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## $A_1^{(1)}$ Admissible Representations-Fusion Transformations and Local Correlators



# $A_1^{(1)}$ ADMISSIBLE REPRESENTATIONS – FUSION TRANSFORMATIONS AND LOCAL CORRELATORS

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## Abstract

We reconsider the earlier found solutions of the Knizhnik-Zamolodchikov (KZ) equations describing correlators based on the admissible representations of  $A_1^{(1)}$ . Exploiting a symmetry of the KZ equations we show that the original infinite sums representing the 4-point chiral correlators can be effectively summed up. Using these simplified expressions with proper choices of the contours we determine the duality (braid and fusion) transformations and show that they are consistent with the fusion rules of Awata and Yamada. The requirement of locality leads to a 1-parameter family of monodromy (braid) invariants. These analogs of the “diagonal” 2-dimensional local 4-point functions in the minimal Virasoro theory contain in general non-diagonal terms. They correspond to pairs of fields of identical monodromy, having one and the same counterpart in the limit to the Virasoro minimal correlators.

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# 1 Introduction

The WZNW models based on the set of integrable representations of an affine Lie algebra  $\hat{g}$  at integer level  $k$  are the best studied conformal models. This is so because on one hand more complicated models can be obtained from them by the coset construction and on the other hand they are simple enough and the full description is probably within reach.<sup>1</sup> The simplicity of these models is most apparent in the equations governing their correlators — the Knizhnik-Zamolodchikov system of equations [1], which has been solved in the general case of integrable representations [2] (see also the earlier works [3], [4], [5] for the case  $\widehat{sl(2)}_k$ ).

Another class of examples is provided by the admissible representations for rational levels (the integrable ones being a particular case of the admissible when the level is a positive integer) introduced and studied in depth by [6]. The study of WZNW conformal models based on admissible representations (see [7] for a lagrangean approach in the case of rational level) is motivated also by the fact that from these models one can obtain others by quantum hamiltonian reduction [8].

In a series of papers [9], [10] we have studied the reduction of the admissible  $\hat{sl}(2)$  models to the Virasoro minimal models on the level of correlators. The first difficulty in describing the correlators of admissible fields comes from the fact that the underlying representations of  $sl(2)$  characterised by rational isospins are infinite dimensional, in general neither lowest nor highest weight representations. In the free field approach this reflects in the necessity to consider screening currents involving rational powers of the ghost fields and a first attempt in this direction was made in [11]. This difficulty we have overcome by passing to a functional realization of  $sl(2)$  in terms of an “isospin coordinate”  $x$  [3] and expanding the correlators in an infinite power series in the differences  $(x_a - z_a)$  with coefficients that are  $z$  dependent multiple contour integrals ( $z_a$  being the “space” coordinates). In this expansion the quantum reduction becomes immediate since the zero order coefficient coincides (for generic values of the isospins) with the minimal model correlators of Dotsenko and Fateev (DF) [12]; thus the reduced theory is obtained by setting  $x_a = z_a$ . The numerical coefficients in the expansion are fixed requiring that the correlators satisfy the KZ equations and the Ward identities accounting for the  $sl(2) \oplus sl(2)$  invariance. (Alternatively the result was reproduced in the free field approach exploiting a particular way of giving meaning to the second screening charge.)

Other approaches to the problem have been proposed [13], [14], [15], leading to different integral representations for the correlators. The relation between them is still not very clear. There is another important issue – namely the braiding properties of the correlators based on the admissible representations, which is not yet systematically investigated. A drawback of our old representation was that the infinite in general power series in  $x - z$  hides a nontrivial analytic behaviour around  $x$ . A naive term by term analysis has led us to an incorrect statement about the fusion rules of admissible fields. After these fusion rules have been found by different means by Awata and Yamada [16] (see also [17] and the recent paper [18]) the question of their consistency with the analytic properties of the correlators still remains open.

We address this problem in the present paper continuing the program started in our earlier work. Recalling in Section 2 some basic definitions and the main results about the solutions of the KZ equations found in [9], [10] (see also [19]), we then show in Section 3 that the infinite sums in the powers of  $(x - z)$  representing the 4-point correlators can be effectively summed up. The resulting simple expressions are precisely the chiral counterparts of the corresponding 2-dimensional physical

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<sup>1</sup>The operator product coefficients (except for the case of  $g = su(2)$ ) are still unknown and the classification of the modular invariants is not yet completed.

4-correlators proposed recently in [15] as a generalisation of the construction of [3]. Our next step in Section 4 is to select a basis in the space of 4-point blocks, characterised by a set of contours (cycles), extending the one in [12]. The blocks reproduce the contribution of all fields described by the fusion rules of Awata and Yamada. Furthermore in Section 5 we show that this set is closed under braid (fusion) transformations, when exchanging pairs of coordinates  $(x_a, z_a)$ , and explicitly compute the matrix elements of these transformations. We point out the existence of another basis of integrals such that a subset of it is closed under the standard fusion transformations of the minimal theory, given (up to signs) by a product of two  $U_q(sl(2))$   $6j$ -symbols. These results are used to construct local 4-point functions. We find basically two types of monodromy (braid) invariants, which can be looked as analogs of the diagonal DF physical correlators recovered in the limit  $x \rightarrow z$ . A particular linear combination of these two basic invariants is identical to the volume integral representation proposed by Andreev in [15] and allows to complete the computation of the corresponding OPE structure constants. The general monodromy invariants however, though containing as labels all the “exponents” in the theory, are not “diagonal” and involve mixed terms. The latter correspond to fields with different values of the isospin label  $J$  and the Sugawara conformal weight  $\Delta_J$ , but identical reduced weight  $h_J = \Delta_J - J$ , which have effectively the same monodromy. This fact is alternatively reflected in the existence of yet another physical 4-point functions with mixed left and right chiral content, corresponding to chiral correlators of the Virasoro minimal and the rational level WZNW models respectively. The technical details of the computations are collected in an Appendix.

## 2 Solutions of the KZ equations with rational isospins

Recall the admissible representations of  $A_1^{(1)}$  [6] labelled by a rational level

$$k + 2 = p/p' \equiv \kappa, \quad (2.1)$$

( $p, p'$  – coprime (positive) integers), and isospins

$$2J_{r,r'} = r - r'\kappa = 2j - 2j'\kappa, \quad (2.2)$$

where  $r, r'$  are nonnegative integers, restricted furthermore to

$$0 \leq r \leq p - 2, \quad 0 \leq r' \leq p' - 1. \quad (2.3)$$

Let  $r' \neq p' - 1$ . Upon quantum hamiltonian reduction the representations labelled by  $J = J_{r,r'}$  and

$$J^{(1)} = \kappa - J - 1, \quad J_{r,r'}^{(1)} = J_{p-r-2, p'-r'-2}, \quad (2.4)$$

both reproduce the Virasoro irreducible representations characterised by a central charge  $c_k = 13 - 6\kappa - 6/\kappa$  and conformal weight

$$h_J = \Delta_J - J = h_{J^{(1)}}, \quad (2.5)$$

where  $\Delta_J$  is the Sugawara scale dimension

$$\Delta_J = \frac{J(J+1)}{k+2} = \Delta_{-J-1}. \quad (2.6)$$

The quantum fields  $\Psi_J(z, x)$  depend on two complex variables and transform with respect to the semidirect product of  $A_1^{(1)}$  and the Virasoro algebra according to

$$[L_n, \Psi_J(x, z)] = (z^{n+1} \partial_z + (n+1) \Delta_J z^n) \Psi_J(x, z), \quad (2.7)$$

$$[X_n^\alpha, \Psi_J(x, z)] = z^n S^\alpha \Psi_J(x, z), \quad (2.8)$$

$$S^0 = -2x \partial_x + 2J, \quad S^- = -\partial_x, \quad S^+ = x^2 \partial_x - 2Jx. \quad (2.9)$$

For real  $x$  (2.9) are the generators of an (infinite dimensional) induced representation of  $SL(2, R)$ . Whenever  $2J$  is a nonnegative integer the representation space has a finite dimensional invariant subspace described by polynomials of  $x$  of maximal degree  $2J$ . Alternatively we can look at them as the generators resulting after analytic continuation to noninteger  $J$  of the induced representations of type  $(J, 0)$  of the algebra  $sl(2, C)$ .

