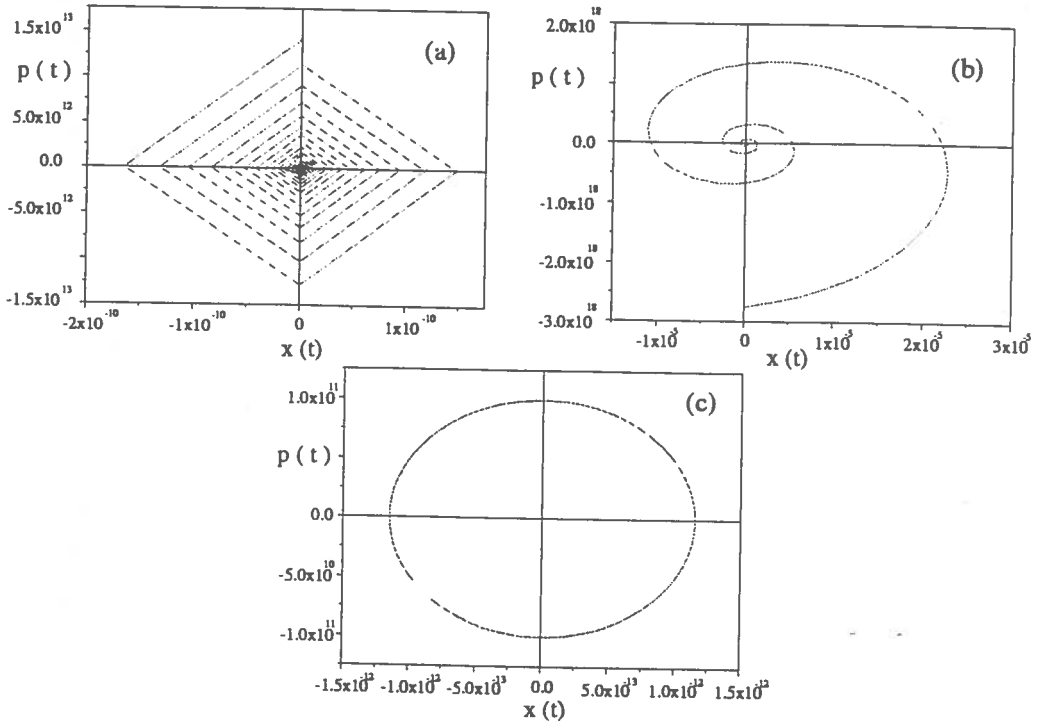


Figure 8: Phase space of the harmonic oscillator when $\omega > \frac{1}{\tau}$ and in the discrete case, with time intervals multiples of τ: in case (a) time is regarded as intrinsically discrete, so that in the picture only the points where the lines touch one another are meaningful, while in case (b) time is regarded as *intrinsically* continuous. In the continuous case no modification is expected with respect to the ordinary case, (c), under the present hypotheses.

$$\langle \hat{\mathbf{x}}(t) \rangle = \langle \hat{\mathbf{x}}(t) \rangle_{\text{cont}} + \frac{\omega^2 \tau^2}{3!m} t \langle \hat{\mathbf{p}}(t) \rangle \ ,$$

in which the term of order τ^2 is expected to be considerably smaller than the mean value for the continuous case. At this point, naturally, the mean values are determined taking for the system a state made up of a superposition of stationary states. For the stationary states $|u_n\rangle$ themselves the mean values of $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ are zero.

For the retarded case the solutions can be obtained using the time evolution operators for the Heisenberg equation (appendix A). As expected, decaying terms come out. The creation and annihilation operators obtained for this case are then given by

$$\hat{A}(t) = \hat{A}(0) \left[1 + i\omega\tau + \tau^2\omega^2\xi\right]^{-\frac{t}{\tau}} \approx \hat{A}(0) \exp\left(-i\omega\tau\right) \exp\left[-\left(\xi + \frac{1}{2}\right)\omega^2\tau t\right],$$

$$\hat{A}^\dagger(t) = \hat{A}^\dagger(0) \left[1 - i\omega\tau + \tau^2\omega^2\xi\right]^{-\frac{t}{\tau}} \approx \hat{A}^\dagger(0) \exp\left(i\omega\tau\right) \exp\left[-\left(\xi + \frac{1}{2}\right)\omega^2\tau t\right],$$

with ξ being a real positive factor. The relation $(\hat{A}^\dagger)^\dagger = \hat{A}$ continues to be valid but the *Number* operator and, consequently, the hamiltonian, is now not constant anymore:

$$\hat{H}(t) = \hat{A}^\dagger(0)\hat{A}(0) \left[\left(1 + \omega^2\tau^2\xi\right)^2 + \omega^2\tau^2\right]^{-\frac{t}{\tau}} = \hat{H}(0) \exp\left[-2\left(\xi + \frac{1}{2}\right)\omega^2\tau t\right],$$

$$\hat{H}(t) = \hbar\omega \left\{ \hat{A}^\dagger(0)\hat{A}(0) \left[\left(1 + \omega^2\tau^2\xi\right)^2 + \omega^2\tau^2\right]^{-\frac{t}{\tau}} + \frac{1}{2} \right\}.$$

Taking into account the terms to the second order in τ, we get that the oscillation frequencies also decays with time. These results are consistent with the fact that the system is emitting radiation, with the consequent reduction of its total energy. However, it is remarkable that the energy of the quanta associated with the creation and annihilation operators is not constant, even with a very tiny variation rate. In the same way, when we calculate the position and momentum operators a damping factor is obtained. Figure (9-a) shows the strange damping factor associated with the *Number* operator. It can be observed that this damping occurs within a period of time which is characteristic for each frequency, being slower and postponed for lower frequencies.

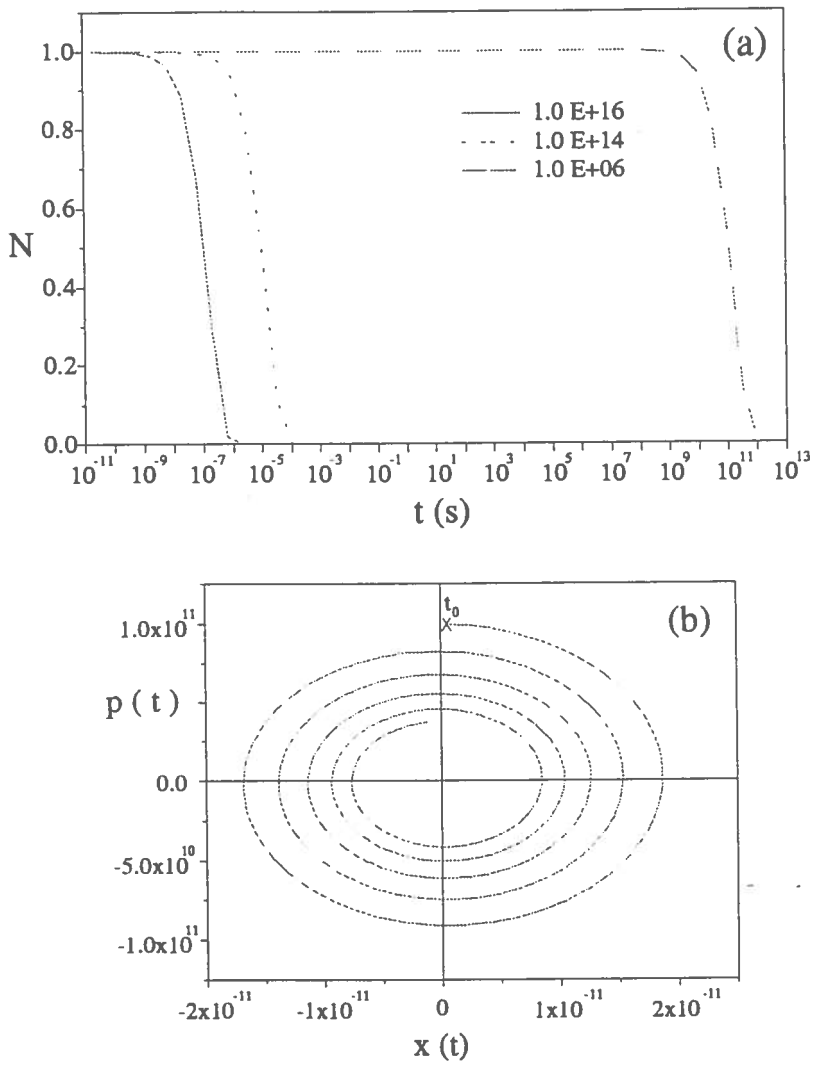


Figure 9: (a) Damping factors associated with the *Number* operator calculated for some frequencies. (b) Damping of the oscillations for the harmonic oscillator described by the *retarded* equation.

