

**BEAM OPTICS OF THE DIRAC PARTICLE WITH ANOMALOUS
MAGNETIC MOMENT**

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Abstract

Beam optics of a spin- $\frac{1}{2}$ particle with anomalous magnetic moment is studied in the monoenergetic and paraxial approximations based on the Dirac equation; the treatment is at the level of single-particle dynamics, considers the electromagnetic field as classical and disregards radiation aspects. The general theory, developed for any magnetic optical element with straight axis, is illustrated by computing the transfer maps for the spin components and the transverse phase-space, including the transverse Stern-Gerlach kicks, in the case of a normal magnetic quadrupole lens. The transfer map for spin components embodies an accelerator optical and quantum mechanical version of the quasiclassical Thomas-Bargmann-Michel-Telegdi (Thomas-BMT) equation. The longitudinal Stern-Gerlach kick in a general inhomogeneous magnetic field is also discussed.

1. Introduction

We present an approach to the quantum theory of accelerator optics for a spin- $\frac{1}{2}$ particle with anomalous magnetic moment, including the spin evolution, at the level of single-particle dynamics and disregarding radiation aspects, based, *ab initio*, on the Dirac equation. Here, we are concerned only with monoenergetic and paraxial beams. The electromagnetic field is treated as classical.

As is well known, the present understanding of accelerator beam optics is based mainly on classical mechanics and electrodynamics (see, *e.g.*, [1] and references therein). The main framework for studying the spin dynamics and beam polarization is essentially based on the well known quasiclassical Thomas-Bargmann-Michel-Telegdi (Thomas-BMT) equation (see, *e.g.*, [2]), though the quantized nature of radiation is taken into account (see, *e.g.*, [3]-[5]) to understand the influence of the radiation effects on the orbital and spin motions (fluctuations from the classical behaviour), particularly for electrons and positrons. The Thomas-BMT equation has been understood on the basis of the Dirac equation, independent of the beam optics, in different ways (see [3] and references therein). Understanding the orbital motion in an axially symmetric focusing magnetic field by solving the Dirac equation has also been attempted [3]. Quantum mechanical implications for low energy polarized (anti)proton beams in a spin-splitter device ([6]-[10]), using the transverse Stern-Gerlach kicks, have been analysed on the basis of the nonrelativistic Schrödinger equation [11],[12]. But, so far, in the realm of accelerator optics there does not seem to have been any serious attempt to understand the quantum mechanics of both the orbital and spin dynamics

of spin- $\frac{1}{2}$ particle beam based on a single unified framework derived from the standard Dirac theory. The main aim of this paper is to initiate the development of such an approach.

Independently, an algebraic approach to the quantum theory of electron optics (or charged-particle optics, in general) has been under systematic development using the Dirac, Klein-Gordon and the nonrelativistic Schrödinger equations ([13]-[18]) (see also [19] for a formal scalar quantum theory of electron optics with a Schrödinger-like basic equation in which the beam emittance plays the role of Planck's constant \hbar , and [20] for a path integral approach to the optics of Dirac particles). In Refs. [13]-[15] and [18], developing the spinor theory of electron optics mainly with applications to high voltage micro-electron-beam devices in mind (see [21] for the traditional approach to the quantum mechanics of electron optics), the spin dynamics has not been explicitly considered. Here, closely following [13]-[15] and [18], we present an approach to understand the accelerator beam optics, at the level of single-particle dynamics, neglecting radiation effects, and treating the evolutions of the transverse phase-space, longitudinal momentum and the spin components within a unified framework : the main point here is that in this approach the spin dynamics, responsible for both the Stern-Gerlach kicks and the Thomas-BMT spin precession, and the orbital dynamics follow uniquely as parts of a single dynamics obtained directly from the canonical Dirac equation without having to add an effective semiclassical spin Hamiltonian [22] to a semiclassical orbital Hamiltonian.

In Section 2 we present the general framework of our theory for any arbitrary magnetic optical element with straight axis; details are given for the

first order (paraxial) approximation. In Section 3 we illustrate the general theory by computing the transfer maps for the spin components and the transverse phase-space, including the transverse Stern-Gerlach kicks, in the case of a normal magnetic quadrupole lens and also discuss briefly the longitudinal Stern-Gerlach kick [23] in a general inhomogeneous magnetic field.

2. Beam optics of the Dirac-particle : General theory for a magnetic optical element with straight axis

We are interested in studying the spin dynamics and optics of an almost monoenergetic paraxial Dirac-particle beam transported through a magnetic optical element with straight axis comprising the static field $\mathbf{B} = \text{curl } \mathbf{A}$ associated with a vector potential \mathbf{A} . Let us consider the Dirac particle to have mass m , charge q and anomalous magnetic moment μ_a . The beam propagation is governed by the stationary Dirac equation

$$H_D |\psi_D\rangle = E |\psi_D\rangle, \quad (2.1)$$

where $|\psi_D\rangle$ is the time-independent 4-component Dirac spinor, E is the energy of the beam particle and the Hamiltonian H_D , including the Pauli term is given by

$$H_D = \beta mc^2 + c\boldsymbol{\alpha} \cdot (-i\hbar\nabla - q\mathbf{A}) - \mu_a\beta\boldsymbol{\Sigma} \cdot \mathbf{B},$$

$$\begin{aligned} \beta &= \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \end{pmatrix}, \\ \mathbf{1} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ \sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (2.2)$$

It should be noted that we are dealing with the scattering states of the time-independent Hamiltonian H_D with conserved positive energy

$$E = +\sqrt{m^2c^4 + c^2p^2}, \quad p = |\mathbf{p}|, \quad (2.3)$$

where \mathbf{p} is the momentum of the beam particle entering the system from the field-free input region. Let the system have its straight optic axis along the z -direction. We shall consider the beam to be paraxial and moving along the positive z -direction such that for any constituent particle of the beam

$$p \approx p_z > 0, \quad |p_x| \ll p, \quad |p_y| \ll p. \quad (2.4)$$

We shall use the right handed Cartesian coordinate system with z pointing along the design trajectory, y as the vertical coordinate and x as the horizontal transverse coordinate.

Since we want to know the changes in the beam parameters along the optic axis of the system (*i.e.*, the $+z$ -direction) we have to study the Dirac equation (2.1) rewritten as

$$i\hbar \frac{\partial}{\partial z} |\psi_D\rangle = \mathcal{H}_D |\psi_D\rangle. \quad (2.5)$$

To this end, we multiply (2.1) from left by α_z/c and rearrange the terms to get the desired form (2.5) : The result is that

$$\mathcal{H}_D = -p\beta\chi\alpha_z - qA_z\mathbb{1} + \alpha_z\alpha_{\perp} \cdot \hat{\pi}_{\perp} + (\mu_a/c)\beta\alpha_z\boldsymbol{\Sigma} \cdot \mathbf{B},$$

$$\begin{aligned} \chi &= \begin{pmatrix} \xi\mathbb{1} & \mathbf{0} \\ \mathbf{0} & -\xi^{-1}\mathbb{1} \end{pmatrix}, \quad \mathbb{1} = \begin{pmatrix} \mathbb{1} & \mathbf{0} \\ \mathbf{0} & \mathbb{1} \end{pmatrix}, \quad \xi = \sqrt{\frac{E + mc^2}{E - mc^2}}, \\ \hat{\pi}_{\perp} &= (-i\hbar\nabla_{\perp} - q\mathbf{A}_{\perp}) = (\hat{\mathbf{p}}_{\perp} - q\mathbf{A}_{\perp}). \end{aligned} \quad (2.6)$$