The state  $|J\rangle = \Psi_J(0, 0)|0\rangle$  represents a highest weight state (h.w.s.) of a Verma module with respect to the generators  $X_n, L_n$ . Similarly  $\Psi_J(x, z)|0\rangle = e^{zL_{-1}-xS^-}|J\rangle$  can be viewed as a Verma module h.w.s. with respect to the generators

$$\mathcal{X}_n^\alpha(x, z) \Psi_J(x, z)|0\rangle := e^{zL_{-1}-xS^-} X_n^\alpha |J\rangle, \quad (2.10)$$

$$\mathcal{L}_n(z) \Psi_J(x, z)|0\rangle := e^{zL_{-1}-xS^-} L_n |J\rangle. \quad (2.11)$$

The chiral correlators of the fields  $\Psi_J(x, z)$  are invariant with respect to the  $sl(2, C) \times sl(2, C)$  subalgebra of (2.7), (2.8). They furthermore satisfy the KZ system of equations

$$\left( \frac{\partial}{\partial z_a} - \frac{1}{\kappa} \sum_{\substack{b=1 \\ b \neq a}}^n \frac{\Omega_{ab}}{(z_a - z_b)} \right) W^{(n)}(x_1, z_1, J_1; \dots; x_n, z_n, J_n) = 0, \quad a = 1, 2, \dots, n, \quad (2.12)$$

$$\Omega_{ab} = S_a^\alpha S_{b,\alpha} = -x_{ab}^2 \frac{\partial}{\partial x_a} \frac{\partial}{\partial x_b} + 2x_{ab} \left( J_a \frac{\partial}{\partial x_b} - J_b \frac{\partial}{\partial x_a} \right) + 2J_a J_b, \quad (2.13)$$

alternatively rewritten as

$$\left( \mathcal{L}_{-1,a} - \frac{1}{\kappa} (\mathcal{X}_{-1,a}^+ \mathcal{X}_{0,a}^- + \frac{1}{2} \mathcal{X}_{0,a}^0 \mathcal{X}_{-1,a}^0) \right) W^{(n)}(x_1, z_1, J_1; \dots; x_n, z_n, J_n) = 0. \quad (2.14)$$

Here for  $n \geq 0$

$$\mathcal{X}_{-n,a}^- = - \sum_{b(\neq a)} \frac{S_b^-}{z_{ba}^n}, \quad \mathcal{X}_{-n,a}^0 = - \sum_{b(\neq a)} \frac{-S_b^0 + 2x_a S_b^-}{z_{ba}^n}, \quad \mathcal{X}_{-n,a}^+ = - \sum_{b(\neq a)} \frac{S_b^+ + x_a S_b^0 - x_a^2 S_b^-}{z_{ba}^n}. \quad (2.15)$$

$$\mathcal{L}_{-n,a} = \sum_{b(\neq a)} \frac{1}{z_{ba}^{n-1}} \left( \frac{(n-1)\Delta_{J_b}}{z_{ba}} - \frac{\partial}{\partial z_b} \right). \quad (2.16)$$

The Ward identities imply that

$$\mathcal{X}_{0,a}^+ W = 0, \quad \mathcal{X}_{0,a}^0 W = 2J_a W, \quad \mathcal{L}_{0,a} W = \Delta_{J_a} W, \quad (2.17)$$

(so that in (2.14)  $\mathcal{X}_{0,a}^0$  can be replaced by  $2J_a$ ) and furthermore  $\mathcal{X}_{0,a}^- W = S_a^- W = -\frac{\partial}{\partial x_a} W$ ,  $\mathcal{L}_{-1,a} W = L_{-1,a} W = \frac{\partial}{\partial z_a} W$ . (Here  $W = W^{(n)}$ .) The operators (2.15) (2.16) (see [21], [3]) are

derived from (2.10), (2.11) and their generalisations for arbitrary number of operators<sup>2</sup>, moving  $X_{-n}$  (or  $L_{-n}$ ) to the left and using the commutators (2.8), (2.7). By the definition in (2.10), (2.11) the positive mode generators vanish.

The correlators furthermore should satisfy two series of algebraic equations resulting from the presence of singular vectors in the Kac–Moody Verma modules labelled by  $J = J_{r,r'} \equiv J_{r-p+2,r'-p'+2}$ ,

$$\mathcal{P}_i(\mathcal{X}_{-1,a}^+, \mathcal{X}_{0,a}^-) W = 0, \quad (2.18)$$

where  $\mathcal{P}_i(X_{-1}^+, X_0^-)$ ,  $i = 1, 2$ , are some monomials in powers of the Kac–Moody generators  $X_{-1}^+, X_0^-$ , [20], realised here in terms of the generators (2.15). If  $J_a$  is of the type  $J_{m,1}$ , the first of these operators is simply  $(\mathcal{X}_{0,a}^-)^m = (-\partial_{x_a})^m$ .

Let us now describe the explicit form of the solutions found in [9], [10]. Up to a standard prefactor the full  $n$ -point correlators are recovered from the correlators “at infinity”

$$\begin{aligned} W^{(n,\infty)}(\{x_a, z_a, J_a\}) &:= \lim_{x_n, z_n \rightarrow \infty} x_n^{-2J_n} z_n^{2\Delta_{J_n}} \langle 0 | \Psi_{J_n}(x_n, z_n) \cdots \Psi_{J_1}(x_1, z_1) | 0 \rangle \\ &= \langle J_n | \Psi_{J_{n-1}}(x_{n-1}, z_{n-1}) \cdots \Psi_{J_1}(x_1, z_1) | 0 \rangle \end{aligned}$$

if furthermore  $x_a, z_a$  are replaced by

$$\underline{x}_a = \frac{x_{1a} x_{n-1,n}}{x_{1,n-1} x_{an}}, \quad \underline{z}_a = \frac{z_{1a} z_{n-1,n}}{z_{1,n-1} z_{an}}, \quad a = 1, 2, \dots, n-1.$$

In the counterpart of (2.12) applied to  $W^{(n,\infty)}$  the indices  $a, b$  run over  $1, 2, \dots, n-1$ ; the same applies to the generators (2.16), (2.15), now defined for  $n > 0$ . The zero mode subalgebra acts as in (2.17) and the surviving Ward identities for  $X_0^0$  and  $L_0$  get accordingly modified by the values  $J_n, \Delta_{J_n}$  at infinity (see e.g., [19]). The operators  $\mathcal{P}_i$  can be also realised in terms of the generators  $\Delta(X_{-1}^-)$  and  $\Delta(X_0^+)$ , (replacing  $X_{-1}^+$  and  $X_0^-$  respectively), where  $\Delta(X_n^\alpha) = \sum_{a=1}^{n-1} X_{n,a}^\alpha$ , resulting from the action on the dual vacuum state  $\langle J_n |$ .

Now assume that the real numbers  $J_a = j_a - j'_a \kappa$ ,  $a = 1, 2, \dots, n$  (with  $2j_a, 2j'_a$  not necessarily satisfying (2.3)) are such that both  $s$  and  $s'$ , defined according to

$$s^{(\prime)} = \sum_{a=1}^{n-1} j_a^{(\prime)} - j_n^{(\prime)},$$

are nonnegative integers. Denoting

$$S = \sum_{a=1}^{n-1} J_a - J_n = s - s' \kappa, \quad (2.19)$$

the correlator  $W^{(n,\infty)}$  reads

$$W^{(n,\infty)}(\{x_a, z_a, J_a\}) = \prod_{1 \leq b < a \leq n-1} (z_a - z_b)^{2J_a J_b / \kappa} \sum_{t=0} \sum_{\tau: |\tau|=t, a=1}^{n-1} \prod (x_a - z_a)^{\tau_a} B_\tau I_{\Gamma, \tau}^{(s, s')}(\{z_a, J_a\}), \quad (2.20)$$

<sup>2</sup>E.g., the generalisation of (2.10) reads

$$\begin{aligned} \mathcal{X}_n^\alpha(x, z) \Psi_J(x, z) \Psi_{J_r}(x_r, z_r) \cdots \Psi_{J_1}(x_1, z_1) | 0 \rangle &:= e^{z L_{-1} - x S^-} \{ [X_n^\alpha, \Psi_J(0, 0)] \Psi_{J_r}(x_r - x, z_r - z) \cdots \\ &\cdots \Psi_{J_1}(x_1 - x, z_1 - z) + \Psi_J(0, 0) \Psi_{J_r}(x_r - x, z_r - z) \cdots \Psi_{J_1}(x_1 - x, z_1 - z) X_n^\alpha \} | 0 \rangle, \end{aligned}$$

etc.

where  $\tau$  is a  $n-1$  vector with nonnegative integer components  $t_a$ ,  $B_\tau$  are numerical coefficients

$$B_\tau = \beta_J \frac{(-1)^t \Gamma(c_t + 1)}{\Gamma(2J_n + 2 - \kappa) \Gamma(S - t + 1)} \prod_{a=1}^{n-1} \binom{2J_a}{\tau_a}, \quad t = |\tau| = \sum_{a=1}^{n-1} \tau_a, \quad (2.21)$$

where  $\binom{a}{k} = \frac{\Gamma(a+1)}{\Gamma(a-k+1) k!}$  and  $\beta_J$  is an overall constant,

$$c_t = c_t(J) = \sum_{a=1}^n J_a - t + 1 - \kappa = S - t + 2J_n + 1 - \kappa, \quad (2.22)$$

while  $I_{\Gamma, \tau}^{(s, s')}$  are contour integrals

$$I_{\Gamma, \tau}^{(s, s')}(\{z_a, J_a\}) = \int_{\Gamma} du_1 \dots du_s dv_1 \dots dv_{s'} \varphi_{\tau}^{(s, s')}(u_i, v_{i'}; z_a) \Phi_J^{(s, s'; \kappa)}(u_i, v_{i'}; z_a). \quad (2.23)$$