Figure (9-b) shows the dampening of the oscillations as described by the retarded equation. Once the expressions for the position and momentum operators are determined, we obtain that, to first order in τ,

$$\langle \hat{\mathbf{x}}(t) \rangle = \left\{ \hat{\mathbf{x}}(0) \cos(\omega\tau) + \frac{\hat{\mathbf{p}}(0)}{m\omega} \sin(\omega\tau) \right\} \exp\left[-\left(\xi + \frac{1}{2} \right) \omega^2 \tau t \right],$$

$$\langle \hat{\mathbf{x}}(t) \rangle = \langle \hat{\mathbf{x}}(t) \rangle_{\text{cont}} \exp\left[-\left(\xi + \frac{1}{2} \right) \omega^2 \tau t \right].$$

Taking into account the higher order terms, we can observe a small variation in the oscillation frequency just as observed in the symmetrical case. The introduction of time independent perturbations does not cause any additional variations apart from those found even in the continuous case. It must be pointed out that the results obtained with the procedure above are in agreement with those obtained following Schrödinger's picture.

4.5 Hydrogen atom

The hydrogen atom is basically a system made up of two particles attracting each other through coulombian force, which is therefore inversely proportional to the square of the distance between them. The basic hamiltonian is given by

$$\hat{H}_0 = \frac{\hat{\mathbf{P}}^2}{2\mu} - \frac{e^2}{R}, \tag{89}$$

and is composed of the kinetic energy of the atom in the centre of mass frame, and of the coulombian electrostatic potential (μ is the reduced mass of the system electron-proton). A more complete description is obtained by adding correction terms (fine structure) to the hamiltonian, including relativistic effects such as the variation of the electron mass with velocity and the coupling of the intrinsic magnetic moment of the electron with the magnetic field due to its orbit (spin–orbit coupling). Besides, there are also the hyperfine (hf) corrections which appear due to the interaction of the electron with the intrinsic magnetic moment of the proton and, finally, the *Lamb shift*, due to the interaction of the electron with the fluctuations of the quantized electromagnetic field. The hamiltonian can finally be written as[56]

$$\hat{H}_{\text{I}} = m_{\text{e}}c^2 + \hat{H}_0 - \frac{\hat{\mathbf{P}}^4}{m_{\text{e}}^2 c^2 R^3} \hat{\mathbf{L}} \cdot \hat{\mathbf{S}} + \hat{H}_{\text{hf}} + \hat{H}_{\text{Lamb}}. \tag{90}$$

The introduction of the magnetic moment of the nucleus through the hyperfine correction causes the total angular momentum to be $\mathbf{F} = \mathbf{J} + \mathbf{I}$. The hamiltonian does not depends explicitly on time such that, for the symmetric Schrödinger equation

$$i\frac{\hbar}{2\tau}\left[\Psi\left(\mathbf{x},\mathbf{t}+\tau\right)-\Psi\left(\mathbf{x},\mathbf{t}-\tau\right)\right]=\hat{H}_{\mathrm{I}}\Psi\left(\mathbf{x},\mathbf{t}\right), \tag{91}$$

we obtain, using the separation of variables, the following uncoupled equations:

$$\hat{H}_{\mathrm{I}}\Phi(\mathbf{x}) = E\Phi(\mathbf{x})$$

$$i\frac{\hbar}{2\tau}\left[T\left(t+\tau\right)-T\left(t-\tau\right)\right]=\hat{H}_{\mathrm{I}}T\left(t\right),$$

with the general solution

$$\Psi\left(\mathbf{x},t\right)=\Phi\left(\mathbf{x}\right)\exp\left[-i\frac{t}{\tau}\sin^{-1}\left(\frac{\tau E}{\hbar}\right)\right]. \tag{92}$$

The difference related to the continuous case appears only in those aspects involving the time evolution of the states. Since the hamiltonian is time independent, its eigenvalues are exactly the same obtained in the continuous case[56]

$$E_{(n,j)} \approx m_0c^2 - \frac{1}{2n^2}m_ec^2\alpha^2 - \frac{m_ec^2}{2n^4}\left(\frac{n}{j+\frac{1}{2}}-\frac{3}{4}\right)\alpha^4 + E_{\mathrm{hf}} + E_{\mathrm{Lamb}}.$$

A situation where a difference between the two cases can appear is when taking into account the probabilities of transition between the eigenstates for an atom submitted to a time dependent potential. In the discrete approach it is possible to use the method of the equivalent hamiltonian in order to obtain the transition probabilities. As mentioned previously (Subsection 3.5), the problem is treated using the conventional approximate methods for time dependent perturbations.

