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HYDRODYNAMICAL REFORMULATION AND QUANTUM LIMIT OF THE BARUT–ZANGHI THEORY[(†)]

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Abstract — One of the most satisfactory pictures for spinning particles is the Barut-Zanghi (BZ) classical theory for the relativistic extended-like electron, that relates spin to *zitterbewegung* (zbw). The BZ motion equations constituted the starting point for recent works about spin and electron structure, co-authored by us, which adopted the Clifford algebra language. This language results to be actually suited for a hydrodynamical reformulation of the BZ theory. Working out a "probabilistic fluid", we are allowed to reinterpret the original classical spinors as quantum wave-functions for the electron. We can pass to "quantize" the BZ theory: by employing this time the tensorial language, more popular in first-quantization. "Quantizing" the BZ theory, however, does *not* lead to the Dirac equation, but rather to a non-linear, Dirac–like equation, which can be regarded as the actual "quantum limit" of the BZ classical theory. Moreover, a new variational approach to the the BZ probabilistic fluid shows that it is a typical "Weyssenhoff fluid", while the Hamilton-Jacobi equation (linking mass, spin and zbw frequency together) appears to be nothing but a special case of the de Broglie energy–frequency relation. Finally, after having discussed the remarkable relation existing between the gauge transformation U(1) and a general rotation on the spin plane, we clarify and comment on the two-valuedness nature of the fermionic wave-function, as well as on the parity and charge conjugation transformations.

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1 The Barut–Zanghi theory for the classical sp ning electron

Since the works by Compton,[1] Uhlenbeck and Goudsmith,[2] and Frenkel,[3] many sical models of spin and classical theories of the electron have been investigate about seventy years.[4–6] For instance, Schrödinger's suggestion[7] that the electron was related to *zitterbewegung* (zbw) did originate a large amount of subsequent including Pauli's. The zbw is actually the spin motion, or "internal" motion —sinc observed in the center-of-mass frame (CMF)— expected to exist for spinning part It arises because the motion of the electrical charge does *not* coincide with the m of the particle CM. In the Dirac theory, indeed, the velocity and impulse operat and $\widehat{\boldsymbol{p}}$ are not parallel:

$$\widehat{\boldsymbol{v}} \neq \widehat{\boldsymbol{p}}/m \; .$$

So a zbw is to be added to the usual drift, translational, or "external", motion o CM with velocity $\boldsymbol{p}/m$ (which is the only one present in the case of scalar parti In the Barut–Zanghi (BZ) theory,[8,9] the classical electron was actually characte besides by the usual pair of conjugate variables (x^μ, p^μ), by a second pair of conj classical *spinorial* variables $(\psi, \overline{\psi})$, representing internal degrees of freedom, whic functions of the (proper) time τ measured in the electron CMF; the CMF bein one —let us remind— in which $\boldsymbol{p} = 0$ at any instant of time. Barut and Zang introduced, namely, a classical lagrangian that in the free case (i.e., when the *exter* electromagnetic potential A^μ vanishes) writes $[c = 1]$

$$\mathcal{L} = \frac{1}{2} i\lambda(\dot{\overline{\psi}}\psi - \overline{\psi}\dot{\psi}) + p_\mu(\dot{x}^\mu - \overline{\psi}\gamma^\mu\psi) \; ,$$

where λ has the dimension of an action, and ψ and $\overline{\psi} \equiv \psi^\dagger\gamma^0$ are ordinary $\mathbb{C}^4$–bispi the dot meaning derivation with respect to τ. The four Euler–Lagrange equations, $-\lambda = \hbar = 1$, yield the following motion equations:

$$\begin{cases} \dot{\psi} + ip_\mu\gamma^\mu\psi = 0 \\ \dot{x}^\mu = \overline{\psi}\gamma^\mu\psi \\ \dot{p}^\mu = 0 \; , \end{cases}$$

besides the hermitian adjoint of eq.(2a), holding for $\overline{\psi}$. From eq.(2) one can also see that

$$H \equiv p_\mu v^\mu = p_\mu \overline{\psi} \gamma^\mu \psi \tag{4}$$

is a constant of the motion [and precisely is the energy in the CMF].[8] Since H is the BZ hamiltonian in the CMF, we can suitably set $H = m$, where m is the particle rest-mass. The general solution of the equations of motion (3) can be shown to be:

$$\psi(\tau) = \left[\cos(m\tau) - i\frac{p_\mu \gamma^\mu}{m}\sin(m\tau)\right]\psi(0) \; , \tag{5a}$$

$$\overline{\psi}(\tau) = \overline{\psi}(0)\left[\cos(m\tau) + i\frac{p_\mu \gamma^\mu}{m}\sin(m\tau)\right] \; , \tag{5b}$$

with $p^\mu =$ constant; $p^2 = m^2$; and finally:

$$v^\mu \equiv \dot{x}^\mu = \frac{p^\mu}{m} + \left[\dot{x}^\mu(0) - \frac{p^\mu}{m}\right]\cos(2m\tau) + \frac{\ddot{x}^\mu(0)}{2m}\sin(2m\tau) \; . \tag{5c}$$

This general solution exhibits the classical analogue of the zbw: in fact, the velocity v^μ contains the (expected) term p^μ/m plus a term describing an oscillating motion with the characteristic zbw frequency $\omega = 2m$. The velocity of the CM will be given by p^μ/m. Let us explicitly observe that the general solution (5c) represents a helical motion of an "internal constituent" $\mathcal{Q}$ in the ordinary 3-space: a result that has been met also by means of other, alternative approaches.[8]

Notice that, instead of adopting the variables ψ and $\overline{\psi}$, we can work in terms of the "spin variables", i.e., in terms of the set of dynamical variables

$$x^\mu \, , \; p^\mu \, ; \; v^\mu \, , S^{\mu\nu} \tag{6}$$

where

$$S^{\mu\nu} \equiv \frac{i}{4}\overline{\psi}[\gamma^\mu, \gamma^\nu]\psi \tag{7}$$

is the particle spin tensor; then, we would get the following motion equations:

$$\begin{cases} \dot{p}^\mu = 0 & (8a) \\ \dot{x}^\mu = v^\mu & (8b) \\ \dot{v}^\mu = 4\, S^{\mu\nu} p_\nu & (8c) \\ \dot{S}^{\mu\nu} = v^\nu p^\mu - v^\mu p^\nu \,. & (8d) \end{cases}$$

The last equation expresses the conservation of the total angular momentum $J^{\mu\nu}$, sum of the orbital angular momentum $L^{\mu\nu}$ and of $S^{\mu\nu}$:

$$\dot{J}^{\mu\nu} = \dot{L}^{\mu\nu} + \dot{S}^{\mu\nu} = 0 \,, \tag{9}$$

it being $\dot{L}^{\mu\nu} = v^\mu p^\nu - v^\nu p^\mu$ from the very definition of L. In the last two refs.[9] it was found that free polarized particles (i.e., with the spin projection s_z equal to $\pm\frac{1}{2}$) are endowed with internal *uniform circular* motions around the z-axis. In such a way, the only *classical* values for s_z corresponding to classical uniform motions in the CM frame (just the ones expected for free particles) belong to the discrete *quantum* spectrum $\pm\frac{1}{2}$. The radius of the orbit in the CM frame was found to be equal to $|\boldsymbol{V}|/2m$ (quantity $\boldsymbol{V}$ being the orbital 3-velocity), which, in the special case of a *light-like* zbw, turns out to be equal to half the Compton wave-length. Subsequently the Euler–Lagrange equations were generalized for the case of an electron in an electromagnetic field, and the analytical solutions of the motion equation in the special case of a uniform magnetic field were written down.

The BZ theory has been recently studied also in the lagrangian and hamiltonian symplectic formulations, both in flat and in curved spacetimes.[8]

2 Hydrodynamics and quantum limit of the BZ theory

The *Multivector* or *Geometric* Algebras are essentially due to the work of great mathematicians of the nineteenth century as Hamilton (1805–1863), Grassmann (1809–1977) and Clifford (1845–1879). Starting from the sixties, some authors, and in particular Hestenes,[10] systematically studied the interesting physical applications of such algebras, and especially of the *Real Dirac Algebra* $\mathbb{R}_{1,3}$, renamed formal *Space-Time Algebra*