The factor  $\Phi_J^{(s, s'; \kappa)}$  is the standard integrand of the DF conformal blocks,

$$\Phi_J^{(s, s'; \kappa)}(u_i, v_{i'}; z_a) = \Phi_J^{(s; 1/\kappa)}(u_i; z_a) \Phi_{-J/\kappa}^{(s'; \kappa)}(v_{i'}; z_a) \prod_{i=1}^s \prod_{i'=1}^{s'} \frac{1}{(u_i - v_{i'})^2}, \quad (2.24)$$

and

$$\Phi_J^{(s; 1/\kappa)}(u_i; z_a) = \prod_{j < i}^s (u_i - u_j)^{2/\kappa} \prod_{i=1}^s \prod_{a=1}^{n-1} (u_i - z_a)^{-2J_a/\kappa}. \quad (2.25)$$

To describe the factor  $\varphi_{\tau}^{(s, s')}$  introduce the  $(n-1)$ -vector  $\mu$  with non-negative integer components  $\mu_a$ , and set for any integer  $m$  and arbitrary  $L$

$$\varphi_{\mu}^{[L, m]}(\{u_i, z_a\}) = \frac{\Gamma(mL - |\mu| + 1)}{\Gamma(mL + 1)} \sum_{\{N_{ia}\}} \prod_{i=1}^m \frac{\Gamma(L + 1)}{\Gamma(L - \sum_{a=1}^{n-1} N_{ia} + 1)} \frac{\prod_{a=1}^{n-1} [\mu_a! \delta(\sum_{i=1}^s N_{ia} - \mu_a)]}{\prod_{i=1}^m \prod_{a=1}^{n-1} [N_{ia}! (u_i - z_a)^{N_{ia}}]}. \quad (2.26)$$

The sums in (2.26) are finite, running over  $N_{ia} = 0, 1, \dots, \mu_a$ , subject to the constraints  $\sum_i N_{ia} = \mu_a$ . For  $L = 1$   $N_{ik}$  get furthermore restricted to  $N_{ik} = 0, 1$ . In general (2.26) is used to define the factor  $\varphi_{\tau}^{(s, s')}(\{u_i, v_{i'}, z_a\})$  in (2.23) according to [9], [10]

$$\varphi_{\tau}^{(s, s')}(\{u_i, v_{i'}, z_a\}) := \frac{1}{\binom{S}{s}} \sum_{\lambda} \prod_{a=1}^{n-1} \binom{\tau_a}{\lambda_a} \binom{S - |\tau|}{s - |\lambda|} \varphi_{\tau - \lambda}^{[-\kappa, s']}(\{v_{i'}, z_a\}) \varphi_{\lambda}^{[1, s]}(\{u_i, z_a\}), \quad (2.27)$$

where the sum runs over  $\{\lambda_a = 0, 1, \dots, \min(s, \tau_a)\}$ ,  $|\lambda| \leq \min(s, |\tau|)$ . Apparently for any  $\tau$  (2.27) is a meromorphic function of  $\{u_i, v_{i'}, z_a\}$ .

In the limit  $x_a \rightarrow z_a$ , which was related to the QHR<sup>3</sup>, (2.20) (taken with the standard DF [12] contours  $\Gamma$ ) recover for generic spins the general DF correlators of the minimal Virasoro theory, represented by the 0<sup>th</sup> term in the (infinite) expansion (2.20). For  $s' = 0$  the sums in (2.20) become finite.

Although a direct check of the general algebraic equations (2.18) is still missing for the above correlators, it was argued in [19] that their validity is related to the fact that the DF counterparts

<sup>3</sup>This limit was generalised and applied to the 2-point functions of the  $sl(n)$  WZNW model in [22].

satisfy the BPZ equations, resulting from the singular vectors in the corresponding Virasoro Verma modules. The latter equations are expressed in terms of Virasoro generators realised by operators similar to (2.16), with the Sugawara dimension  $\Delta_J$  replaced by  $h_J$ . These arguments were further supported by the proof, done within the algebraic BRST framework, that the MFF singular vectors reproduce the corresponding Virasoro ones modulo trivial terms in the  $Q_{\text{BRST}}$  cohomology [23].

Applied to the 3-point functions the equations (2.18) yield the fusion rules, computed explicitly in this way in [16].

### 3 Simplified expression for the 4-point correlators due to a symmetry of the KZ equations.

We now turn to the case of 4-point correlators. In the system  $(0, z = z_2, 1)$  the summation in (2.20) reduces to a single sum,  $\tau = (0, t, 0)$ , and the KZ equation applied to the series (2.20) turns to an infinite set of equations for the integral coefficients  $I_{\Gamma, t}^{(s, s')}(J; z) := I_{\Gamma, (0, t, 0)}^{(s, s')}(\{z_a, J_a\})$

$$(\kappa \partial_z + a_t(J, z)) I_{\Gamma, t}^{(s, s')}(J; z) + \frac{t c_{t-1}(J)}{z(z-1)} I_{\Gamma, t-1}^{(s, s')}(J; z) = (2J_2 - t)(S - t) I_{\Gamma, t+1}^{(s, s')}(J; z), \quad (3.1)$$

where

$$a_t(J, z) = \frac{t(2J_1 + 2J_2 - t + 1)}{z} + \frac{t(2J_3 + 2J_2 - t + 1)}{z-1}, \quad \text{and } t = 0, 1, 2, \dots \quad (3.2)$$

Introduce the new set of isospins <sup>4</sup>

$$\tilde{J}_1 = \frac{1}{2}(J_1 + J_2 - J_3 - J_4 - 1 + \kappa), \quad (3.3)$$

$$\tilde{J}_2 = \frac{1}{2}(J_1 + J_2 + J_3 + J_4 + 1 - \kappa), \quad (3.4)$$

$$\tilde{J}_3 = \frac{1}{2}(-J_1 + J_2 + J_3 - J_4 - 1 + \kappa), \quad (3.5)$$

$$\tilde{J}_4 = \frac{1}{2}(-J_1 + J_2 - J_3 + J_4 - 1 + \kappa). \quad (3.6)$$

Note that  $J_i \rightarrow \tilde{J}_i$  is an involution. This change of variables (3.3) keeps invariant  $s$  and  $s'$  as well as the numerical coefficient  $a_t(J, z)$  in (3.2) since  $\tilde{J}_1 + \tilde{J}_2 = J_1 + J_2$ ,  $\tilde{J}_3 + \tilde{J}_4 = J_3 + J_4$ . Replacing everywhere in (3.1) the isospins  $J_a$  by  $\tilde{J}_a$  as defined in (3.3) we can identify the set of integrals  $I_{\Gamma, t}^{(s, s')}(\tilde{J}; z)$  serving as solutions of the new equation with

$$I_{\Gamma, t}^{(s, s')}(\tilde{J}; z) = \text{const} (-1)^t \frac{B_t}{\binom{S}{t}} I_{\Gamma, t}^{(s, s')}(J; z), \quad t = 0, 1, 2, \dots, \quad B_t := B_{(0, t, 0)}, \quad (3.7)$$

since the equation (3.1) remains invariant under this change. To see this multiply (3.1) by the  $t$ -dependent constant in (3.7) and use that

$$\frac{B_t}{\binom{S}{t}} c_{t-1}(J) = \frac{B_t}{\binom{S}{t}} (2\tilde{J}_2 - t + 1) = \frac{B_{t-1}}{\binom{S}{t-1}} c_{t-1}(\tilde{J}),$$

<sup>4</sup>There is a misprint on page 683 of [10] in the analogous formula for  $\tilde{j}_t$  ( $j_1$  and  $j_4$  should be reversed), as well as in eq. (3.14), in which  $c_t$  should be replaced by  $c_{t-1}$ .

$$\frac{B_t}{\binom{S}{t}} (2J_2 - t) = \frac{B_{t+1}}{\binom{S}{t+1}} (2\tilde{J}_2 - t).$$

For  $t = 0$  (3.7) reduces to a relation for the DF integrals of the minimal Virasoro theory. This in particular allows to compute the overall,  $t$ -independent, constant in (3.7) – taking  $t = 0$  and using the results of [12] for the normalisation constants of the integrals  $I_{\Gamma,0}^{(s,s')}(J; z)$  and  $I_{\Gamma,0}^{(s,s')}(\tilde{J}; z)$ . Note that due to the invariance of the combinations  $J_1 + J_2$  (or  $J_3 - J_4$ ),  $J_2 + J_3$  (or  $J_1 - J_4$ ) under the transformation (3.7), and due to the relation  $\tilde{J}_2 - \tilde{J}_4 + \kappa - 1 = J_1 + J_3$  (or  $\tilde{J}_1 - \tilde{J}_3 = J_2 - J_4 + \kappa - 1$ ), these two types of integrals admit an identical asymptotic behaviour for a fixed contour in the sets corresponding to any of the  $s$ -,  $t$ -, and  $u$ -channels.