If we consider, for example, the non-relativistic interaction of an atom with an electromagnetic field described by the vector potential $\mathbf{A}(\mathbf{x},t)$, we have for the low intensity limit, in the Coulomb gauge, the hamiltonian

$$\hat{H}(t) = \hat{H}_{\mathrm{I}} - \hat{V}(t) = \hat{H}_{\mathrm{I}} - \frac{e}{m_ec}\hat{\mathbf{A}}\left(\hat{\mathbf{R}},t\right)\cdot\hat{\mathbf{P}}, \tag{93}$$

where the potential term is taken as being the perturbation. If we consider that the potential describes a monochromatic field of a plane wave, then

$$\mathbf{A}(\mathbf{x},t) = A_0\,\hat{\epsilon}\left[\exp\left(i\omega\frac{\hat{\mathbf{n}}\cdot\mathbf{x}}{c} - i\omega t\right) + \exp\left(-i\omega\frac{\hat{\mathbf{n}}\cdot\mathbf{x}}{c} + i\omega t\right)\right] \tag{94}$$

where $\hat{\epsilon}$ is the linear polarization of the field and $\hat{\mathbf{n}}$ is the propagation direction. The term depending on $(-i\omega t)$ corresponds to the absorption of a quantum of radiation $\hbar\omega$ and the $(i\omega t)$ term to stimulated emission. Let us assume that the system is initially in an eigenstate $|\Phi_i\rangle$ of the time independent hamiltonian. Keeping only the perturbations to the first order in $\hat{V}(t)$, we obtain that

$$c_n^1(t) = -\frac{i}{\hbar}\int_0^t \exp\left(i\omega_{ni}t'\right)V_{ni}(t')\mathrm{d}t' \ ,$$

where ω_{ni} in the discrete case is given by

$$\omega_{ni} = \frac{1}{\tau}\left[\sin^{-1}\left(\frac{\tau E_n}{\hbar}\right) - \sin^{-1}\left(\frac{\tau E_i}{\hbar}\right)\right].$$

Working with the absorption term, we get by contrast that

$$c_n^{(1)}(t) = \frac{ieA_0}{m_e^2 c\hbar}\left\langle\Phi_n|e^{i\omega\hat{\mathbf{n}}\cdot\mathbf{x}/c}(\hat{\epsilon}\cdot\mathbf{p})|\Phi_i\right\rangle\int_0^t \exp\left[i(\omega_{ni}-\omega)t'\right]\mathrm{d}t' \ .$$

Thus, the probability of transition from the initial state $|\Phi_i\rangle$ to the final state $|\Phi_f\rangle$ is given by

$$P_{fi}(t) = \left|c_f^{(1)}(t)\right|^2 = \frac{e^2|A_0|^2}{m_e^2c^2\hbar^2}\left|\left\langle\Phi_f|e^{i\omega\hat{\mathbf{n}}\cdot\mathbf{x}/c}(\hat{\epsilon}\cdot\mathbf{p})|\Phi_i\right\rangle\right|^2\left|\int_0^t \exp\left[i(\omega_{fi}-\omega)t'\right]\mathrm{d}t'\right|^2 ,$$

or

$$P_{fi}(t) = \frac{4e^2|A_0|^2}{m_e^2c^2\hbar^2}\left|\left\langle\Phi_f|e^{i\omega\hat{\mathbf{n}}\cdot\mathbf{x}/c}(\hat{\epsilon}\cdot\mathbf{p})|\Phi_i\right\rangle\right|^2\frac{\sin^2\left[(\omega_{fi}-\omega)t/2\right]}{(\omega_{fi}-\omega)^2},$$

so that the determination of the matrix elements of the spacial term, using the electric dipole approximation, provides the selection rules for the transitions. What is remarkable in this expression is the presence of a *resonance* showing a larger probability for the transition when

$$\omega = \omega_{fi} = \frac{1}{\tau}\left[\sin^{-1}\left(\frac{\tau E_f}{\hbar}\right) - \sin^{-1}\left(\frac{\tau E_i}{\hbar}\right)\right]. \tag{95}$$

This expression is formally different from the one obtained for the continuous approach. When we expand this expression in powers of τ, we get that

$$\omega \approx \frac{E_f - E_i}{\hbar} + \frac{1}{6}\frac{E_f^3 - E_i^3}{\hbar^3}\tau^2. \tag{96}$$