Noting that, with

$$M = \frac{1}{\sqrt{2}}(\mathbb{1} + \chi\alpha_z), \quad M^{-1} = \frac{1}{\sqrt{2}}(\mathbb{1} - \chi\alpha_z), \quad (2.7)$$

one has

$$M(\beta\chi\alpha_z)M^{-1} = \beta, \quad (2.8)$$

we define

$$|\psi_D\rangle = M^{-1}|\psi'\rangle. \quad (2.9)$$

This turns (2.5) into

$$i\hbar \frac{\partial}{\partial z} |\psi'\rangle = \mathcal{H}' |\psi'\rangle, \quad \mathcal{H}' = M\mathcal{H}_D M^{-1} = -p\beta + \hat{\mathcal{E}} + \hat{\mathcal{O}}, \quad (2.10)$$

with the matrix elements of $\hat{\mathcal{E}}$ and $\hat{\mathcal{O}}$ given by

$$\begin{aligned} \hat{\mathcal{E}}_{11} &= -qA_z\mathbf{1} - (\mu_a/2c) \left\{ (\xi + \xi^{-1}) \boldsymbol{\sigma}_\perp \cdot \mathbf{B}_\perp + (\xi - \xi^{-1}) \sigma_z B_z \right\}, \\ \hat{\mathcal{E}}_{12} &= \hat{\mathcal{E}}_{21} = \mathbf{0}, \\ \hat{\mathcal{E}}_{22} &= -qA_z\mathbf{1} - (\mu_a/2c) \left\{ (\xi + \xi^{-1}) \boldsymbol{\sigma}_\perp \cdot \mathbf{B}_\perp - (\xi - \xi^{-1}) \sigma_z B_z \right\}, \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} \hat{\mathcal{O}}_{11} &= \hat{\mathcal{O}}_{22} = \mathbf{0}, \\ \hat{\mathcal{O}}_{12} &= \xi \left[\boldsymbol{\sigma}_\perp \cdot \hat{\boldsymbol{\pi}}_\perp - (\mu_a/2c) \left\{ i (\xi - \xi^{-1}) (B_x\sigma_y - B_y\sigma_x) \right. \right. \\ &\quad \left. \left. - (\xi + \xi^{-1}) B_z \mathbf{1} \right\} \right], \\ \hat{\mathcal{O}}_{21} &= -\xi^{-1} \left[\boldsymbol{\sigma}_\perp \cdot \hat{\boldsymbol{\pi}}_\perp + (\mu_a/2c) \left\{ i (\xi - \xi^{-1}) (B_x\sigma_y - B_y\sigma_x) \right. \right. \\ &\quad \left. \left. + (\xi + \xi^{-1}) B_z \mathbf{1} \right\} \right]. \end{aligned} \quad (2.12)$$

The significance of the transformation (2.9) is that for a paraxial Dirac spinor propagating in the $+z$ -direction $|\psi'\rangle$ is such that its lower spinor components are very small compared to the upper spinor components. To see

this, let us consider the standard free Dirac plane-wave associated with positive energy E , namely,

$$\begin{aligned} \langle \mathbf{r}_\perp | \psi_D \rangle_F(z) &= \begin{pmatrix} \langle \mathbf{r}_\perp | \psi_{D1} \rangle_F(z) \\ \langle \mathbf{r}_\perp | \psi_{D2} \rangle_F(z) \\ \langle \mathbf{r}_\perp | \psi_{D3} \rangle_F(z) \\ \langle \mathbf{r}_\perp | \psi_{D4} \rangle_F(z) \end{pmatrix} \\ &= \frac{1}{4} \sqrt{\frac{\xi c p}{\pi^3 \hbar^3 E}} \begin{pmatrix} s_+ \\ s_- \\ \{s_- p_- + s_+ p_z\} / \xi p \\ \{s_+ p_+ - s_- p_z\} / \xi p \end{pmatrix} \\ &\quad \times \exp \left\{ \frac{i}{\hbar} (\mathbf{p}_\perp \cdot \mathbf{r}_\perp + p_z z) \right\}, \end{aligned}$$

$$p_+ = p_x + i p_y, \quad p_- = p_x - i p_y, \quad \mathbf{r}_\perp = (x, y),$$

$$|s_+|^2 + |s_-|^2 = 1. \quad (2.13)$$

Correspondingly,

$$\begin{aligned} \langle \mathbf{r}_\perp | \psi' \rangle_F(z) &= \begin{pmatrix} \langle \mathbf{r}_\perp | \psi'_1 \rangle_F(z) \\ \langle \mathbf{r}_\perp | \psi'_2 \rangle_F(z) \\ \langle \mathbf{r}_\perp | \psi'_3 \rangle_F(z) \\ \langle \mathbf{r}_\perp | \psi'_4 \rangle_F(z) \end{pmatrix} \\ &= \frac{1}{4} \sqrt{\frac{\xi c p}{2\pi^3 \hbar^3 E}} \begin{pmatrix} \{s_+(p + p_z) + s_- p_-\} / p \\ \{s_-(p + p_z) - s_+ p_+\} / p \\ -\{s_+(p - p_z) - s_- p_-\} / \xi p \\ \{s_-(p - p_z) + s_+ p_+\} / \xi p \end{pmatrix} \\ &\quad \times \exp \left\{ \frac{i}{\hbar} (\mathbf{p}_\perp \cdot \mathbf{r}_\perp + p_z z) \right\}, \quad (2.14) \end{aligned}$$

and for a paraxial plane-wave moving in the positive z -direction, satisfying the condition (2.4), the upper spinor components of $|\psi'\rangle_F$, namely, $|\psi'_1\rangle_F$ and $|\psi'_2\rangle_F$, are obviously very large compared to its lower spinor components $|\psi'_3\rangle_F$ and $|\psi'_4\rangle_F$. We can take this to be generally true for any paraxial beam.

Then, in the paraxial situation, the even operator $\hat{\mathcal{E}}$ in (2.11) does not couple the large upper components and the small lower components while the odd operator $\hat{\mathcal{O}}$ in (2.12) couples them. This is exactly like in the nonrelativistic situation obtained in the standard Dirac theory with respect to time evolution. This, and the striking resemblance of (2.10) with the standard Dirac equation (2.1) make us turn to the Foldy-Wouthuysen (FW) transformation technique [24] (see also, *e.g.*, [25]) to analyse (2.10) further; note that in (2.10) the analogue of mc^2 is $-p$ since $i\hbar\frac{\partial}{\partial z}$ corresponds to $-\hat{p}_z$.

Let us recall that the FW-technique is useful in analysing the Dirac equation systematically as a sum of the nonrelativistic part and a series of relativistic correction terms. The FW-technique is essentially based on the fact that β commutes with any even operator with off-diagonal 2×2 block elements equal to \mathbf{o} , and anticommutes with any odd operator with diagonal 2×2 block elements equal to \mathbf{o} . So, applying this technique to (2.10) should help us analyse it as a sum of the paraxial part and a series of nonparaxial (aberration) correction terms. To this end, we substitute in (2.10)

$$|\psi'\rangle = \exp\left(\frac{1}{2p}\beta\hat{\mathcal{O}}\right)|\psi^{(1)}\rangle. \quad (2.15)$$

The resulting equation for $|\psi^{(1)}\rangle$ is

$$\begin{aligned} i\hbar\frac{\partial}{\partial z}|\psi^{(1)}\rangle &= \mathcal{H}^{(1)}|\psi^{(1)}\rangle, \\ \mathcal{H}^{(1)} &= \exp\left(-\frac{1}{2p}\beta\hat{\mathcal{O}}\right)\mathcal{H}'\exp\left(\frac{1}{2p}\beta\hat{\mathcal{O}}\right) \\ &\quad -i\hbar\exp\left(-\frac{1}{2p}\beta\hat{\mathcal{O}}\right)\frac{\partial}{\partial z}\left\{\exp\left(\frac{1}{2p}\beta\hat{\mathcal{O}}\right)\right\} \\ &= -p\beta + \hat{\mathcal{E}}^{(1)} + \hat{\mathcal{O}}^{(1)}, \\ \hat{\mathcal{E}}^{(1)} &= \hat{\mathcal{E}} - \frac{1}{2p}\beta\hat{\mathcal{O}}^2 + \dots, \end{aligned}$$

$$\hat{\mathcal{O}}^{(1)} = -\frac{1}{2p}\beta \left\{ [\hat{\mathcal{O}}, \hat{\mathcal{E}}] + i\hbar \frac{\partial}{\partial z} \hat{\mathcal{O}} \right\} + \dots \quad (2.16)$$