The observation about the invariance of the KZ-ZF system of equations (3.1) under the transformation (3.7) was made in [10] in the particular case  $s' = 0$  in order to compare with the solutions of the system as presented in [3], where the combinations of isospins appear. Applied however in the more general case  $s' \neq 0$  it allows to effectively sum the infinite power in the 4-point function (2.20). Indeed for  $\tau = (0, t, 0)$  the factor (2.27) simplifies to

$$\begin{aligned} \varphi_{(0,t,0)}^{(s,s')}(\{u_i, v_{i'}, z\}) &= \frac{1}{\binom{S}{t}} \sum_{l=0}^{\min(s,t)} \sum_{\{N'_{i'}\}} \delta\left(\sum_{i'} N'_{i'} - t + l\right) \prod_{i'=1}^{s'} \binom{-\kappa}{N'_{i'}} \frac{1}{(v_{i'} - z)^{N'_{i'}}} \\ &\quad \sum_{\{N_i\}} \delta\left(\sum_i N_i - l\right) \prod_{i=1}^s \binom{1}{N_i} \frac{1}{(u_i - z)^{N_i}}, \end{aligned}$$

and it is straightforward to show that

$$\begin{aligned} \varphi^{(s,s';\kappa)}(\{u_i, v_{i'}; \mathbf{x}, z\}) &:= \sum_{t=0}^{\infty} (-1)^t \binom{S}{t} (x - z)^t \varphi_{(0,t,0)}^{(s,s')}(\{u_i, v_{i'}, z\}) \\ &= \prod_{i=1}^s \left(\frac{u_i - \mathbf{x}}{u_i - z}\right) \prod_{i'=1}^{s'} \left(\frac{v_{i'} - \mathbf{x}}{v_{i'} - z}\right)^{-\kappa}. \end{aligned} \quad (3.8)$$

Hence taking into account (3.7) we can rewrite the 4-point correlator (2.20) as

$$W^{(4,\infty)}(\mathbf{x}, z, \mathbf{J}) = \text{const } z^{2J_1 J_2 / \kappa} (1 - z)^{2J_3 J_2 / \kappa} I_{\Gamma}^{(s,s')}(\tilde{\mathbf{J}}; \mathbf{x}, z), \quad (3.9)$$

$$I_{\Gamma}^{(s,s')}(\tilde{\mathbf{J}}; \mathbf{x}, z) := \int_{\Gamma} du_1 \dots du_s dv_1 \dots dv_{s'} \varphi^{(s,s';\kappa)}(\{u_i, v_{i'}; \mathbf{x}, z\}) \Phi_{\tilde{\mathbf{J}}}^{(s,s';\kappa)}(u_i, v_{i'}; z). \quad (3.10)$$

In the limit  $\mathbf{x} \rightarrow z$  (3.9) provides an alternative expression for the DF conformal blocks in which in the integrand  $J_a$  are replaced by  $\tilde{J}_a$ .

The expression (3.9), (3.10) extends the infinite sum representing the correlator beyond its range of convergence. (Expanding instead in powers of  $(u_i - z_a)/(x_a - z_a)$  we recover the solutions with inverse powers of  $(x_a - z_a)$ , mentioned in [10].) The explicit singularity for noninteger  $\kappa$  of the new factor (3.8) suggests that the set of contours (cycles) considered in [12] has to be enlarged. Our objective in the next section will be to study the braiding properties of the integrals  $I_{\Gamma}^{(s,s')}(\tilde{\mathbf{J}}; \mathbf{x}, z)$  for some choices of the contours  $\Gamma$ .

**Remark:** The infinite sum in (2.20) can be taken also in a different way for special values of the isospins. E.g., for  $n = 4$ ,  $s' = 1$  and  $2J_4 + 1 = \kappa$  we have  $c_t = S - t$ , the constant  $B_t$  simplifies to

$B_t = (-1)^t \binom{2J_2}{t}$ , and the infinite sum produces  $(\frac{v-x}{v-z})^{2J_2}$ , without other changes in the integrand. In this form this example coincides with the corresponding integral in [13]. However, unless we misunderstand the prescription there, this identification holds under different restrictions on the isospins – the condition (2.19) above,  $J_1 + J_2 + J_3 - J_4 = -\kappa$  seems to be replaced in [13] by  $J_1 + J_2 + J_3 + J_4 + 1 = 0$  and the two coincide only for the special choice  $2J_4 + 1 = \kappa$ .

## 4 Basic set of chiral blocks and fusion rules

Recall the well known fusion rules of the WZNW conformal field theory for  $A_1^{(1)}$  at integer level  $p-2$ . If the highest weights of the integrable representations are labelled by  $r(\equiv 2j) = 0, 1, \dots, p-2$  then the fusion multiplicities  $N_{r_1 r_2}^r(p)$  are equal to 1 if

$$r \in \{|r_1 - r_2|, |r_1 + r_2| + 2, \dots, p-2 - |r_1 + r_2 - p + 2|\}$$

and  $N_{r_1 r_2}^r(p) = 0$  otherwise. The algebra of matrices  $N_r(p)$ ,  $(N_r)_{r_1}^{r_2}(p) = N_{r r_1}^{r_2}(p)$  is  $\mathbb{Z}_2$  graded, namely if  $r \mapsto \tau(r) = 0, 1$  if  $r$  – even, odd, respectively, and if  $N_{r_1 r_2}^r = 1$  then  $\tau(r_1) + \tau(r_2) = \tau(r) \bmod 2$ , and thus it possesses a subalgebra  $\{N_r(p), \tau(r) = 0\}$ . Note that  $N_{r p - r_1 - 2}^{p - r_2 - 2}(p) = N_{r r_1}^{r_2}(p)$ . We shall furthermore use the notation  $J^{(t)}$ ,  $t = 0, 1 \bmod 2$ , where  $J^{(0)} \equiv J$ , and  $J_{r r'}^{(1)} = J_{p-r-2, p'-r'-2}$ , for  $r' \neq p-1$ , as defined in (2.4) (i.e., the weight obtained from  $J$  by the shifted action of the affine root Weyl reflection  $w_0$ ).

The fusion rules for the admissible representations, first derived by Awata-Yamada [16] can be written in the following form, suggested by the consideration in the recent paper [18]

$$J_{r_1, r_1'} \otimes J_{r_2, r_2'} = \sum_{r, r'} N_{r_1 r_2}^r(p) N_{2r_1' 2r_2'}^{2r'}(2p') J_{r, r' - \tau(r_1' + r_2' - r')}^{(\tau(r_1' + r_2' - r'))}, \quad (4.1)$$

where  $N_{2r_1' 2r_2'}^{2r'}(2p')$  are the fusion coefficients of the fusion subalgebra  $\{N_m(2p'), \tau(m) = 0\}$  of a  $A_1^{(1)}$  theory at level  $2(p'-1)$ . Thus in (4.1)  $r'$  runs along all integers between  $|r_1' - r_2'|$  and  $p'-1 - |p'-1 - r_1' - r_2'| \leq p'-1$ . Both multiplicities  $N_{r_1 r_2}^r(p)$  and  $N_{2r_1' 2r_2'}^{2r'}(2p')$ , can be expressed by a Verlinde type formula in terms of the modular matrices at level  $p-2$  and  $2p'-2$  respectively. In [18] Feigin and Malikov gave a ‘‘supersymmetric’’ interpretation of the fusion rule of Awata and Yamada, showing that it coincides with the product of the truncated tensor products of  $U_q(sl(2))$  and  $U_{q'}(osp(1|2))$  for  $q^p = 1, (q')^{p'} = 1$ , respectively. Note that in (4.1) the sum in  $r$  runs by two while the sum in  $r'$  – by one, the new terms counted by the ‘‘odd’’ isospins  $J^{(t=1)}$ .

Now we turn to the basis of 4-point conformal blocks. Such a basis can be labelled by the isospin running in the intermediate channel or equivalently is given by an appropriate choice of the integration contours  $\mathcal{C}$ .