The first term supplies the Bohr frequencies as in the continuous case; the second, the deviation in the frequencies caused by the introduction of the time discretization:

$$\Delta\omega_{fi} = \frac{1}{6}\frac{E_f^3 - E_i^3}{\hbar^3}\tau^2.$$

If we consider the chronon of the classical electron, $\tau \approx 6.26 \times 10^{-24}$ s, it is possible to estimate the deviation in frequency due to the time discretization. Then, for the hydrogen atom,

$$\Delta\omega_{\mathrm{fi}} \approx 2.289 \times 10^{-2}\left(E_{\mathrm{f}}^3 - E_{\mathrm{i}}^3\right).$$

If we take into account, for example, the transitions corresponding to the first lines of the series of Lyman and Balmer, i.e., of the non-disturbed states $n = n_{\mathrm{i}} \longrightarrow n = n_{\mathrm{f}}$, we have

n_i	n_f	ΔE (eV)	ν (Hz)	$\Delta\nu_D$ (Hz)
1	2	10.2	2.465×10^{15}	~ 10
1	3	12.1	2.922×10^{14}	~ 10
1	4	12.75	3.082×10^{14}	~ 10
2	3	1.89	4.566×10^{14}	< 1

where ΔE is the difference of energy between the states, ν is the frequency of the photon emitted in the transition and $\Delta\nu_{\mathrm{D}}$ is the frequency deviation due to the discretization. Such deviation is always very tiny. We must remember that the hyperfine corrections and those due to the Lamb shift are of order of a Gigahertz. For the transition $n = 1 \longrightarrow n = 2$, e.g., the correction due to the Lamb shift is approximately 1.06 GHz.

Larger deviations caused by the discretization occurs for mono-electronic atoms with larger atomic numbers. For the first transition the deviation is of approximately 90 Hz for the ^2He, 1.1 kHz for the ^3Li, and 420 kHz for the ^6C. However, these deviations are still quite smaller than that one due to the Lamb shift. That is also the case for the *muonic* atoms. For a muonic atom with a proton as nucleus, using for the chronon a value derived from the classical expression for the electron ($\tau_\mu = 3.03 \times 10^{-26}$ s) the deviation is of about 1.4 kHz for the transition $n = 1 \longrightarrow n = 2$. For that transition the frequency of the emitted radiation is approximately 4.58×10^{17} Hz.

For the *retarded* equation, a difference with respect to the symmetrical case is present in the time evolution of the states. The procedure is identical to the one used above and the general solution is now given by

$$\Psi(\mathbf{x},t) = \Phi(\mathbf{x})\left[1 + i\frac{\tau E}{\hbar}\right]^{-t/\tau},$$

so that the transitions now occurs with frequencies given by

$$\omega = \omega_{fi} = \frac{-i\hbar}{\tau}\left[\ln\left(1 + \frac{i\tau E_f}{\hbar}\right) - \ln\left(1 + \frac{i\tau E_i}{\hbar}\right)\right]. \tag{97}$$

As results from the characteristics of the retarded equation, this is a complex frequency. The real component of such frequency can be approximated by

$$\mathrm{Re}(\omega_{fi}) \approx \frac{E_f - E_i}{\hbar} + \frac{1}{3}\frac{E_\mathrm{f}^3 - E_\mathrm{i}^3}{\hbar^3}\tau^2,$$

where the first term is the expression for the continuous case. For the particular transition $n = 1 \rightarrow n = 2$ we have that the deviation due to the discretization is of about 18 Hz.

The imaginary component, on the other hand, can be approximated by

$$\mathrm{Im}(\omega_\mathrm{fi}) \approx -\frac{i}{2}\frac{E_\mathrm{f}^2 - E_\mathrm{i}^2}{\hbar^2}\tau.$$

In the expression for the probability of transition we have the module of an integral involving the time dependency of the general solution. In this case, the characteristic damping causes the probability to tend to a fixed, non zero value. An example of such behaviour is showed in figure (10), which shows the variation of the time dependent term between a initial instant $t_0 = 0$ and some hundreds chronons later. In order to observe the decay of amplitude factor we have used a larger value for the chronon, of about 10^{-18} s. When the chronon is of the order of the one we have been considering for the electron the decay is slower.

As it can be observed, the effect of the time discretization on the emission spectrum of the hydrogen is extremely small. Using the expressions obtained above we can estimate that, in order that the effect of the time discretization is of the same order of the Lamb shift, the chronon associated with the electron should be of about 10^{-18} s, far above the classical value (but close to the typical interval of the electromagnetic interactions). In any case, it should be remembered that the Lamb shift measurements do not seem to be in full agreement[61] with quantumelectrodynamics.