The effect of this transformation is to eliminate from the odd part of \mathcal{H}' the terms of zeroth order in $1/p$; note that $\hat{\mathcal{O}}^{(1)}$ of $\mathcal{H}^{(1)}$ contains only terms of first and higher orders in $1/p$ (not shown explicitly above). By a series of successive transformations with the same recipe (2.15) one can eliminate odd parts up to any desired order in $1/p$. We shall stop with the above first step which would correspond to the paraxial approximation. Let us write down explicitly, for later use, the 11-block element of $\mathcal{H}^{(1)}$:

$$\begin{aligned} \mathcal{H}_{11}^{(1)} &= -p\mathbf{1} + \hat{\mathcal{E}}_{11}^{(1)} \\ &= \left\{ -p - qA_z + \frac{1}{2p}\hat{\pi}_\perp^2 - \frac{\epsilon\hbar^2}{4p^2}(\text{curl } \mathbf{B})_z + \frac{\epsilon^2\hbar^2}{8p^3}(B_\perp^2 + \gamma^2 B_z^2) \right\} \mathbf{1} \\ &\quad - \frac{1}{p} \{ (q + \epsilon)B_z S_z + \gamma\epsilon \mathbf{B}_\perp \cdot \mathbf{S}_\perp \} \\ &\quad + \frac{\epsilon}{2p^2} \{ \gamma (B_z \mathbf{S}_\perp \cdot \hat{\pi}_\perp + \mathbf{S}_\perp \cdot \hat{\pi}_\perp B_z) - (\mathbf{B}_\perp \cdot \hat{\pi}_\perp + \hat{\pi}_\perp \cdot \mathbf{B}_\perp) S_z \}, \end{aligned}$$

$$\hat{\pi}_\perp^2 = \hat{\pi}_x^2 + \hat{\pi}_y^2, \quad \epsilon = 2m\mu_a/\hbar, \quad \gamma = E/mc^2, \quad \mathbf{S} = \frac{1}{2}\hbar\boldsymbol{\sigma}. \quad (2.17)$$

Before proceeding further, let us find out the nature of $|\psi^{(1)}\rangle$ by looking at the field-free case again. For $|\psi\rangle_F$ in (2.13)

$$|\psi^{(1)}\rangle_F = \exp\left(-\frac{1}{2p}\beta\chi\boldsymbol{\alpha}_\perp \cdot \mathbf{p}_\perp\right) |\psi'\rangle_F \approx \left(\mathbf{1} - \frac{1}{2p}\beta\chi\boldsymbol{\alpha}_\perp \cdot \mathbf{p}_\perp\right) |\psi'\rangle_F,$$

$$\langle \mathbf{r}_\perp | \psi^{(1)} \rangle_F(z) = \begin{pmatrix} \langle \mathbf{r}_\perp | \psi_1^{(1)} \rangle_F(z) \\ \langle \mathbf{r}_\perp | \psi_2^{(1)} \rangle_F(z) \\ \langle \mathbf{r}_\perp | \psi_3^{(1)} \rangle_F(z) \\ \langle \mathbf{r}_\perp | \psi_4^{(1)} \rangle_F(z) \end{pmatrix}$$

$$\approx \frac{1}{4} \sqrt{\frac{\xi c p}{2\pi^3 \hbar^3 E}} \begin{pmatrix} s_+ \left\{ 1 + \frac{p_z}{p} - \frac{p_\perp^2}{2p^2} \right\} + \frac{1}{2} s_- \left\{ \left(1 + \frac{p_z}{p} \right) \frac{p_-}{p} \right\} \\ s_- \left\{ 1 + \frac{p_z}{p} + \frac{p_\perp^2}{2p^2} \right\} - \frac{1}{2} s_+ \left\{ \left(1 + \frac{p_z}{p} \right) \frac{p_+}{p} \right\} \\ -\frac{1}{\xi} \left[s_+ \left\{ 1 - \frac{p_z}{p} - \frac{p_\perp^2}{2p^2} \right\} + \frac{1}{2} s_- \left\{ \left(1 - \frac{p_z}{p} \right) \frac{p_-}{p} \right\} \right] \\ \frac{1}{\xi} \left[s_- \left\{ 1 - \frac{p_z}{p} - \frac{p_\perp^2}{2p^2} \right\} + \frac{1}{2} s_+ \left\{ \left(1 - \frac{p_z}{p} \right) \frac{p_+}{p} \right\} \right] \end{pmatrix} \\ \times \exp \left\{ \frac{i}{\hbar} (\mathbf{p}_\perp \cdot \mathbf{r}_\perp + p_z z) \right\}, \quad (2.18)$$

showing clearly that the transformation (2.15) keeps the upper spinor components of $|\psi^{(1)}\rangle$ large compared to its lower spinor components.

Since the lower pair of components of $|\psi^{(1)}\rangle$ ($|\psi_3^{(1)}\rangle$ and $|\psi_4^{(1)}\rangle$) are almost vanishing compared to the the upper pair ($|\psi_1^{(1)}\rangle$ and $|\psi_2^{(1)}\rangle$) and the odd part of $\mathcal{H}^{(1)}$ is negligible compared to its even part we can effectively introduce a Pauli-like two-component spinor formalism based on the representation (2.16). Naming the two-component spinor comprising the upper pair of components of $|\psi^{(1)}\rangle$ as $|\tilde{\psi}\rangle$ and calling $\mathcal{H}_{11}^{(1)}$ as $\tilde{\mathcal{H}}$ it is clear from (2.16) and (2.17) that we can write

$$i\hbar \frac{\partial}{\partial z} |\tilde{\psi}\rangle = \tilde{\mathcal{H}} |\tilde{\psi}\rangle, \quad |\tilde{\psi}\rangle = \begin{pmatrix} |\tilde{\psi}_1\rangle \\ |\tilde{\psi}_2\rangle \end{pmatrix}, \\ \tilde{\mathcal{H}} \approx \left(-p - qA_z + \frac{1}{2p} \hat{\pi}_\perp^2 \right) - \frac{1}{p} \{ (q + \epsilon) B_z S_z + \gamma \epsilon \mathbf{B}_\perp \cdot \mathbf{S}_\perp \}, \quad (2.19)$$

where $\tilde{\mathcal{H}}$ has been approximated by keeping only terms up to first order in $1/p$ (see (2.17)), consistent with the assumption of paraxiality condition (2.4) for the beam. Throughout the paper we shall approximate the various expressions by keeping only up to the lowest order nontrivial terms consistent

with the paraxiality condition for the beam and the approximation symbol (\approx) will usually imply this, even if not stated explicitly.