The  $s$ -channel basis is labelled by an isospin  $J$  that is the result of the fusion of  $J_1$  and  $J_2$ . The  $t$ -channel basis is labelled by an isospin  $J$  that is the result of the fusion of  $J_3$  and  $J_2$ , and the  $u$ -channel — by isospins  $J$  in the fusion of  $J_4$  and  $J_2$ . The dimension of any of them is

$$\sum_{r r'} N_{(r_1 r_1')(r_2 r_2')}^{(r r')} N_{(r r')(r_3 r_3')}^{(r_4 r_4')} = \sum_{r r'} N_{(r_3 r_3')(r_2 r_2')}^{(r r')} N_{(r r')(r_1 r_1')}^{(r_4 r_4')} = \sum_{r r'} N_{(r_1 r_1')(r_3 r_3')}^{(r r')} N_{(r r')(r_2 r_2')}^{(r_4 r_4')}$$

where

$$N_{(r_1 r_1')(r_2 r_2')}^{(r r')} := N_{r_1 r_2}^r(p) N_{2r_1' 2r_2'}^{2r'}(2p').$$

We describe in more details the “even” part of the  $s$ -basis. Let  $m, m'$  be a pair of nonnegative integers,  $m = 0, 1, \dots, s, m' = 0, 1, \dots, s'$ . The “even”  $s$ -channel 4-point blocks (3.10) are given by

$$\begin{aligned}
I_{mm'}(a, b, c, d; x, z) = & \int_{\mathcal{C}_{m,m'}} \prod_{i=1}^m u_i^{-a/\kappa} (z - u_i)^{-1-c/\kappa} (x - u_i) (1 - u_i)^{-b/\kappa} \\
& \prod_{i=m+1}^s u_i^{-a/\kappa} (u_i - z)^{-1-c/\kappa} (u_i - x) (u_i - 1)^{-b/\kappa} \\
& \prod_{i'=1}^{m'} v_{i'}^a (z - v_{i'})^{c+\kappa} (x - v_{i'})^{-\kappa} (1 - v_{i'})^b \\
& \prod_{i'=m'+1}^{s'} v_{i'}^a (v_{i'} - z)^{c+\kappa} (v_{i'} - x)^{-\kappa} (v_{i'} - 1)^b \\
& \prod_{1 \leq j < i \leq s} (u_i - u_j)^{2/\kappa} \prod_{1 \leq j' < i' \leq s'} (v_{i'} - v_{j'})^{2\kappa} \prod_{i=1}^s \prod_{i'=1}^{s'} (u_i - v_{i'})^{-2} du_i dv_{i'}
\end{aligned} \tag{4.2}$$

where  $a = 2\tilde{J}_1, b = 2\tilde{J}_3, c = 2\tilde{J}_2$ , and  $d = 2(\kappa - 1 - \tilde{J}_4)$ . Since  $a + b + c + d + 2 - 2\kappa - 2S = 0$  the integrals can be parametrized also by  $a, b, c$  and  $s, s'$ . The above integral is defined for real  $z, x, 0 < z < x < 1$ . The cycle over which the variables  $u_i, v_{i'}$  are integrated can be described as

$$\begin{aligned}
\mathcal{C}_{m,m'} = & \{0 < u_1 < \dots < u_m < z\} \cup \{1 < u_{m+1} < \dots < u_s < \infty\} \\
& \cup \{0 < v_1 < \dots < v_{m'} < z\} \cup \{1 < v_{m'+1} < \dots < v_{s'} < \infty\},
\end{aligned} \tag{4.3}$$

and in this region all powers in (4.2) can be chosen real, positive (for real parameters).

Alternatively one can use

$$\mathcal{C}(m, m') = \{C_i : i = 1, \dots, s\} \cup \{C_{i'} : i' = 1, \dots, s'\} \tag{4.4}$$

where  $C_{i^{(l)}}$  are contours from 0 to  $z$  for  $i^{(l)} = 1, \dots, m^{(l)}$  and contours from 1 to  $\infty$  for  $i^{(l)} = m^{(l)} + 1, \dots, s^{(l)}$  with  $C_{i^{(l)}+1}^{(l)}$  above  $C_{i^{(l)}}^{(l)}$ . Moreover the integrals with contour  $\mathcal{C}(m, m')$  are taken with an overall phase

$$\exp -\frac{i\pi}{2} [(m(m-1) + (s-m)(s-m-1))/\kappa + (m'(m'-1) + (s'-m')(s'-m'-1))\kappa]. \tag{4.5}$$

The contribution from the two alternative cycles differs by a numerical factor

$$\int_{\mathcal{C}(m,m')} = [m]! [s-m]! [m']! [s'-m']! \int_{\mathcal{C}_{m,m'}}, \tag{4.6}$$

where

$$[k] = \sin(\pi k/\kappa) / \sin(\pi/\kappa) \quad \text{and} \quad [k]' = \sin(\pi k\kappa) / \sin(\pi\kappa)$$

are  $q = \exp(2i\pi/\kappa)$  and  $q' = \exp(2i\pi\kappa)$  numbers respectively. In the limit  $x \rightarrow z$  we recover the DF basis of integrals (our notation differs from that in [12] by a shift in the indices  $m, m'$  and  $s, s'$  by one).

The “even” integrals in the  $t$ -basis we will denote by  $I^{m,m'}$ . The contours of integration are obtained from the above by interchanging the roles of the points 1 and 0, i.e., we have  $(m, m')$

integrations from  $z$  to 1 and  $(s - m, s' - m')$  integrations from  $-\infty$  to 0. Also the integrand is modified so that again all differences (when the variables are continued to the real line) are positive real numbers and for the  $t$ -channel we assume that  $0 < x < z < 1$ .

The  $u$ -channel integrals correspond to contours from 0 to 1 and from  $z$  to  $\infty$ . We will not introduce a special notation for these integrals.

The “even” integrals are not closed under crossing transformations. If we employ the well known technique of moving contours, e.g., to express the  $s$ -channel in terms of the  $t$ -channel (see the Appendix for details), because the integrand has a nontrivial singularity at  $x$  (with respect to the  $v_i$  variables) we will generate also “odd” integrals that will involve contours  $\Gamma_x$  ending at, or going around  $x$ . On the other hand the singularity at  $x$  is like the insertion of a vertex operator of isospin  $j' = 1/2$ . Because of this if we have one contour to  $x$  and add a second encircling the previous one the result will be zero. Thus we have only two possibilities — no contour to  $x$  which corresponds to the “even” integrals and one contour to  $x$  which corresponds to the “odd” integrals.

Now we describe a convenient choice for the “odd” integrals of the  $s$ -basis which we will denote with  ${}^{(1)}I_{m,m'}$  with  $m = 0, 1, \dots, s; m' = 1, 2, \dots, s'$ . The cycle  ${}^{(1)}C_{m,m'}$  consists of integrating  $v_{i'}$ ,  $i' = 1, \dots, m' - 1$  from 0 to  $z$  while  $v_{m'}$  is integrated along a contour starting at  $x$  going around all the contours from 0 to  $z$  in the counterclockwise direction and returning back to  $x$  (i.e., we have a Felder type contour for  $v_{m'}$ ). The other integration contours are unchanged. The integrand is the same as for  $I_{m,m'}$  except for the order of the arguments in the factor  $(v_{m'} - z)^{c+\kappa}$ . It is convenient moreover to include in the definition an overall numerical factor

$${}^{(1)}I_{m,m'} = \frac{e^{-i\pi(a+c+(2m'-1)\kappa)}}{\sin(\pi(a+c+(2m'-1)\kappa))} \int_{{}^{(1)}C_{m,m'}} \dots$$

The limit  $x \rightarrow z$  of the “odd” integrals  ${}^{(1)}I_{m,m'}$  coincides (up to numerical prefactors) with that of the “even” ones  $I^{m,m'}$  (for  $m = 0, 1, \dots, s; m' = 1, 2, \dots, s'$ ).

Analogously, the “odd”  $t$ -basis is given by  ${}^{(1)}I^{m,m'}$  which differ from the “even” ones by trading one integration from  $z$  to 1 for an integration along a Felder type contour from  $x$  around all the contours from  $z$  to 1 and back to  $x$  and by appropriate change of the integrand. Also in the numerical factor above  $a$  is changed to  $b$ .

One can check that the proposed contours correctly correspond to the relevant basis by taking the asymptotics of the 4-point blocks  $W^{(4)}(x, z)$  as  $z, x \rightarrow 0$  for the  $s$ -channel and  $z, x \rightarrow 1$  for the  $t$ -channel and comparing with the corresponding 3-point blocks. We emphasise that in order to get the correct asymptotics in both channels we have to take the limit first in  $z$  and then in  $x$  and this explains why the  $s$ -channel integrals we define for  $0 < z < x < 1$  while the  $t$ -channel ones are defined for  $0 < x < z < 1$ .