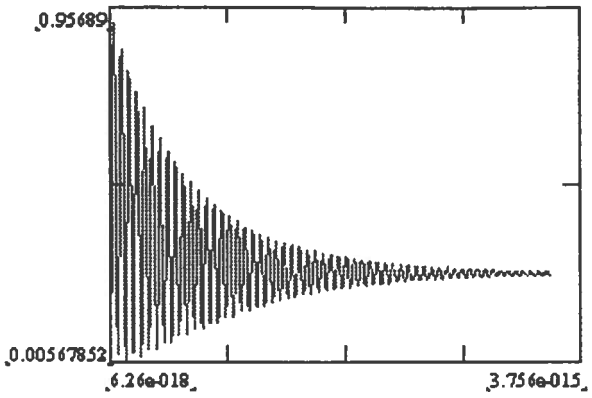


Figure 10: Behaviour of the time dependent component of the probability of transition

Closing this Subsection, it is worthwhile to remark that, for a time independent hamiltonian, the outputs obtained in the discrete formalism using the symmetric equation are very similar to those from the continuous case. For such hamiltonians, the effect of the discretization appears basically in the frequencies associated with the time dependent term of the wave function. As already observed, the difference in the time dependency is of the kind

$$\exp\left[-iE_n(t-t_0)/\hbar\right] \quad \longrightarrow \quad \exp\left[-i\sin^{-1}\left(\frac{\tau E_n}{\hbar}\right)(t-t_0)/\tau\right]$$

The discretization causes a change in the phase of the eigenstate, which can be quite large. The eigenfunctions individually describe stationary states, so that the time evolution appears when we have a linear combination of such functions, this way describing the state of the system. This state evolves according to

$$|\Psi(t)\rangle = \sum_n c_n(0)\exp\left[-i\sin^{-1}\left(\frac{\tau E_n}{\hbar}\right)(t-t_0)/\tau\right]|\phi_n\rangle\,,$$

considering that $H|\phi_n\rangle = E_n|\phi_n\rangle$ is the eigenvalue equation associated with the hamiltonian.

When the stationary states of a particle under, e.g., one-dimensional squared potentials are studied, the same reflection and transmission coefficients and the same tunnel effect are obtained, since they are calculated starting from the stationary states. When we consider a linear superposition of these stationary states, building a wave packet, the time dependent terms have to be taken into account, resulting in some differences with respect to the continuous case. Some attempts have been carried out in order to find out significant measurable differences between the two formalisms[57, 58] but no encouraging case has been found yet.

5 Density Operators and the "Coarse Graining" Hypothesis

5.1 The "coarse graining" hypothesis

First of all, it is convenient to present a brief review of some topics related to the introduction of the *coarse grained* description of a physical system. This hypothesis is then going to be used to obtain a discretized form of the Liouville equation, which represents the evolution law of the density operators in the usual QM.

An important point to be remarked is that the introduction of a fundamental interval of time is perfectly compatible with a coarse grained description. The basic premise of such description, in statistical physics, is the impossibility of a precise determination of the position and momentum of each particle forming the system, in a certain instant of time. Let us consider, for the sake of simplicity, a system composed by N similar pointlike particles, each of them with three degrees of freedom described by the coordinates (q_1, q_2, q_3). We can associate with this ensemble of particles an individual phase space (named μ-space) defined by the six coordinates $(q_1, q_2, q_3; p_1, p_2, p_3)$ so that the system as a whole is represented by a crowd of points in this space.

Since the macroscopic observation is unable to precisely determine the six coordinates for each particle, let us assume that it is only possible to know if a given particle has its coordinates inside the intervals $(q_i + dq_i)$ and $(p_i + dp_i)$, with $i = 1, 2, 3$. In order to describe the state of the system in the μ-space, we divide it into cells corresponding to the macroscopic uncertainties δq_i and δp_i, each one occupying in the μ -space a volume

$$w_i = \delta q_1 \, \delta q_2 \, \delta q_3 \, \delta p_1 \, \delta p_2 \, \delta p_3 \, . \tag{98}$$

These cells must be sufficiently small related to the macroscopically measurable dimensions but also sufficiently large so as to contain a great number of particles.