Up to now, all the observables, the field components, time, etc., are defined with reference to the laboratory frame. But, as is well known, in the covariant description the spin of the Dirac particle has simple operator representation in terms of the Pauli matrices only in a frame at which the particle is at rest. So, as is usual, we shall prefer to define spin with reference to the instantaneous rest frame of the particle while keeping the other observables, field components, time, etc., defined with reference to the laboratory frame. To this end, we transform the two-component $|\tilde{\psi}\rangle$ to an ‘accelerator optics representation’ $|\psi^{(A)}\rangle$ defined by

$$|\tilde{\psi}\rangle = \exp\left\{\frac{i}{2p}(\hat{\pi}_x\sigma_y - \hat{\pi}_y\sigma_x)\right\}|\psi^{(A)}\rangle. \quad (2.20)$$

The reason for the choice of this transformation will become clear shortly. Now, the z -evolution equation for $|\psi^{(A)}\rangle$ is

$$i\hbar\frac{\partial}{\partial z}|\psi^{(A)}\rangle = H^{(A)}|\psi^{(A)}\rangle, \\ H^{(A)} \approx \left(-p - qA_z + \frac{1}{2p}\hat{\pi}_\perp^2\right) + \frac{\gamma m}{p}\boldsymbol{\Omega}_s \cdot \mathbf{S},$$

$$\text{with } \boldsymbol{\Omega}_s = -\frac{1}{\gamma m}\left\{q\mathbf{B} + \epsilon(\mathbf{B}_\parallel + \gamma\mathbf{B}_\perp)\right\}, \quad (2.21)$$

where \mathbf{B}_\parallel is the field along the z -direction (or the longitudinal component of \mathbf{B} on the design trajectory). When $q = \pm e$ we can write $\epsilon = qa = q(g - 2)/2$ where g and a are, respectively, the gyromagnetic ratio and the magnetic anomaly of the particle; for neutron $\epsilon = g|e|/2$. It may be noted that the accelerator optical quantum Hamiltonian $H^{(A)}$ is hermitian though

\mathcal{H}_D in (2.6) is nonhermitian. The nonunitary similarity transformations we have made have resulted in this change and the hermiticity of $H^{(A)}$ implies the approximate constancy of the total intensity of the beam in any transverse plane along the optic axis.

Since the z -evolution of $|\psi^{(A)}\rangle$ is unitary we can associate the beam with a wavefunction normalized in such a way that, at any z ,

$$\langle \psi^{(A)}(z) | \psi^{(A)}(z) \rangle = \sum_{i=1}^2 \int d^2 \mathbf{r}_\perp \langle \psi_i^{(A)}(z) | \mathbf{r}_\perp \rangle \langle \mathbf{r}_\perp | \psi_i^{(A)}(z) \rangle = 1. \quad (2.22)$$

When the beam is described by a 2×2 statistical (density) matrix

$$\rho^{(A)} = \begin{pmatrix} \rho_{11}^{(A)} & \rho_{12}^{(A)} \\ \rho_{21}^{(A)} & \rho_{22}^{(A)} \end{pmatrix}, \quad (2.23)$$

with the normalization

$$\text{Tr}(\rho^{(A)}(z)) = \sum_{i=1}^2 \int d^2 \mathbf{r}_\perp \langle \mathbf{r}_\perp | \rho_{ii}^{(A)}(z) | \mathbf{r}_\perp \rangle = 1, \quad (2.24)$$

at any z , the accelerator optical z -evolution equation is

$$i\hbar \frac{\partial}{\partial z} \rho^{(A)} = [H^{(A)}, \rho^{(A)}]. \quad (2.25)$$

If the beam can be described as a pure state we would have $\rho^{(A)} = |\psi^{(A)}\rangle \langle \psi^{(A)}|$.

Let us now define the average of any observable O at the transverse plane at z to be given by

$$\begin{aligned} \langle \hat{O}^{(A)} \rangle(z) &= \text{Tr}(\rho^{(A)}(z) \hat{O}^{(A)}) \\ &= \sum_{i,j=1}^2 \int \int d^2 \mathbf{r}_\perp d^2 \mathbf{r}'_\perp \langle \mathbf{r}_\perp | \rho_{ij}^{(A)}(z) | \mathbf{r}'_\perp \rangle \langle \mathbf{r}'_\perp | \hat{O}_{ji}^{(A)} | \mathbf{r}_\perp \rangle, \end{aligned} \quad (2.26)$$

where $\hat{O}^{(A)}$ is the operator representing O in the accelerator optical representation.

For any observable O , associated with the operator \hat{O}_D in the standard Dirac representation (2.1) the corresponding $\hat{O}^{(A)}$ can be obtained as follows :

$$\begin{aligned} \hat{O}^{(A)} = & \text{the hermitian part of the 11 – block element of} \\ & \left(\exp \left\{ -\frac{i}{2p} (\hat{\pi}_x \Sigma_y - \hat{\pi}_y \Sigma_x) \right\} \right. \\ & \times \exp \left(-\frac{1}{2p} \beta \hat{O} \right) M \hat{O}_D M^{-1} \exp \left(\frac{1}{2p} \beta \hat{O} \right) \\ & \left. \times \exp \left\{ \frac{i}{2p} (\hat{\pi}_x \Sigma_y - \hat{\pi}_y \Sigma_x) \right\} \right). \end{aligned} \quad (2.27)$$

In the Dirac representation the operator for the spin unit vector corresponding to the spin as defined in the instantaneous rest frame of the particle (see [3]) is given by

$$\mathbf{S}_R = \frac{\hbar}{2} \begin{pmatrix} \boldsymbol{\sigma} - \frac{c^2(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{\pi}} + \hat{\boldsymbol{\pi}} \cdot \boldsymbol{\sigma})}{2E(E+mc^2)} & \frac{c\hat{\boldsymbol{\pi}}}{E} \\ \frac{c\hat{\boldsymbol{\pi}}}{E} & -\boldsymbol{\sigma} + \frac{c^2(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{\pi}} + \hat{\boldsymbol{\pi}} \cdot \boldsymbol{\sigma})}{2E(E+mc^2)} \end{pmatrix}. \quad (2.28)$$

If we now compute the corresponding operator $\mathbf{S}_R^{(A)}$ in the accelerator optical representation, using the formula (2.27), it is found that up to first order (paraxial) approximation

$$\mathbf{S}_R^{(A)} \approx \frac{\hbar}{2} \boldsymbol{\sigma} \quad (2.29)$$

as is desired. In the Dirac representation the position operator in free space can be taken to be given by the mean position operator as indicated by the FW-theory (or what is same as the Newton-Wigner position operator). In presence of the magnetic field we can extend this position operator by the replacement $\hat{\mathbf{p}} \rightarrow \hat{\boldsymbol{\pi}}$ and symmetrization (to make it hermitian). Then, the operator for the transverse position coordinate in the accelerator optical representation becomes just the canonical position operator \mathbf{r}_\perp in the first

order approximation. From these considerations it is clear that in the accelerator optical evolution equation (2.21) \mathcal{S} represents the spin as defined in the instantaneous rest frame of the particle; the field components and other operators are all defined with respect to the laboratory frame. It should be noted that in this formalism, with z as the evolution parameter (analogous to time t), $-H^{(A)}$ corresponding to $-i\hbar\frac{\partial}{\partial z}$, will represent \hat{p}_z , the z -component of canonical momentum operator (analogous to the energy operator); hence, the operator $-(H^{(A)} + qA_z)$ will represent π_z the z -component of the kinetic momentum.

If we now work out the equations of motion for the average values of \mathbf{r}_\perp using (2.25), they have to be consistent, *à la* Ehrenfest, with the traditional transfer map for the phase-space, including the transverse Stern-Gerlach kicks (see, *e.g.*, [8,23]), in the paraxial approximation. The transfer map for the averages of spin components, in the lowest order approximation, has to be consistent with the Thomas-BMT equation. This is confirmed easily by a preliminary analysis as follows. From (2.25) and (2.26) we have, in general,

$$\frac{d}{dz} \langle \hat{O}^{(A)} \rangle (z) = -\frac{i}{\hbar} \langle [\hat{O}^{(A)}, H^{(A)}] \rangle (z) + \left\langle \frac{\partial}{\partial z} \hat{O}^{(A)} \right\rangle (z). \quad (2.30)$$

To compare (2.30) with the time evolution of classical O we can use the correspondence

$$\frac{d}{dt} O \longrightarrow \frac{d}{dt} \langle \hat{O}^{(A)} \rangle \approx v_z \frac{d}{dz} \langle \hat{O}^{(A)} \rangle \approx \frac{p}{\gamma m} \frac{d}{dz} \langle \hat{O}^{(A)} \rangle, \quad (2.31)$$

since

$$v_z = \frac{1}{\gamma m} \langle \hat{\pi}_z \rangle = -\frac{1}{\gamma m} \langle H^{(A)} + qA_z \rangle$$

$$= \frac{p}{\gamma m} - \frac{1}{2\gamma m p} \langle \hat{\pi}_\perp^2 \rangle - \frac{1}{p} \langle \boldsymbol{\Omega}_s \cdot \mathbf{S} \rangle \approx \frac{p}{\gamma m}, \quad (2.32)$$