To find the asymptotics of the “even” integrals one makes a change of variables  $u \rightarrow zu, v \rightarrow zv$  for those variables that are integrated from 0 to  $z$  and takes first  $z \rightarrow 0$  and then  $x \rightarrow 0$  obtaining

$$I_{m,m'}(a, b, c, d; z, x) \propto z^{-(\Delta_{J_1} + \Delta_{J_2} - \Delta_J) - 2J_1 J_2} x^{J_1 + J_2 - J} J_{m,m'}(a, c + \kappa; \kappa) J_{s-m, s'-m'}(d, b; \kappa), \quad (4.7)$$

where the intermediate isospin  $J$  is given by

$$m - m'\kappa = J_1 + J_2 - J \quad (4.8)$$

and  $J_{m,m'}(\alpha, \beta; \rho)$  are the Selberg integrals giving the normalization of the 3-point functions which have been calculated explicitly in [12], formula (A.35). In particular

$$J_{m,m'}(a, c + \kappa; \kappa) = \frac{\Gamma(1 + c - m + m'\kappa) \Gamma(2 + a + c - m + (m - 1)\kappa)}{\Gamma(1 + c) \Gamma(2 + a + c - 2m + (2m' - 1)\kappa)} J_{m,m'}(a, c; \kappa). \quad (4.9)$$

Recall that the chiral correlators (3.9) differ from the integrals by the overall factor  $z^{2J_1 J_2} (1-z)^{2J_2 J_3}$ , which will compensate  $-2J_1 J_2$  in the power of  $z$  above, and thus producing exactly the functional dependence of a 3-point function.

The asymptotics of the “odd” integrals is performed as follows. With the integrals from 0 to  $z$  proceed as above. The integral over  $v_{m'}$  express as a combination of two integrals from 0 to  $x$ , one going above, the other below  $z$ , and in each of them make the change of variables  $v_{m'} \rightarrow xv_{m'}$  and again take first  $z \rightarrow 0$  and then  $x \rightarrow 0$ . The result is

$${}^{(1)}I_{m,m'}(a, b, c, d; z, x) \propto z^{-(\Delta_{J_1} + \Delta_{J_2} - \Delta_J^{(1)}) - 2J_1 J_2} x^{J_1 + J_2 - J^{(1)}} J_{0,1}(2J - \kappa, -\kappa; \kappa) J_{m,m'-1}(a, c + \kappa; \kappa) J_{s-m, s'-m'}(d, b; \kappa), \quad (4.10)$$

where  $J^{(1)} = \kappa - 1 - J$  with  $J$  as in (4.8) and  $\Delta_J^{(1)} \equiv J^{(1)}(J^{(1)} + 1)/\kappa = \Delta_J + \kappa - 2J - 1$ . Note that the prefactor we had included in the definition of  ${}^{(1)}I_{m,m'}$  has cancelled on the r.h.s. above. The normalisation constants can be again expressed through the ones in [12]

$$\begin{aligned} & J_{0,1}(2J - \kappa, -\kappa; \kappa) J_{m,m'-1}(a, c + \kappa; \kappa) \\ &= \frac{\Gamma(\kappa) \Gamma(1 - \kappa) \Gamma(1 + a + c - 2m + (2m - 1)\kappa)}{\Gamma(1 + c) \Gamma(-m + m'\kappa) \Gamma(1 + a - m + (m' - 1)\kappa)} J_{m,m'}(a, c; \kappa). \end{aligned} \quad (4.11)$$

We conclude that the asymptotics of the “even” and “odd” integrals is qualitatively in agreement with the fusion rules (4.1).

For the asymptotics of the  $t$ -basis integrals analogously first we take  $z \rightarrow 1$  and then  $x \rightarrow 1$ . Again one checks that the identification between choice of contours and intermediate isospins is correct.

Note that the above choices of “odd” integrals is not unique. Instead of the Felder type contours one could also take e.g., an arbitrary combination of the constituent contours going from 0 to  $x$ , above or below  $z$ , the corresponding integrals to be denoted  ${}^{(1)}I_{m,m'}^+$  and  ${}^{(1)}I_{m,m'}^-$  respectively. Furthermore there is a certain ambiguity in the way the limit “first  $z$ , then  $x$  goes to 0”, is taken.

## 5 Braid and fusion transformations, local correlators and structure constants

The duality transformations describing the exchange (braiding) of several punctures or the connection between the different possible sewings of the  $n$ -punctured sphere out of 3-punctured spheres translate into matrix transformations acting on the basis of conformal blocks introduced in the previous section. The physical correlators are built as quadratic forms of left and right chiral blocks. The principle of locality (i.e., the symmetry of the euclidean functions) demands that these quadratic forms are invariant under the duality transformations. Once the local two dimensional 4-point functions have been determined and the normalizations of the 3-point functions have been fixed one can easily read off the operator product coefficients.

It is important to emphasise that our sets of 4-point blocks realise a representation of the braid group on 4-strands, each strand or puncture corresponding to a pair  $(x_a, z_a)$ ,  $a = 1, \dots, 4$ . Thus we are moving each pair as a whole, e.g., if we consider the monodromy of the second puncture with respect to the first we should move  $z_2$  and  $x_2$  simultaneously around the pair  $(x_1, z_1)$  along homotopic loops (equivalently we should move the ratios  $z$  and  $x$  simultaneously around 0 along homotopic loops).

To describe the duality matrices it is enough to specify the diagonal braidings in the different channels and the crossing (fusion) matrices between two channels.

For example the diagonal braiding in the  $s$ -channel is the exchange of the first two punctures. The result is

$$\begin{aligned} W_J(z_1, x_1, J_1; z_2, x_2, J_2; z_3, x_3, J_3; z_4, x_4, J_4) \\ = \exp(\pm i\pi(h_{J_1} + h_{J_2} - h_J)) W_J(z_2, x_2, J_2; z_1, x_1, J_1; z_3, x_3, J_3; z_4, x_4, J_4). \end{aligned}$$

Here  $W_J$  are the full chiral correlators, including the prefactors. The  $(-)+$  sign corresponds to (counter) clockwise exchange of  $(x_1, z_1)$  and  $(x_2, z_2)$ , and  $h_J$  are the “reduced” conformal weights (2.5),  $h_J = \Delta_J - J$ . As already pointed out the weights  $h_J$  are invariant under the transformation  $J \rightarrow \kappa - 1 - J$ , hence the “even” and “odd” blocks have the same diagonal braidings. The appearance of the “reduced” conformal weights and not the Sugawara weights  $\Delta$  in the phases is due to the fact that the analytic continuation of the integrals is done moving the pairs  $(x_a, z_a)$  as a whole.

Now we describe the crossing matrices expressing the  $s$ -basis in terms of the  $t$ -basis. All details and explicit expressions are left for the appendix. The standard  $z$ -dependent prefactor relating the blocks and the integrals plays no role in the crossing matrices so we define them in terms of the integrals introduced in the previous section. It is convenient first to introduce a new basis of integrals in which the fusion transformation takes a block diagonal form. Let

$$Y_{mm'} = h_{m'} I_{mm'} - f {}^{(1)}I_{mm'} \quad (5.1)$$

$${}^{(1)}Y_{mm'} = g_{m'} I_{mm'} + {}^{(1)}I_{mm'}$$

for  $m' = 1, 2, \dots, s'$ , while  $Y_{m0} \equiv I_{m0}$ , where

$$h_m = \frac{\sin(\pi(a + c + (m-1)\kappa)) \sin(\pi(c + m\kappa))}{\sin(\pi(a + c + (2m-1)\kappa)) \sin(\pi c)}, \quad f = \frac{\sin(\pi\kappa)}{\sin(\pi c)}, \quad (5.2)$$

$$g_m = -\frac{[m]' \sin(\pi(a + (m-1)\kappa))}{\sin(\pi(a + c + (2m-1)\kappa))} = \frac{1 - h_m}{f}. \quad (5.3)$$

The inverse transformation is

$$I_{mm'} = Y_{mm'} + f {}^{(1)}Y_{mm'} \quad (5.4)$$

$${}^{(1)}I_{mm'} = -g_{m'} Y_{mm'} + h_{m'} {}^{(1)}Y_{mm'}.$$

For the  $t$ -channel analogs (the integrals with upper indices  $m, m'$ ) we have analogous relations with  $h^m$  and  $g^m$  being the same as  $h_m$  and  $g_m$  only  $a$  is substituted with  $b$ . The integrals  ${}^{(1)}Y$  are described by cycles in which instead of the Felder type contour starting and ending at  $x$  we have simply a contour from  $z$  to  $x$  and all the rest is as in  ${}^{(1)}I$ .