When considering the system as a whole, its macroscopic state is given by a collection of points n_i corresponding to the number of particles inside each cell. Now, if we take into account the 6N-dimensional phase space Γ, in which each of the states assumed by the system is represented by a point, to each configuration n_i corresponds in Γ a cell with volume given by

$$(\delta V)_\Gamma = \prod_{n=1}^{N} (w_i)^{n_i} \, .$$

Considering that the permutation of the particles inside the cells of the Γ space does not change the macroscopic state of the system, then to each collection of numbers n_i corresponds a volume Ω_n in the Γ space [§] given by

$$W(\Omega_n) = \frac{N!}{\prod_i n_i!} \prod_i (w_i)^{n_i} \qquad \left(\sum_i n_i = N \right) .$$

[§] Jancel[62] calls it a *star*.

The state of the system is determined by the star occupied by the representative point of the system in the Γ space. This way, macroscopically, it is only possible to distinguish in which star the system is, such that any point in this star corresponds to a same macroscopic state. When we consider a system which is not in equilibrium, a change in its macroscopic state can only be observed when the point describing the system changes star. The crossing time is small but finite. During this period of time the macroscopic state of the system does not change notwithstanding its microscopic state is continuously changing.

Thus, from the point of view of statistical physics, the introduction of a fundamental interval of time appears in a very natural way. That is still more significant when we remember that the predictions of QM are always obtained as mean values of observables. The uncertainty relations, according to the usual interpretation of QM —the Copenhagen interpretation—, are independent of the arguments above. If we accept that they play a fundamental role in the microscopic world —and this is postulated by Copenhagen—, then the concept of chronon, as a fundamental interval of time, must be related to them.

5.2 Discretized Liouville equation and the time–energy uncertainty relation

An attempt to set up a relationship between the chronon and the time–energy uncertainty relation has been put forward by Bonifacio (1983)[63], extended the *coarse graining* hypothesis to the time coordinate. In the conventional QM the density operator evolves according to the Liouville–von Neumann equation

$$\frac{\partial \hat{\rho}}{\partial t} = -i\mathcal{L}\hat{\rho}(t) = -\frac{i}{\hbar}\left[\hat{H}, \hat{\rho}\right], \tag{99}$$

where $\mathcal{L}$ is the Liouville operator. It is immediate to observe that, if H is time independent, the solution is given by

$$\hat{\rho}(T) = \exp\left(-i\frac{H}{\hbar}\right)\hat{\rho}(0)\ \exp\left(i\frac{H}{\hbar}\right), \tag{100}$$

which gives the time evolution of the density operator starting from an initial time t_0, such that $T = t - t_0$ is the evolution time.

When we build a coarse grained description of the time evolution, by introducing a graining of value τ such that the evolution time is now given by $T =$

$k\tau$ $(k = 1, 2, \ldots, \infty)$, we have that the resulting density operator ρ does not satisfy the continuous equation (99) but a discretized form of it given by

$$\frac{\hat{\rho}(t) - \hat{\rho}(t-\tau)}{\tau} = -i\mathcal{L}\hat{\rho}(t), \tag{101}$$

with $t = k\tau$, which reduces to the Liouville–von Neumann equation when $\tau \to 0$. In the energy representation $|n\rangle$, once satisfied certain conditions which ensure that $\rho(k)$ is a density operator, we have that (101) rules for ρ an evolution which preserves trace, obeys the semigroup law and is an irreversible evolution towards a stationary diagonal form. In other words, we observe a *reduction of state* in the same sense as in the measurement problem of QM. This *reduction* is not instantaneous and depends on the characteristic value τ:

$$\rho(t) \overset{t\to 0}{\to} \sum_n \rho_{nn}(0)|n\rangle\langle n| \ .$$

It is important to observe that the non diagonal terms tend exponentially to zero according to a factor which, to the first order, is given by

$$\exp\left|\frac{-\omega_{nm}^2 \tau t}{2}\right|.$$

Thus, the reduction to the diagonal form occurs provided we have a finite value for τ, no matter how small, and provided we do not have $\omega_{nm}\tau \ll 1$ for every n,m, where $\omega_{nm} = (E_n - E_m)/\hbar$ are the transition frequencies between the different energy eigenstates. This later condition is always satisfied for systems not bounded.