Then, for \mathbf{r}_\perp we get

$$\frac{d}{dt} \langle \mathbf{r}_\perp \rangle \approx -\frac{i}{\hbar} \frac{p}{\gamma m} \langle [\mathbf{r}_\perp, H^{(A)}] \rangle (z) = \frac{1}{\gamma m} \langle \hat{\pi}_\perp \rangle, \quad (2.33)$$

identifying $\hat{\pi}_\perp$ as the transverse kinetic momentum. For $\hat{\pi}_\perp$

$$\begin{aligned} \frac{d}{dt} \langle \hat{\pi}_\perp \rangle &\approx -\frac{i}{\hbar} \frac{p}{\gamma m} \langle [\hat{\pi}_\perp, H^{(A)}] \rangle - \frac{pq}{\gamma m} \left\langle \frac{\partial}{\partial z} \mathbf{A}_\perp \right\rangle \\ &\approx \frac{q}{\gamma m} \left\langle \frac{1}{2} (\hat{\pi} \times \mathbf{B} - \mathbf{B} \times \hat{\pi})_\perp \right\rangle - \langle \nabla_\perp (\boldsymbol{\Omega}_s \cdot \mathbf{S}) \rangle \\ &= \frac{q}{\gamma m} \left\langle \frac{1}{2} (\hat{\pi} \times \mathbf{B} - \mathbf{B} \times \hat{\pi})_\perp \right\rangle \\ &\quad + \frac{1}{\gamma m} \langle \nabla_\perp \{ (q + \epsilon) B_z S_z + (q + \gamma \epsilon) \mathbf{B}_\perp \cdot \mathbf{S}_\perp \} \rangle, \end{aligned}$$

$$\text{with } \hat{\pi}_z \approx p. \quad (2.34)$$

Equation (2.34) is just in accordance with the quasiclassical equation for motion under the Lorentz and Stern-Gerlach forces up to the approximations considered. From this it is clear that $\langle \mathbf{r}_\perp \rangle$ and $\langle \hat{\pi}_\perp \rangle / p$ ($\langle \hat{\mathbf{p}}_\perp \rangle / p$ in the field-free regions) can be identified with the transverse position and slope of the classical ray corresponding to the wavepacket represented by $\rho^{(A)}$.

In the case of spin

$$\frac{d}{dt} \langle \mathbf{S} \rangle \approx -\frac{i}{\hbar} \frac{p}{\gamma m} \langle [\mathbf{S}, H^{(A)}] \rangle = -\frac{i}{\hbar} \langle [\mathbf{S}, \boldsymbol{\Omega}_s \cdot \mathbf{S}] \rangle = \langle \boldsymbol{\Omega}_s \times \mathbf{S} \rangle, \quad (2.35)$$

as should be expected from the Thomas-BMT equation. The vector \mathbf{P} characterizing the polarization of the beam is given by the relation

$$\langle \mathbf{S} \rangle = \frac{\hbar}{2} \langle \boldsymbol{\sigma} \rangle = \frac{\hbar}{2} \mathbf{P}. \quad (2.36)$$

To obtain the required maps for transfer of the averages ($\langle \mathbf{r}_\perp \rangle, \langle \hat{\boldsymbol{\pi}}_\perp \rangle, \langle \mathbf{S} \rangle$) across an optical element we can employ the quantum mechanical version [18] of the technique developed by Dragt *et al.* (see [26,27]) in the context of classical accelerator optics. We shall explain this in the next section through the example of a normal quadrupolar magnetic lens. Though we have taken $\langle \hat{\pi}_z \rangle \approx p$ in the above preliminary analysis, following (2.32), to understand the small variations in the longitudinal kinetic momentum, including the Stern-Gerlach kicks [23], a more careful analysis of the evolution of $\langle \hat{\pi}_z \rangle(z)$ along the z -axis is needed. We shall discuss this in the next section by examining the case of a general inhomogeneous magnetic field.

Before closing this section let us note that the Pauli-like two-component spinor formalism developed above is valid for all p , from the nonrelativistic to the extreme relativistic case; it will, of course, become identical to Pauli's two-component formalism in the nonrelativistic case when we can take $p \approx \sqrt{2m(E - mc^2)}$.

3. Transfer maps for phase-space and spin

First, let us consider an ideal normal magnetic quadrupole lens field given by

$$\mathbf{B} = (-Gy, -Gx, 0), \quad (3.1)$$

associated with the vector potential

$$\mathbf{A} = \left(0, 0, \frac{1}{2}G(x^2 - y^2) \right), \quad (3.2)$$

where G is assumed to be a constant in the lens region and zero outside. Let the z -coordinates of the xy -planes at the entrance and exit of the quadrupole

magnet of length ℓ be z_n and z_x (the subscripts ‘n’ and ‘x’ denoting e’n’tance and e’x’it, respectively, and $\ell = z_x - z_n$). Throughout the present section we shall be working with the accelerator optical representation and shall omit the superscript (A).

Now, the basic accelerator optical Hamiltonian of the system is

$$H(z) = \begin{cases} H_F = -p + \frac{1}{2p}\hat{p}_\perp^2, & \text{for } z < z_n \text{ and } z > z_x, \\ H_L(z) = -p + \frac{1}{2p}\hat{p}_\perp^2 - \frac{1}{2}qG(x^2 - y^2) + \frac{\eta p}{\ell}(y\sigma_x + x\sigma_y), & \text{for } z_n \leq z \leq z_x, \text{ with } \eta = (q + \gamma\epsilon)G\ell\hbar/2p^2. \end{cases} \quad (3.3)$$

The subscripts F and L indicate, respectively, the field-free and the lens regions. Let us write H as a core part \bar{H} plus a perturbation part \tilde{H} :

$$H(z) = \bar{H}(z) + \tilde{H}(z),$$

$$\bar{H}(z) = \begin{cases} \bar{H}_F \equiv H_F, & \text{for } z < z_n \text{ and } z > z_x, \\ \bar{H}_L(z) = -p + \frac{1}{2p}\hat{p}_\perp^2 - \frac{1}{2}qG(x^2 - y^2), & \text{for } z_n \leq z \leq z_x. \end{cases}$$

$$\tilde{H}(z) = \begin{cases} \tilde{H}_F = 0, & \text{for } z < z_n \text{ and } z > z_x, \\ \tilde{H}_L(z) = \frac{\eta p}{\ell}(y\sigma_x + x\sigma_y), & \text{for } z_n \leq z \leq z_x. \end{cases} \quad (3.4)$$

A formal integration of the basic z -evolution equation (2.25) for ρ leads, in general, to

$$\rho(z) = U(z, z_0)\rho(z_0)U^\dagger(z, z_0), \quad z \geq z_0, \quad (3.5)$$

with the unitary z -propagator U given by

$$U(z, z_0) = \wp \left[\exp \left\{ -\frac{i}{\hbar} \int_{z_0}^z d\zeta H(\zeta) \right\} \right], \quad (3.6)$$

where \wp indicates the path-ordering of the exponential. Further, U is such that

$$i\hbar \frac{\partial}{\partial z} U(z, z_0) = H(z)U(z, z_0), \quad U(z_0, z_0) = I, \quad (3.7)$$

where I is the identity operator and for any set of points $\{z_1, z_2, \dots, z_j\}$ in the interval $[z_0, z]$ with $z > z_j > z_{j-1} > \dots > z_2 > z_1 > z_0$,

$$U(z, z_0) = U(z, z_j)U(z_j, z_{j-1}) \cdots U(z_2, z_1)U(z_1, z_0). \quad (3.8)$$