(STA).[10] In microphysics, we can meet applications for the case of space-time [O(3), Lorentz] transformations, gauge [SU(2), SU(5), strong and electroweak isospin] transformations, chiral [$SU(2)_L$] transformations, Maxwell equations, magnetic monopoles[11], and so on. The most rigorous application is probably the (formal and conceptual) analysis of the geometrical, kinematical and *hydrodynamical* content of the Pauli and Dirac equations, performed by means of the Real Pauli and the Dirac Algebras, respectively. We are now going to see that, not only for the Dirac probabilistic "fluid", but also for the BZ probabilistic fluid, the Clifford Algebras language results to be actually suited and fruitful. We shall obtain in the next Sections the hydrodynamical (often said *local*, or *field*) formulation in the STA of the BZ theory. In this Section we first get the field equations for the BZ electron; then, by translating from the STA into the standard algebra, we work out a "quantization" of the BZ theory. As a consequence, we arrive at a non-linear Dirac-like wave-equation, which can be actually regarded as the "quantum limit" of the BZ classical theory. Finally, in the next Section, a variational STA approach to the BZ "fluid" will lead us to the conclusion that it is a typical "Weyssenhoff fluid".[12]

The *translation* of the single terms of lagrangian (1) into the STA language can be performed as follows:[9]

$$
\begin{array}{lcl}
\frac{1}{2}i(\dot{\overline{\psi}}\psi - \overline{\psi}\dot{\psi}) & \longrightarrow & \langle\widetilde{\psi}\dot{\psi}\gamma_1\gamma_2\rangle_0 \\
\pi_\mu(\dot{x}^\mu - \overline{\psi}\gamma^\mu\psi) & \longrightarrow & \langle\pi(\dot{x} - \psi\gamma_0\widetilde{\psi})\rangle_0 \\
eA_\mu\overline{\psi}\gamma^\mu\psi & \longrightarrow & e\,\langle A\psi\gamma_0\widetilde{\psi}\rangle_0 \ ,
\end{array}
$$

where, as usual, $\pi \equiv p - eA$ is the kinetic momentum, ψ indicates the so-called *Dirac real spinor*, whilst $\langle\ \ \rangle_0$ represents the so-called *scalar part of the Clifford product.*[9,10] In such a way, our lagrangian becomes:

$$\mathcal{L} \ = \ \langle\widetilde{\psi}\dot{\psi}\gamma_1\gamma_2 \ + \ \pi(\dot{x} - \psi\gamma_0\widetilde{\psi}) \ + \ eA\psi\gamma_0\widetilde{\psi}\rangle_0 \ . \qquad (10)$$

The Eulero–Lagrange equations now read:

$$\dot{\psi}\gamma_1\gamma_2 + \pi\psi\gamma_0 \ = \ 0 \qquad (11a)$$

$$\dot{x} = \psi\gamma_0\widetilde{\psi} \tag{11b}$$

$$\dot{\pi} = eF\cdot\dot{x} \tag{11c}$$

where $F \equiv \partial\wedge A$ is the electromagnetic field *bivector*.

In view of a quantum interpretation of the BZ theory, we need a formulation and analysis of it in terms of spinors, which be no longer functions of τ, but instead of x^μ (spinorial *fields* $\psi(x)$). At the same time the BZ theory gets a conceptual (and not merely formal) "extension". In fact, it will result to describe a "fluid" which admits a *probabilistic interpretation*; its integral stream-lines# coinciding with the single (semi-)classical world-lines (up to now parametrized in terms of τ) of the point-like electron charge $\mathcal{Q}$. Beside the spinorial field $\psi(x), \overline{\psi}(x)$, we shall meet also the fields $v(x)$, $p(x)$, $\boldsymbol{s}(x)$, $S^{\mu\nu}(x)$, which will replace the corresponding functions of τ. And the spinorial field $\psi(x)$ will be such that its *restriction* $\psi(x)_{|\sigma}$ to the world-line σ (along which the particle moves) coincides with $\psi(\tau)$. At the same time, the velocity distribution $V(x)$ is required to be such that its restriction $V(x)_{|\sigma}$ to the world-line σ results to be just the ordinary 4-velocity $v(\tau)$ of the considered particle. Therefore, for the tangent vector along any line σ, the relevant relation holds:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\partial}{\partial x^\mu} \equiv \dot{x}^\mu\partial_\mu\ . \tag{12}$$

Eq.(12) is nothing but the relativistic form of the well-known equation found in the non-relativistic theory of fluids, and linking each other the "eulerian" (or "spatial") and the "lagrangian" ("material") velocities:

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla}\ .$$

Inserting the total derivative (12) into the Euler–Lagrange equation (11a), we get:

$$v\cdot\partial\psi\gamma_1\gamma_2 + \pi\psi\gamma_0 = 0\ , \tag{13}$$

We refer to *congruence* of world-lines.