These results, together with an analysis of the discrete Heisenberg equation defined in terms of the average values of observables

$$\bar{A}(t) = \mathrm{Tr}\left(\rho(t)\ \hat{A}\right)$$

in the *coarse grained* description, suggest an interpretation of τ in terms of the uncertainty relation $\Delta E \Delta t \geq \hbar/2$ such that τ is a characteristic interval of time satisfying the inequality

$$\tau \geq \tau_E \equiv \frac{\hbar}{2\Delta E} \qquad \text{with} \qquad \Delta E = \sqrt{\langle H^2\rangle - \langle H\rangle^2}\ , \tag{102}$$

$$\hat{U}_{\text{ret}}(t,t_0) =$$

$$= \exp\left[-\frac{(t-t_0)}{2\tau}\ln\left(1+\frac{\tau^2\hat{H}^2}{\hbar^2}\right)\right]\exp\left[-\frac{i(t-t_0)}{\tau}\tan^{-1}\left(\frac{\tau\hat{H}}{\hbar}\right)\right]. \tag{124}$$

Comparing (122) and (124) we obtain the equivalence of the hamiltonians once $\hat{\nu}$ and $\hat{\kappa}$ are given by

$$\hat{\nu} = \frac{\hbar}{\tau}\tan^{-1}\left(\frac{\tau\hat{H}}{\hbar}\right),$$

$$\hat{\kappa} = -\frac{\hbar}{2\tau}\ln\left(1+\frac{\tau^2\hat{H}^2}{\hbar^2}\right).$$

For the advanced case we obtain the same expressions except by a minus sign for $\hat{\kappa}$. For the symmetric case, below the critical limit, we have

$$\hat{\nu} = \frac{\hbar}{\tau}\sin^{-1}\left(\frac{\tau\hat{H}}{\hbar}\right),$$

$$\hat{\kappa} = 0.$$

Above that limit $\hat{\nu}$ ceases to be hermitian and, in this case, the evolution operator can be written as

$$\hat{U}_{\text{sym}}(t,t_0) = \exp\left[-\frac{i\pi}{2\tau}(t-t_0)\right]\exp\left\{-\frac{(t-t_0)}{\tau}\ln\left[\left|\frac{\tau\hat{H}}{\hbar}\right| + \sqrt{\left(\frac{\tau\hat{H}}{\hbar}\right)^2 - 1}\right]\right\}$$

so that

$$\hat{\nu} = \frac{\hbar\pi}{2\tau},$$

$$\hat{\kappa} = -\frac{\hbar}{\tau}\ln\left[\left|\frac{\tau\hat{H}}{\hbar}\right| + \sqrt{\left(\frac{\tau\hat{H}}{\hbar}\right)^2 - 1}\right],$$

with $\hat{\nu}$ being now independent of the hamiltonian and $\hat{\kappa}$ ceases to be zero.

The expressions obtained above show the characteristics that $\hat{\nu}$ and $\hat{\kappa}$ must fulfil in order that the continuous equation reproduces the outputs of the discretized equations. By observing the continuous evolution operator we have that the anti-hermitian part of $\tilde{H}$ shows a non-stationary behaviour, resulting in a damping or amplifying term associated with the evolution of the quantum state. Thus, the stationary solutions appear only for the symmetric case below the critical limit. In all the other cases the transient term always appears.

In QM, the non-hermitian operators have been used only as mathematical shortcuts, as in the case of the Lippmann–Schwinger equation in the scattering theory. It has already been observed that the introduction of such operators could make possible the description of unstable states, by phenomenologically linking the transient factor to the lifetime of the considered stated.[56] If in a certain instant $t_0 = 0$ the system is in one of the eigenstates $|n\rangle$ of the hamiltonian $\hat{H}$ then, if such state is unstable, the probability of the system to be found in the same state in a later instant t is

$$P_n(t) = \left|\langle n|\hat{U}^\dagger\hat{U}|n\rangle\right| = \exp\left(-t/\tau_L\right),$$

and that allows us to specify a lifetime τ_L, for the retarded case, as being

$$\tau_L = \frac{\tau}{\ln\left(1 + \frac{\tau^2 E_n^2}{\hbar^2}\right)}, \tag{125}$$

and for the symmetric case, above the critical energy,

$$\tau_L = \frac{\tau}{2\ln\left(\left|\frac{\tau E_n}{\hbar}\right| + \sqrt{\frac{\tau^2 E_n^2}{\hbar^2} - 1}\right)}.$$

Such lifetimes are connected with states that, in the discretized formalism, are intrinsically unstable. Only the retarded equation seems to be associated with quantum states which decay with time. If that is really valid, we have an expression which could be used for phenomenologically determining the value of the chronon. Finally, we can conclude that the time discretization brings forth a formalism which, even if only hermitian hamiltonians are involved, is equivalent to the introduction of non-hermitian operators in the continuous QM.

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