In the  $\{Y, {}^{(1)}Y\}$  basis the fusion transformation takes a particularly simple form

$$Y_{mm'} = \sum_{nn'} \alpha_{mm', nn'} Y^{nn'} \quad (5.5)$$

$${}^{(1)}Y_{mm'} = \sum_{nn'} \gamma_{mm', nn'} {}^{(1)}Y^{nn'},$$

where

$$\alpha_{mm',nn'} = \alpha_{mn}(-a/\kappa, -b/\kappa, -c/\kappa, -d/\kappa; 1/\kappa) \alpha_{m'n'}(a, b, c, d; \kappa) \quad (5.6)$$

$$\gamma_{mm',nn'} = \theta \alpha_{mn}(-a/\kappa, -b/\kappa, -c/\kappa, -d/\kappa; 1/\kappa) \alpha_{m'-1, n'-1}(a, b, c + 2\kappa, d; \kappa) \quad (5.7)$$

The matrices  $\alpha$  coincide with the crossing matrices between the  $s$ - and  $t$ -channels in the Virasoro minimal models. The derivation of (5.5) is outlined in the Appendix. Note that the second factor in  $\gamma$  differs from that in  $\alpha$  by a simple shift of the parameters. The phase  $\theta = \exp i\pi(c + 1)$  appears because transforming the  $s$ -channel “odd” integral into  $t$ -channel “odd” ones we have also to “braid”  $x$  and  $z$ , exchanging their places. The properly normalized matrices  $\alpha$  are given up to signs by a product of  $q$ - and  $q'$ - $6j$ -symbols. Note that these symbols (see [24] for an explicit expression) are invariant (up to sign) under the change  $j_1 \rightarrow \frac{1}{2}(j_1 + j_2 - j_3 - j_4 - 1)$ , etc., implied by the “tilde” transformation (3.3); the multiples of  $\kappa$  lead to an overall sign. To see this one has to use at an intermediate step the property

$$\left\{ \begin{matrix} j_1 & j_2 & j_5 \\ j_3 & \bar{j}_4 & j_6 \end{matrix} \right\}_q = (-1)^{2j_4+1} \left\{ \begin{matrix} j_1 & j_2 & j_5 \\ j_3 & j_4 & j_6 \end{matrix} \right\}_q, \quad \bar{j}_4 = -1 - j_4,$$

derived in [25].

In the limit  $x \rightarrow z$  the elements  $Y$  of the basis, “block diagonalising” the fusion transformations, reduce to the corresponding DF integrals while  $(^1)Y$  map to zero.

Now we discuss the possible 2-dimensional local 4-point correlators. They are invariant with respect to braid and fusion transformations quadratic forms of the chiral blocks. Since according to (5.5) the subset of integrals  $Y$  behave under braid and fusion transformations as the minimal model DF integrals we immediately have (omitting the  $z$  dependent prefactors) an invariant quadratic form

$$\sum_{m=0}^s \sum_{m'=0}^{s'} X_{mm'} |Y_{mm'}|^2 \quad (5.8)$$

where  $X_{mm'} = X_{m+1}^{(s+1)}(-a/\kappa, -b/\kappa, -c/\kappa; 1/\kappa) X_{m'+1}^{(s'+1)}(a, b, c; \kappa)$  are the same as the ones found in [12]

$$X_{m+1}^{(s+1)}(a, b, c; \rho) = [m]! [s - m]! \prod_{k=0}^{m-1} \frac{\sin(\pi(1 + a + k\rho)) \sin(\pi(1 + c + k\rho))}{\sin(\pi(2 + a + c + (m - 1 + k)\rho))} \prod_{k=0}^{s-m-1} \frac{\sin(\pi(1 + b + k\rho)) \sin(\pi(1 + d + k\rho))}{\sin(\pi(2 + b + d + (s - m - 1 + k)\rho))}. \quad (5.9)$$

Since the crossing matrices  $\gamma$  for the “odd” integrals  $(^1)Y$  are obtained from the “even” crossing matrices  $\alpha$  by a certain shift of the parameters the “odd” invariant

$$\sum_{m=0}^s \sum_{m'=1}^{s'} (^1)X_{mm'} |(^1)Y_{mm'}|^2 \quad (5.10)$$

is obtained by the same shift. From the explicit form of  $X_{mm'}$  we have

$$(^1)X_{mm'} = -\frac{h_{m'}}{g_{m'} \sin(\pi(c + \kappa))} X_{mm'}, \quad (5.11)$$

where  $h_m$  and  $g_m$  were defined in (5.2), (5.3).

Having two independent invariants we obtain a 1-parameter family of monodromy invariants

$$M(\xi) \propto \sum_{mm'} X_{mm'} \left( |Y_{mm'}|^2 + \xi \frac{f h_{m'}}{g_{m'}} |(1)Y_{mm'}|^2 \right). \quad (5.12)$$

Going back to the basis of  $I$  integrals this invariant is nondiagonal except for  $\xi = 1$  in which case we get

$$\begin{aligned} M(1) &\propto \sum_{mm'} (h_{m'} X_{mm'}) |I_{mm'}|^2 + \sum_{mm'} \left( \frac{f}{g_{m'}} X_{mm'} \right) |(1)I_{mm'}|^2 \\ &= \sum_{mm'} \frac{\sin(\pi(2+a+c-m+(m'-1)\kappa)) \sin(\pi(1+c-m+m'\kappa))}{\sin(\pi(2+a+c-2m+(2m'-1)\kappa)) \sin(\pi(1+c))} X_{mm'} |I_{mm'}|^2 \\ &\quad + \sum_{mm'} \frac{\sin^2(\pi\kappa) \sin(\pi(1+a+c-2m+(2m'-1)\kappa))}{\sin(\pi(-m+m'\kappa)) \sin(\pi(1+c)) \sin(\pi(1+a+(m'-1)\kappa))} X_{mm'} |(1)I_{mm'}|^2. \end{aligned} \quad (5.13)$$

(The summation over  $m'$  in the second terms in (5.12) and (5.13) starts from  $m' = 1$ .) The numerical coefficients in the  $t$ -channel ( $u$ -channel) are obtained exchanging  $a \leftrightarrow b$  ( $a \leftrightarrow d$ ) respectively.

A straightforward though rather lengthy computation, which we omit, shows that the invariant  $M(1)$  coincides precisely, up to the standard prefactors, with the (explicitly local) volume integral for the 4-point correlator [15]; the derivation of this result is based on the technique explained in [26]. The choice of “odd” integrals  $(1)I$  as described above is thus selected by their property to diagonalise the invariant of [15] and this is the way we have originally arrived at it.

The expression (5.13) is still formal since in general the summation may include terms violating the fusion rules (4.1). The cancellation of the unwanted terms amounts in general to some linear relations for the set of integrals (see [27] for the analogous consideration in the Virasoro case); we shall not address this problem here.

From (5.13) one can read the product  $D_{J_1 J_2}^{J_5^{(t)}} D_{J_5^{(t)} J_3}^{J_4}$  of OPE structure constants. Here as above  $t = 0$ , or 1, corresponding to the intermediate isospin  $J_5 = J_5^{(0)}$  or  $J_5^{(1)}$  respectively. (To simplify the notation we write  $J_i$  instead of the pair  $(J_i, J_i)$  of left-right isospins in these 2-dimensional constants.) We shall assume that the even constants are normalised according to  $D_{JJ}^0 = 1 = D_{J_0}^J$ . Normalizing the two point functions to one together with the locality of the 3-point correlators imply that the (even) structure constants are symmetric with respect to all indices; in particular  $D_{J_1 J_2}^{J_5} = D_{J_1 J_5}^{J_2}$ . To recover the constants one has to compare the  $s$ - and  $t$ -channels of (5.13) in the case of two by two coinciding fields  $J_1 = J_4, J_2 = J_3$  – hence  $a = \kappa - 1, c = 1 - \kappa + 2J_1 + 2J_2, b = \kappa - 1 + 2J_2 - 2J_1, S = 2J_2$ . Then the identity operator appears in the  $t$ -channel and one multiplies the correlator by an overall constant determined by the normalisation requirement  $D_{J_2 J_2}^0 = 1 = D_{J_1 0}^{J_1}$ , i.e., this constant is given by the inverse of the square of the normalisation constant in (4.7) times the constant in front of the first term in (5.13), both taken for  $m = s, m' = s'$ , and with  $a \leftrightarrow b$ . This determines the structure constants for any intermediate isospin. Taking into account (4.7), (4.9), (4.10), (4.11), (5.13) one obtains

$$D_{J_1 J_2}^{J_5^{(t)}} D_{J_1 J_5^{(t)}}^{J_2} = \frac{P^2(2 - \kappa + J_1 + J_2 + J_5^{(t)}) P(2 - \kappa)}{P(2 - \kappa + 2J_1) P(2 - \kappa + 2J_2) P(2 - \kappa + 2J_5^{(t)})} (D_{J_1 J_2}^{J_5; DF})^2, \quad (5.14)$$

where  $P(a) := \Gamma(a)/\Gamma(1-a)$ . The structure constant  $D_{J_1 J_2}^{J_5; DF}$  in the r.h.s. is precisely the corresponding DF constant, see [28]. The r.h.s. of (5.14) has sense also for the values of  $J_a$  beyond the

fusion table (and the fusion rules) of the minimal models; in particular the zeros of the DF constant are compensated by singularities of the prefactor. From (5.14) it is clear that we can also require for the odd constants the symmetry properties mentioned above, and hence replace the l.h.s. with the square of the constant  $D_{J_1 J_2}^{J^{(t)}}$ ; in other words, we can identify in averages the physical operators  $\Psi_{(J_{r,r'}, J_{r,r'}^{(1)})}^{(1)}$  and  $\Psi_{(J_{p-r-2, p'-r'-2}, J_{p-r-2, p'-r'-2})}^{(0)}$ , cf. (2.4). In the even case  $t = 0$  the above expression (which can be looked as the analytic continuation of the expression for the constants of Fateev and Zamolodchikov [3] in the case of (half)integer isospins) first appeared in [15]. The constants  $D_{J_1 J_2}^{J^{(t)}}$  are not real in general.