and, it being $\dot{x} = \psi\gamma_0\widetilde{\psi}$ because of eq.(11b), we finally obtain the following noticeable equation:

$$(\psi\gamma_0\widetilde{\psi}) \cdot \partial\psi\gamma_1\gamma_2 + \pi\psi\gamma_0 \; = \; 0 \; . \tag{14}$$

Equation (14) expresses the "field" content of the BZ theory. Incidentally, let us notice that, differently from eq.(11a), equation (14) can be valid a priori even for massless spin $\frac{1}{2}$ particles, since the CMF proper time does not enter it any longer. We see also that, since the restriction of $\psi(x)$ to the world-line σ coincides with $\psi(\tau)$, the velocity field $V(x) \equiv \psi(x)\gamma_0\widetilde{\psi}(x)$ results to be endowed with the same zbw as found for $v(\tau)$ (with the oscillation $m\tau$ suitably replaced by the equivalent quantity $p \cdot x$, in the free-case).

The "quantum reformulation" of the classical BZ theory essentially consists in:

i) reinterpreting the spinor field $\psi(x)$ in eq.(14) as the proper *wave-function* Ψ for the spinning particle (or also, in a second quantization formulation, as the creation operator $\widehat{\Psi}$ for the particle quantum field);

ii) reinterpreting the original BZ theory as the *classical limit* of the non-linear *quantum wave-equation* we are going to get (see below).

Obviously, for the probabilistic reinterpretation i), we must consider the bilinear quantity $\psi\widetilde{\psi}$ as a *probability density*. Let us now translate eq.(14) into the ordinary tensorial language (limiting ourselves for simplicity to the free case), so that ψ does loose the direct geometrical meaning owned within the STA, playing on the contrary the customary rôle of a wave-function in quantum mechanics. By means of the usual rules linking the STA and the standard algebra,[9,10]

$$\psi\gamma_0\widetilde{\psi}\gamma_\mu \;\longrightarrow\; \overline{\Psi}\gamma_\mu\Psi \tag{15a}$$

$$\partial_\mu\psi\gamma_2\gamma_1 \;\longrightarrow\; i\partial_\mu\Psi \tag{15b}$$

$$p\gamma_0\psi \;\longrightarrow\; m\Psi \tag{15c}$$

we straightforwardly get the following *non-linear* Dirac–like equation, the "quantum limit" of the BZ theory:

$$i\overline{\Psi}\gamma^{\mu}\Psi\partial_{\mu}\Psi \;=\; m\Psi \; . \qquad (16)$$

Let us remark that the non-linearity with respect to Ψ was already present in the original Euler–Lagrange equations, because of eq.(11b), in which the bilinear quantity $\psi\gamma_0\widetilde{\psi}$ first appeared. "Quantizing" the BZ theory, therefore, does *not* lead to the Dirac equation, but rather to such a non-linear, Dirac–like equation.

3 Variational approach to the BZ fluid

By means of Clifford algebras, we can now work out, in a variational (lagrangian) context, the set of equations necessary for a complete description of the BZ–fluid hydrodynamics. In STA a Dirac spinor may be written as follows:

$$\psi \;=\; \sqrt{\rho}\,\mathrm{e}^{\frac{1}{2}\beta\gamma_5}R_0\,\mathrm{e}^{\hbar\gamma_1\gamma_2\varphi}$$

The scalar ρ, the "proper density", works as a normalization factor; the scalar β is the "Takabayasi angle"; γ_5 indicates the pseudoscalar unit of the STA; and the scalar φ may be considered as a "generalized" spinor phase. The bivector $R_0 \equiv R_0(S)$ depends only on the bivector S. The constant bivector $\hbar\gamma_2\gamma_1$ is generally associated with "the oriented *spin plane*", so that in a generic frame we have $S = \frac{1}{2}R_0\hbar\gamma_2\gamma_1\widetilde{R}_0$, where by $\widetilde{R}_0$ we indicate the result of the "reversion" operation on R_0. [10] Starting from lagrangian (2), after some algebra, we obtain:

$$\mathcal{L} \;=\; \rho\cos\beta(\Omega\cdot S - \dot{\varphi}) + p\cdot(\dot{x} - \rho v) \qquad (17)$$

where (as functions of the chosen lagrangian variables $\rho, \varphi, \beta, R_0$) the following relations hold

$$\Omega \equiv 2\dot{R}\widetilde{R} \equiv 2\dot{R}_0\widetilde{R}_0 \qquad \Omega\cdot S \equiv \dot{R}_0\hbar\gamma_2\gamma_1\widetilde{R}_0 \qquad v \;\equiv\; R_0\gamma_0\widetilde{R}_0 \; , \qquad (18)$$

Ω being the so-called *angular velocity* bivector.[10] Notice that lagrangian (17) results to be the sum of two expressions which vanish (yielding in this way the equations of the motion), both multiplied by suitable "Lagrange multipliers" ($\rho\cos\beta$ and p).