A convenient expression for U is given by the Magnus formula [28]: for any $z'' \geq z'$,

$$\begin{aligned} U(z'', z') = \exp \left\{ -\frac{i}{\hbar} \int_{z'}^{z''} d\zeta H(\zeta) \right. \\ + \frac{1}{2} \left(-\frac{i}{\hbar} \right)^2 \int_{z'}^{z''} d\zeta_2 \int_{z'}^{\zeta_2} d\zeta_1 [H(\zeta_2), H(\zeta_1)] \\ + \frac{1}{6} \left(-\frac{i}{\hbar} \right)^3 \int_{z'}^{z''} d\zeta_3 \int_{z'}^{\zeta_3} d\zeta_2 \int_{z'}^{\zeta_2} d\zeta_1 \{ [[H(\zeta_3), H(\zeta_2)], H(\zeta_1)] \\ \left. + [[H(\zeta_1), H(\zeta_2)], H(\zeta_3)] \} \cdots \right\}. \quad (3.9) \end{aligned}$$

Let us now compute $\rho(z)$ *via* an interaction picture. Defining

$$\rho_i(z) = \bar{U}^\dagger(z, z_0) \rho(z) \bar{U}(z, z_0), \quad \bar{U}(z, z_0) = \wp \left[\exp \left\{ -\frac{i}{\hbar} \int_{z_0}^z d\zeta \bar{H}(\zeta) \right\} \right], \quad (3.10)$$

we have

$$i\hbar \frac{\partial}{\partial z} \rho_i = [\bar{H}_i, \rho_i], \quad \bar{H}_i = \bar{U}^\dagger(z, z_0) \bar{H} \bar{U}(z, z_0). \quad (3.11)$$

Then, since $\rho_i(z_0) = \rho(z_0)$,

$$\rho_i(z) = \tilde{U}_i(z, z_0) \rho_i(z_0) \tilde{U}_i^\dagger(z, z_0) = \tilde{U}_i(z, z_0) \rho(z_0) \tilde{U}_i^\dagger(z, z_0),$$

$$\tilde{U}_i(z, z_0) = \wp \left[\exp \left\{ -\frac{i}{\hbar} \int_{z_0}^z d\zeta \tilde{H}_i(\zeta) \right\} \right]. \quad (3.12)$$

Now, from (3.10) and (3.12), we see that

$$\rho(z) = \bar{U}(z, z_0) \tilde{U}_i(z, z_0) \rho(z_0) \tilde{U}_i^\dagger(z, z_0) \bar{U}^\dagger(z, z_0). \quad (3.13)$$

Hence, for the average of any observable O we have

$$\begin{aligned} \langle \hat{O} \rangle(z) &= \text{Tr} \langle \rho(z) \hat{O} \rangle = \text{Tr} \langle \bar{U}(z, z_0) \tilde{U}_i(z, z_0) \rho(z_0) \tilde{U}_i^\dagger(z, z_0) \bar{U}^\dagger(z, z_0) \hat{O} \rangle \\ &= \text{Tr} \langle \rho(z_0) \{ \tilde{U}_i^\dagger(z, z_0) \bar{U}^\dagger(z, z_0) \hat{O} \bar{U}(z, z_0) \tilde{U}_i(z, z_0) \} \rangle. \end{aligned} \quad (3.14)$$

This equation (3.14) provides the general basic formula to compute the transfer map for $\langle \hat{O} \rangle$ across the system as will be seen below in the case of the present example.

Let us take z_0 and z to be respectively in the field-free input and output regions of the quadrupole magnet : $z_0 < z_n$, $z > z_x$. Using (3.8) and (3.9), and after some straightforward algebra we get

$$\bar{U}(z, z_0) = \bar{U}_F(z, z_x) \bar{U}_L(z_x, z_n) \bar{U}_F(z_n, z_0),$$

$$\tilde{U}_i(z, z_0) = \tilde{U}_{i,F}(z, z_x) \tilde{U}_{i,L}(z_x, z_n) \tilde{U}_{i,F}(z_n, z_0) \equiv \tilde{U}_{i,L}(z_x, z_n),$$

$$\bar{U}_F(z, z_x) = \exp \left\{ \frac{i}{\hbar} \Delta z_{>} \left(p - \frac{1}{2p} \hat{p}_\perp^2 \right) \right\}, \quad \text{with } \Delta z_{>} = z - z_x$$

$$\bar{U}_L(z_x, z_n) = \exp \left\{ \frac{i}{\hbar} \ell \left[\left(p - \frac{1}{2p} \hat{p}_\perp^2 \right) + \frac{1}{2} p K (x^2 - y^2) \right] \right\},$$

with $K = qG/p$,

$$\bar{U}_F(z_n, z_0) = \exp \left\{ \frac{i}{\hbar} \Delta z_{<} \left(p - \frac{1}{2p} \hat{p}_\perp^2 \right) \right\}, \quad \text{with } \Delta z_{<} = z_n - z_0,$$

$$\begin{aligned}
 & \tilde{U}_{i,L}(z_x, z_n) \\
 &= \exp \left\{ -\frac{i}{\hbar} \eta \left[\left(\left(\frac{\sinh(\sqrt{K}\ell)}{\sqrt{K}\ell} \right) px + \left(\frac{\cosh(\sqrt{K}\ell) - 1}{K\ell} \right) \hat{p}_x \right) \sigma_y \right. \right. \\
 & \quad \left. \left. + \left(\left(\frac{\sin(\sqrt{K}\ell)}{\sqrt{K}\ell} \right) py - \left(\frac{\cos(\sqrt{K}\ell) - 1}{K\ell} \right) \hat{p}_y \right) \sigma_x \right] \right\}, \\
 &\approx \exp \left\{ -\frac{i}{\hbar} \eta \left[\left(\left(1 + \frac{K\ell^2}{6} \right) px + \frac{\ell}{2} \left(1 + \frac{K\ell^2}{12} \right) \hat{p}_x \right) \sigma_y \right. \right. \\
 & \quad \left. \left. + \left(\left(1 - \frac{K\ell^2}{6} \right) py + \frac{\ell}{2} \left(1 - \frac{K\ell^2}{12} \right) \hat{p}_y \right) \sigma_x \right] \right\}. \tag{3.15}
 \end{aligned}$$

Now, using (3.14) and (3.15) the transfer maps for $\langle \mathbf{r}_\perp \rangle$ and $\langle \hat{\mathbf{p}}_\perp \rangle$ ($\equiv \langle \hat{\boldsymbol{\pi}}_\perp \rangle$ in this case) are obtained as follows : with $\lambda = h/p$, the de Broglie wavelength,