Let us now take the variations with respect to $\rho, \varphi, \beta, R_0$; we shall obtain in the same order the following equations:

A) *The "Hamilton–Jacobi" equation:*

$$p \cdot v \;=\; \Omega \cdot S \cos\beta \;.$$

As it is easy to verify, it holds[9] $\beta = 0$ for the solution (5a) of the BZ theory (suitably translated into the STA), so that we may take $\cos\beta = 1$. In such a way, our first equation shows the *kinematico–geometrical content of the* celebrated *de Broglie's relation* $E = \hbar\omega$:

$$p \cdot v = E_{\mathrm{CMF}} = \Omega \cdot S \equiv \boldsymbol{\omega} \cdot \boldsymbol{s} = \frac{1}{2}\hbar\omega = \hbar\omega_\psi \;, \tag{19}$$

were we indicated by $\omega = 2m$ the zbw motion frequency [that is, the frequency appearing in the motion equation (5c)]; and by $\omega_\psi = m$ the frequency of the spinor ψ, appearing in the solution (5a) [and corresponding to the frequency of the wave-function, after our "quantum reinterpretation"].

A wave-plane is a mathematical device found in the quantum formalism which is not endowed with a direct, intuitive physical meaning; and the Planck constant, appearing throughout the quantum theory, is not deduced from a physical context, but is required *a priori* and inserted "by hand". *It is therefore noticeable the possibility of replacing, in the de Broglie relation, the wave-plane frequency by the zbw motion frequency, as well as the Planck constant by the spin.* Starting from the interpretation of $\hbar/2$ as actually meaning $|\boldsymbol{s}|$, one of the present authors has recently deduced the so-called "quantum potential" of the Madelung fluid as being the kinetical energy of the zbw.[13] Let us here only notice that expressing the mass m in the form $\boldsymbol{\omega} \cdot \boldsymbol{s}$ seems to denounce the origin of the particle mass as due to a sort of "rotational kinetic–energy".[10]

B) *The continuity equation:*

$$\dot{\rho} \;=\; 0 \;.$$

As expected, along a stream-line the flux density is constant in time.

C) *A correlation between spinorial phase and angular velocity:*

$$\dot{\varphi} \; = \; \Omega \cdot S \equiv \tfrac{1}{2}\omega$$

By integrating this equation, we obtain a simple proportionality relation between the variations of the spinorial phase angle and of the zbw-plane rotation angle:

$$\Delta\varphi \; = \; \frac{1}{2}\Delta\vartheta \; . \tag{20}$$

In such a way, the so-called U(1) *gauge transformations get a straightforward and clear kinematico-geometrical meaning.* They indeed may be regarded not just as rotations in an abstract space (namely, the Gauss plane of the complex spinors), but actually as spatial rotations in the *physical* spin plane. Thus the *electromagnetic gauge invariance* —global or local as it be— owned by the currents, the wave-equations, and the interaction lagrangians, *means*, as a matter of fact, *that currents, energies and forces are independent of the instantaneous angular position of the point-like charge.*

D) *The total angular momentum conservation:*

$$(\Omega \wedge S)\cos\beta = \dot{S}\cos\beta = p \wedge v$$

(the Lagrange equation obtained by the variation with respect to R_0 has been multiplied on the left by R_0, so that it has been singled out the 2-vectorial part)
This is equivalent in Clifford Algebra to the tensorial equation (8d).

In our Hamilton–Jacobi equation (case A)) it *does not appear a "quantum potential"*, which is quite present, on the contrary, in the Dirac fluid.[10] Moreover, the total angular momentum *is locally conserved*, whilst this is not the case for the Dirac fluid. In conclusion, we can state that the present hydrodynamics is that of a typical Weyssenhoff fluid.

4 Operations on spinors and rotations in the spin plane

In this section we are going to point out the interesting relation existing between some important transformations (acting on spinors) and the orbital zbw motion of the electric pointlike charge $\mathcal{Q}$ in the spin plane. In the previous section, we discussed about the remarkable relation existing between the gauge transformation U(1) and a general rotation on the spin plane; in that follows we shall deal with the two-valuedness nature of the fermionic wave-function, and with the parity and charge conjugation transformations.

As is well-known, in the standard framework of quantum wave-mechanics the sign of the fermion wave-function —at variance with the scalar particles case— does change if we make a 360°-rotation of the reference frame around an arbitrary axis. We can really get *a quite simple and natural classical interpretation* of this fact in the framework of the present BZ theory. Without any recourse to fibre-bundles or to other topological tools, we shall succeed in understanding why the phase of the quantum final state varies even if the final particle position remains unchanged. As seen above, in the BZ model the phase of the wave-function is strictly connected to a kinematico-geometrical quantity: the phase angle of the position of Q in the spin plane. Now, a 360°-rotation around the z-axis of the coordinate frame ("passive point of view") is fully equivalent to a 360°-rotation, around the same axis, of our microsystem, and therefore of the rotating charge ("active point of view"). On the other hand, as a consequence of the last-mentioned transformation, the zbw angle $2m\tau$ in $x(\tau)$, eq.(5c), suffers a variation of 360° and the proper time τ increases by a zbw period $T_{zbw} = \pi/m$, which is exactly what happens when Q performs a complete circular orbit around the z-axis. But, because the period

$T_\psi = 2\pi/m$ of the spinor $\psi(\tau)$ in (5a) is twice as big as the period T_{zbw} of the zbw motion, such a spinor *results to suffer a phase increment of* 180° only, and then does change sign ($e^{i\pi} \equiv -1$); so as it does in standard quantum mechanics.

Analogous considerations can be performed in connection with the *parity transformation.* In this case the "active" operation consists in the mirror reflection of $\mathcal{Q}$ around the origin of the cartesian axes. In the CMF it is equivalent, once the motion–plane is the xy-plane, to a 180°-rotation of $\mathcal{Q}$ around the z-axis. Once again, a 180°-rotation of $\mathcal{Q}$ implies a 90° spinor-phase variation only, since $T_\psi = 2T_{zbw}$. Now, if we take $m\tau = \pi/2$ in the spinor solution (5a), we immediately get (in the CMF)

$$\psi(\pi/2) = -i\gamma_0\psi(0) \; . \tag{21}$$

Let us recall[14], at this point, that the *parity operator* in Dirac wave-mechanics is nothing but $\widehat{P} = \pm i\gamma_0$ (the choice of the sign being actually arbitrary). In this way, once again, the formal features of such a quantum operator get a simple meaning in the classical context of the BZ theory. The (previously) intrinsic non-intuitive property of the electron state vector, for which the double application of a parity operation is *not* an identity —indeed we have $\widehat{P}^2 \equiv -\mathbb{1}$—, *is now related to the peculiar fact that parity really corresponds to a 90°-rotation for* $\mathcal{Q}$.

Finally, let us consider the basic transformation of relativistic quantum mechanics: charge conjugation. If we associate as usual the negative energies (when assuming $m < 0$ in the present CM frame) with antiparticles, we shall have for the particle case a simple *inversion of the rotation direction*, the other kinematical features of the motion remaining unchanged. All this is an immediate consequence of the motion equation (5c), where only the sign of the odd function $\sin(2m\tau)$ changes when we make the transformation $m \longrightarrow -m$, whilst the even function $\cos(2m\tau)$ does not change. Therefore, for free polarized particles,[9] the condition $s_z = +\frac{1}{2}$ does imply anti-clockwise and clockwise circular uniform motions, for electrons and positrons respectively. This result seems to agree with the interpretation of the antiparticle states as "time-inverted" (and "energy inverted"!) states (for such an interpretation within the classical context see refs.[15] and refs. therein).

Analogously to what seen above, it is possible in the present approach to forward a simple classical deduction of the *relative fermion-antifermion parity* P_r, which is known to be equal to -1. In fact, the phase of the state vector for the electron-positron system, which is a factorization of the electron and positron wave-functions, suffers a total variation of 180°. This because, while the particle state vector, under parity, results rotated by a +90°-angle, the antiparticle vector is instead rotated by the same magnitude: but, as we have seen before, *in the opposite direction*, that is by a -90°-angle.

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