$$\begin{aligned}
 \begin{pmatrix} \langle \mathbf{x} \rangle(z) \\ \langle \hat{\mathbf{p}}_x \rangle(z)/p \\ \langle \mathbf{y} \rangle(z) \\ \langle \hat{\mathbf{p}}_y \rangle(z)/p \end{pmatrix} &\approx \begin{pmatrix} T_{11}^x & T_{12}^x & 0 & 0 \\ T_{21}^x & T_{22}^x & 0 & 0 \\ 0 & 0 & T_{11}^y & T_{12}^y \\ 0 & 0 & T_{21}^y & T_{22}^y \end{pmatrix} \begin{pmatrix} \langle \mathbf{x} \rangle(z_0) \\ \langle \hat{\mathbf{p}}_x \rangle(z_0)/p \\ \langle \mathbf{y} \rangle(z_0) \\ \langle \hat{\mathbf{p}}_y \rangle(z_0)/p \end{pmatrix} \\
 &+ \eta \begin{pmatrix} \frac{\ell}{2} \left(1 + \frac{K\ell^2}{12} \right) \langle \sigma_y \rangle(z_0) \\ - \left(1 + \frac{K\ell^2}{6} \right) \langle \sigma_y \rangle(z_0) \\ \frac{\ell}{2} \left(1 - \frac{K\ell^2}{12} \right) \langle \sigma_x \rangle(z_0) \\ - \left(1 - \frac{K\ell^2}{6} \right) \langle \sigma_x \rangle(z_0) \end{pmatrix} \\
 &\begin{pmatrix} T_{11}^x & T_{12}^x \\ T_{21}^x & T_{22}^x \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} 1 & \Delta z_{>} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh(\sqrt{K} \ell) & \frac{1}{\sqrt{K}} \sinh(\sqrt{K} \ell) \\ \sqrt{K} \sinh(\sqrt{K} \ell) & \cosh(\sqrt{K} \ell) \end{pmatrix} \begin{pmatrix} 1 & \Delta z_{<} \\ 0 & 1 \end{pmatrix} \\
&\begin{pmatrix} T_{11}^y & T_{12}^y \\ T_{21}^y & T_{22}^y \end{pmatrix} \\
&= \begin{pmatrix} 1 & \Delta z_{>} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\sqrt{K} \ell) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} \ell) \\ -\sqrt{K} \sin(\sqrt{K} \ell) & \cos(\sqrt{K} \ell) \end{pmatrix} \begin{pmatrix} 1 & \Delta z_{<} \\ 0 & 1 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\langle S_x \rangle(z) &\approx \langle S_x \rangle(z_0) + \\
&\frac{4\pi\eta}{\lambda} \left(\left(1 + \frac{K\ell^2}{6}\right) \langle x S_z \rangle(z_0) + \frac{\ell}{2p} \left(1 + \frac{K\ell^2}{12}\right) \langle \hat{p}_x S_z \rangle(z_0) \right),
\end{aligned}$$

$$\begin{aligned}
\langle S_y \rangle(z) &\approx \langle S_y \rangle(z_0) - \\
&\frac{4\pi\eta}{\lambda} \left(\left(1 - \frac{K\ell^2}{6}\right) \langle y S_z \rangle(z_0) + \frac{\ell}{2p} \left(1 - \frac{K\ell^2}{12}\right) \langle \hat{p}_y S_z \rangle(z_0) \right),
\end{aligned}$$

$$\begin{aligned}
\langle S_z \rangle(z) &\approx \langle S_z \rangle(z_0) - \\
&\frac{4\pi\eta}{\lambda} \left\{ \left(1 + \frac{K\ell^2}{6}\right) \langle x S_x \rangle(z_0) - \left(1 - \frac{K\ell^2}{6}\right) \langle y S_y \rangle(z_0) \right. \\
&\quad \left. + \frac{\ell}{2p} \left(\left(1 + \frac{K\ell^2}{12}\right) \langle \hat{p}_x S_x \rangle(z_0) - \left(1 - \frac{K\ell^2}{12}\right) \langle \hat{p}_y S_y \rangle(z_0) \right) \right\}.
\end{aligned} \tag{3.16}$$

So, we have got a fully quantum mechanical derivation of the traditional transfer map for the transverse phase-space, including the Stern-Gerlach effect (see [8]), in the case of a spin- $\frac{1}{2}$ particle beam propagating through a normal magnetic quadrupole lens : the lens is focusing (defocusing) in the yz -plane and defocusing (focusing) in the xz -plane when $K > 0$ ($K < 0$). The transverse Stern-Gerlach kicks to the trajectory slope ($\delta\langle \hat{p}_\perp \rangle/p \sim \eta$) are

seen to disappear at relativistic energies, varying like $\sim 1/\gamma$. At nonrelativistic energies, with $\gamma \approx 1$, the kicks are $\sim G\ell\mu/mv^2$ where μ is the total magnetic moment. These results are in general agreement with the conclusions reached earlier [8],[23] based on semiclassical treatments. The spin map obtained above contains the Thomas-BMT map plus the lowest order corrections; it should be also noted that the polarization transfer map is linear in the polarization components only when there is no spin-space correlation.

Using the general theory, let us now understand the longitudinal Stern-Gerlach kicks [23] in a general inhomogeneous magnetic field. For $\hat{\pi}_z = -(H^{(A)} + qA_z)$ we get, from (2.30) and (2.31),

$$\begin{aligned}
 \frac{d}{dt}\langle\hat{\pi}_z\rangle &\approx \frac{p}{\gamma m} \left\{ \frac{i}{\hbar} \langle [H^{(A)} + qA_z, H^{(A)}] \rangle - \left\langle \frac{\partial}{\partial z} (H^{(A)} + qA_z) \right\rangle \right\} \\
 &= \frac{p}{\gamma m} \left\langle \frac{i}{\hbar} [qA_z, H^{(A)}] - \frac{1}{2p} \frac{\partial}{\partial z} \hat{\pi}_\perp^2 \right\rangle - \left\langle \frac{\partial}{\partial z} (\boldsymbol{\Omega}_s \cdot \mathbf{S}) \right\rangle \\
 &= \frac{q}{\gamma m} \left\langle \frac{1}{2} (\hat{\boldsymbol{\pi}} \times \mathbf{B} - \mathbf{B} \times \hat{\boldsymbol{\pi}})_z \right\rangle - \left\langle \frac{\partial}{\partial z} (\boldsymbol{\Omega}_s \cdot \mathbf{S}) \right\rangle \\
 &= \frac{q}{\gamma m} \left\langle \frac{1}{2} (\hat{\boldsymbol{\pi}} \times \mathbf{B} - \mathbf{B} \times \hat{\boldsymbol{\pi}})_z \right\rangle \\
 &\quad + \frac{1}{\gamma m} \left\langle \frac{\partial}{\partial z} \{ (q + \epsilon) B_z S_z + (q + \gamma\epsilon) \mathbf{B}_\perp \cdot \mathbf{S}_\perp \} \right\rangle. \quad (3.17)
 \end{aligned}$$

The first term of the r.h.s. of (3.17) obviously represents the Lorentz force and the rest of it accounts for the Stern-Gerlach force due to the longitudinal gradient of the field (*i.e.*, gradient in the z -direction). Multiplying both sides of (2.34) and (3.17) by $1/v_z \approx \gamma m/p$ and collecting them together we get for the z -evolution of $\langle\hat{\boldsymbol{\pi}}\rangle$

$$\begin{aligned}
 \frac{d}{dz}\langle\hat{\boldsymbol{\pi}}\rangle &\approx \frac{q}{p} \left\langle \frac{1}{2} (\hat{\boldsymbol{\pi}} \times \mathbf{B} - \mathbf{B} \times \hat{\boldsymbol{\pi}}) \right\rangle \\
 &\quad + \frac{1}{p} \langle \boldsymbol{\nabla} \{ (q + \epsilon) B_z S_z + (q + \gamma\epsilon) \mathbf{B}_\perp \cdot \mathbf{S}_\perp \} \rangle. \quad (3.18)
 \end{aligned}$$

For any given field configuration \mathbf{B} , with a specified \mathbf{A} , the solution of this equation (3.18) is given by $\langle \hat{\pi} \rangle(z) = \text{Tr} \langle \rho(z_0) U^\dagger(z, z_0) \hat{\pi} U(z, z_0) \rangle$, for any $z > z_0$, and hence the spin-dependent Stern-Gerlach kick to the kinetic momentum and the resultant spin-dependent splitting of the kinetic energy at any $z > z_0$ can be calculated.

From (2.34) and (3.17), or (3.18), it is clear that

$$\begin{aligned} \frac{d}{dt} \langle \hat{\pi} \rangle &\approx \frac{q}{\gamma m} \left\langle \frac{1}{2} (\hat{\pi} \times \mathbf{B} - \mathbf{B} \times \hat{\pi}) \right\rangle \\ &\quad + \frac{1}{\gamma m} \langle \nabla \{ q\mathbf{B} \cdot \mathbf{S} + \epsilon(B_z S_z + \gamma \mathbf{B}_\perp \cdot \mathbf{S}_\perp) \} \rangle \\ &= \frac{q}{\gamma m} \left\langle \frac{1}{2} (\hat{\pi} \times \mathbf{B} - \mathbf{B} \times \hat{\pi}) \right\rangle - \langle \nabla \Omega_s \cdot \mathbf{S} \rangle, \end{aligned} \quad (3.19)$$

in which the first term represents the Lorentz force and the second term represents the Stern-Gerlach force. This equation for orbital motion (3.19), accounting for both the Lorentz and the Stern-Gerlach forces, and the Thomas-BMT spin evolution equation (2.35), together, justify the term $\Omega_s \cdot \mathbf{S}$ as the effective spin Hamiltonian [22] to be added to the orbital Hamiltonian, on the basis of the Dirac equation.

In the instantaneous rest frame of the particle with $\gamma = 1$ the second term in (3.19) is seen to correspond to the familiar Stern-Gerlach force

$$\mathbf{F}_{SG} = -\nabla U, \quad U = -\boldsymbol{\mu} \boldsymbol{\sigma} \cdot \mathbf{B}, \quad (3.20)$$

where $\boldsymbol{\mu}$ is the total magnetic moment of the particle; note that in (3.20), apart from the spin, the field components, the coordinates, etc., are also defined in the rest frame of the particle. Having derived the classical Stern-Gerlach force (3.20) from the Dirac equation, of course up to the paraxial approximation in the context of beam dynamics, one can use the standard classical relativistic dynamics [29,30] to compare the relative merits and

demerits of spin-splitter devices employing the transverse and longitudinal Stern-Gerlach kicks. Such a comparison (see [23] for details) seems to suggest that, at high energies, while the transverse kick decreases as γ increases, as we have seen already above, the longitudinal kick has a much more favourable feature of becoming almost independent of energy as γ increases; of course, the kicks are larger at lower energies in both the cases.

4. Conclusion

In fine, we have demonstrated how one can obtain a fully quantum mechanical understanding of the accelerator beam optics for a spin- $\frac{1}{2}$ particle, with anomalous magnetic moment, starting *ab initio* from the Dirac-Pauli equation. To this end, we have used a beam optical representation of the Dirac theory, following [13]-[15] and [18], and have shown that such an approach, in the lowest order approximation, leads naturally to a picture of orbital and spin dynamics based on the Lorentz force, the Stern-Gerlach force and the Thomas-BMT equation for spin evolution, as is to be expected. Only the lowest order (paraxial) approximation has been considered in detail. To illustrate the general theory we have considered the computation of the transfer maps for the spin components and the transverse phase-space, including the transverse Stern-Gerlach kicks, in the case of a normal magnetic quadrupole lens, and a brief understanding of the longitudinal Stern-Gerlach kicks in a general inhomogeneous magnetic field. It is found that the above theory supports the spin-splitter concepts based on transverse and longitudinal Stern-Gerlach kicks ([6]-[12],[23]). It is clear from the general theory, presented briefly here, that the approach is suitable to handle any magnetic

optical element with straight axis and computations can be carried out to any order of accuracy desired by easily extending the order of approximation. In fact, even the lowest order approximation considered reveals some correction terms not usually realized in the traditional quasiclassical theory. Extension of the theory to include electric fields is straightforward (see [14,18]). Inclusion of the effects of radiation in this approach should be possible by following a procedure like in [5].

References

- [1] M. Conte and W.W. MacKay, *An Introduction to the Physics of Particle Accelerators* (World Scientific, Singapore, 1991).
- [2] B.W. Montague, *Phys. Rep.* **113**, 1 (1984).
- [3] A.A. Sokolov and I.M. Ternov, *Radiation from Relativistic Electrons*, Translated by S. Chomet and edited by C.W. Kilmister (American Institute of Physics, New York, 1986).
- [4] J.S. Bell and J.M. Leinass, *Nucl. Phys. B* **284**, 488 (1987).
- [5] L.N. Hand and A. Skuja, *Proc. Conf. on Spin and Polarization Dynamics in Nuclear & Particle Physics (Trieste, January 1988)* (eds. A.O. Barut, Y. Onel and A. Penzo) p.185 (World Scientific, Singapore, 1990).
- [6] Y. Onel, A. Penzo and R. Rossmanith, *AIP Conf. Proc.* 150 (eds. R.G. Lernerand and D.F. Geesaman) p.1229 (American Institute of Physics, New York, 1986).
- [7] T. Niinikoski and R. Rossmanith, *Nucl. Instrum. Methods A* **255**, 46 (1987).
- [8] M. Conte, A. Penzo, A. Pisent and M. Pusterla, *Analytical treatment of the spin splitter*, Internal Report : INFN/TC-88/25.
- [9] M. Conte, *Proc. Conf. on Spin and Polarization Dynamics in Nuclear & Particle Physics (Trieste, January 1988)* (eds. A.O. Barut, Y. Onel and A. Penzo) p.316 (World Scientific, Singapore, 1990).

- [10] A. Pisent, Proc. Conf. on Spin and Polarization Dynamics in Nuclear & Particle Physics (Trieste, January 1988) (eds. A.O. Barut, Y. Onel and A. Penzo) p.327 (World Scientific, Singapore, 1990).
- [11] M. Conte and M. Pusterla, *Il Nuovo Cimento A* **103**, 1087 (1990).
- [12] M. Conte, Y. Onel, A. Penzo, A. Pisent, M. Pusterla and R. Rossmannith, The spin-splitter concept, Internal Report : INFN/TC-93/04.
- [13] R. Jagannathan, R. Simon, E.C.G. Sudarshan and N. Mukunda, *Phys. Lett. A* **134**, 457 (1989).
- [14] R. Jagannathan, *Phys. Rev. A* **42**, 6674 (1990); Corrigendum : *ibid* **44**, 7856 (1991).
- [15] S.A. Khan and R. Jagannathan, Theory of relativistic electron beam transport based on the Dirac equation, presented at the 3rd National Seminar on Physics and Technology of Particle Accelerators and their Applications, 1993, Calcutta, India (Preprint : IMSc-93/53).
- [16] S.A. Khan and R. Jagannathan, Quantum mechanics of charged-particle optics : An operator approach, presented at the JSPS-KEK International Spring School on High Energy Ion Beams - Novel Beam Techniques and their Applications, 1994, Japan (Preprint : IMSc-94/11).
- [17] S.A. Khan and R. Jagannathan, *Phys. Rev. E* **51**, 2510 (1995).
- [18] R. Jagannathan and S.A. Khan, Quantum theory of the optics of charged particles, to appear in *Advances in Imaging and Electron Physics* (Preprint : IMSc-95/31).

- [19] G. Dattoli, A. Reneiri and A. Torre, *Lectures on the Free Electron Laser Theory and Related Topics* (World Scientific, Singapore, 1993).
 - [20] J. Liñares, *Lectures on Path Integration : Trieste* (eds. A. Cerdeira et al.) p.378 (World Scientific, Singapore, 1993).
 - [21] P.W. Hawkes and E. Kasper, *Principles of Electron Optics - Vol.3 : Wave Optics* (Academic Press, San Diego, 1994).
 - [22] J.D. Jackson, *Rev. Mod. Phys.* **48**, 417 (1976).
 - [23] M. Conte, A. Penzo and M. Pusterla, *Il Nuovo Cimento A* **108**, 127 (1995).
 - [24] L.L. Foldy and S.A. Wouthuysen, *Phys. Rev.* **78**, 29 (1950).
 - [25] J.D. Bjorken and S.D. Drell *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
 - [26] A.J. Dragt, F. Neri, G. Rangarajan, D.R. Douglas, L.M. Healy and R.D. Ryne, *Ann. Rev. Nucl. Part. Sci.* **38**, 455 (1988).
 - [27] E. Forest and K. Hirata, A contemporary guide to beam dynamics, KEK Report 92-12.
 - [28] W. Magnus, *Comm. Pure. Appl. Math.* **7**, 649 (1954).
 - [29] J.D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, New York, 1962).
 - [30] K. Hagedorn, *Relativistic Kinematics* (W.A. Benjamin, New York, 1963).
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