Another important special case of the invariants (5.12) is provided by  $M(0)$ , i.e., (5.8), which allows furthermore for a “heterotic” interpretation having  $A_1^{(1)}$  as chiral algebra for the left movers and pure Virasoro for the right movers. Indeed the braid and monodromy properties of the integrals  $Y_{mm'}(\mathbf{x}, z)$  imply that they can be coupled with the corresponding (complex conjugate) DF integrals  $I_{mm'}^{(DF)}(z)$  to produce an invariant analogous to (5.8). The resulting structure constants can be computed as above. Such local correlators might be of relevance for the program outlined in [29].

We conclude with a few remarks. One has to check that the obtained crossing matrices satisfy the polynomial (in particular the pentagon) equations determined by the Awata-Yamada fusion rules. This in particular could fix the ambiguity in the physical interpretation of the “odd” correlators. As mentioned at the end of Section 4, instead of  ${}^{(1)}I_{m,m'}$  we could interpret  ${}^{(1)}I_{m,m'}^+$  or  ${}^{(1)}I_{m,m'}^-$  as “odd” correlators. The set  $\{I_{m,m'}, {}^{(1)}I_{m,m'}^+\}$  transforms via a fusion matrix whose “even - even” block has a lot of vanishing matrix elements. E.g., for coinciding isospins  $J_1 = J_2 = J_3 = J_4$  and  $J_5 < J_1 + J_2$  there is no contribution of “even” isospins  $J_6$  apart from  $J_6 = J_2 + J_3$  – this behaviour is partially in agreement with the observation in the early paper [11] – only unlike [11] we have in addition nonvanishing fusion matrix elements corresponding to the “odd” isospins.<sup>5</sup>

The supersymmetric interpretation of the fusion rules [18] raises also the question whether the crossing matrices could be related to the  $6j$ -symbols of the  $q$ -analog of the relevant superalgebra.

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Note added:

While this paper was being written down<sup>6</sup> a paper by J.L. Petersen, J. Rasmussen and M. Yu [31], has appeared, the results of which overlap partially with ours.

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<sup>5</sup>Note that this property of the fusion matrix in the basis  $\{I_{m,m'}, {}^{(1)}I_{m,m'}^+\}$  still does not contradict the fusion rules (4.1), since there are also nonvanishing matrix elements corresponding to the admissible triples  $(J_2, J_3, J_6)$  with  $J_6 \leq J_2 + J_3$ . The phenomena of vanishing of some fusion matrix elements for triples consistent with the fusion rules is known to occur already in the Virasoro case.

<sup>6</sup>A preliminary version has been reported by one of us (A.Ch.G.) in May, 1996, at the program on Topological, Conformal and Integrable Quantum Field Theory of ESI, Vienna

## A Appendix

In this appendix we will derive the formulae for the crossing matrices. As in [12] the fusion transformation in the general case  $s \neq 0$   $s' \neq 0$  factorises due to the trivial monodromy of the mixing factor  $\prod (u_i - v_{i'})^{-2}$  in (2.24) into two pieces. The derivation of the first repeats the one relevant for the correlators in the Virasoro models because of the trivial monodromy of the modifying factors,  $\prod_{i=1}^s (u_i - z)^{-1} (u_i - x)$ , in (3.8). So we will concentrate on the derivation of the second piece assuming furthermore for simplicity of notation that  $s = 0$ . In order to make the formulae more readable we will skip the primes that we have consistently put on all variables related to this case until now.

Define a generalization of the “even” integrals  $I$  in section 4

$$I[m_1, m_2, m_3, m_4] = \int_{\mathcal{C}(m_1, m_2, m_3, m_4)} \dots$$

Here the contour is a generalization of the contour (4.4) (including an analog of the phase (4.5)) in which there are  $m_1$  contours from  $-\infty$  to 0,  $m_2$  — from 0 to  $z$ ,  $m_3$  — from  $z$  to 1, and  $m_4$  from 1 to  $\infty$ . A word of caution — these integrals we will need only when either  $m_2 = 0$  or  $m_3 = 0$  and our convention (in order not to overburden the notation) is that if  $m_2 = 0$  then the integral is defined for  $0 < x < z < 1$  while if  $m_3 = 0$  then the integral is defined for  $0 < z < x < 1$ . Note that the integral  $I[m_1, 0, 0, m_2]$  is well defined (and identical) in both cases because there is no relative monodromy between the points  $z$  and  $x$ . Up to the  $q$ -factorials in (4.6) we have that  $I[0, m, 0, s - m]$  is equal to  $I_{0m}$  and  $I[s - m, 0, m, 0]$  is equal to  $I^{0m}$ .

Now we define “odd” integrals  $Z$  that are generalizations of the  $(^1)Y$  ones. For  $Z[m_1, 0, m_3, m_4]$  we assume that  $0 < x < z < 1$  and that there are  $m_1$  contours from  $-\infty$  to 0, one contour from  $x$  to  $z$ ,  $(m_3 - 1)$  contours from  $z$  to 1, and  $m_4$  contours from 1 to  $\infty$ . For  $Z[m_1, m_2, 0, m_4]$  we assume that  $0 < z < x < 1$  and that there are  $m_1$  contours from  $-\infty$  to 0,  $(m_2 - 1)$  contours from 0 to  $z$ , one contour from  $z$  to  $x$ , and  $m_4$  contours from 1 to  $\infty$ . Up to the  $q$ -factorial prefactors we have that  $Z[0, m, 0, s - m]$  is equal to  $(^1)Y_{0m}$  while  $Z[s - m, 0, m, 0]$  is equal to  $(^1)Y^{0m}$ . Now the monodromy between  $x$  and  $z$  is nontrivial and we have

$$Z[m_1, 1, 0, m_4] = \theta Z[m_1, 0, 1, m_4] \quad (\text{A.1})$$

where  $\theta = \exp(\pm i\pi(c + 1))$  depending on whether we exchange  $x$  and  $z$  in a clockwise or counter-clockwise direction.

The transformation of  $s$ -channel integrals (the ones with subscripts) into  $t$ -channel integrals (the ones with superscripts) is performed in two steps. The first step consists in reducing  $m_2$  keeping  $m_3 = 0$  fixed until we get  $m_2 = 0$  (or  $m_2 = 1$  for the “odd” integrals). The second step consists in reducing  $m_4$  keeping  $m_2 = 0$  fixed.

In order to obtain recursion formulae allowing us to move contours from one interval to another one adds a closed contour above/below the real axis and subtracting the resulting two equations with appropriate coefficients one gets the relations

$$\begin{aligned} & \sin(\pi(a + c + (m_1 + 2m_2)\kappa)) I[m_1 + 1, m_2, 0, m_4] + \sin(\pi(c + m_2\kappa)) I[m_1, m_2 + 1, 0, m_4] \\ & - \sin(\pi(b + m_4\kappa)) I[m_1, m_2, 0, m_4 + 1] - \sin(\pi\kappa) Z[m_1, m_2 + 1, 0, m_4] = 0, \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} & \sin(\pi(a + c + (m_1 + 2m_2)\kappa)) Z[m_1 + 1, m_2, 0, m_4] + \sin(\pi(c + (1 + m_2)\kappa)) Z[m_1, m_2 + 1, 0, m_4] \\ & - \sin(\pi(b + m_4\kappa)) Z[m_1, m_2, 0, m_4 + 1] = 0, \end{aligned} \quad (\text{A.3})$$

which are relevant for the first step and the relations

$$\begin{aligned} & \sin(\pi(a + m_1\kappa)) I[m_1 + 1, 0, m_3, m_4] - \sin(\pi(c + m_3\kappa)) I[m_1, 0, m_3 + 1, m_4] \\ & - \sin(\pi(b + c + (2m_3 + m_4)\kappa)) I[m_1, 0, m_3, m_4 + 1] + \sin((\pi\kappa)) Z[m_1, 0, m_3 + 1, m_4] = 0, \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} & \sin(\pi(a + m_1\kappa)) Z[m_1 + 1, 0, m_3, m_4] - \sin(\pi(c + (1 + m_3)\kappa)) Z[m_1, 0, m_3 + 1, m_4] \\ & - \sin(\pi(b + c + (2m_3 + m_4)\kappa)) Z[m_1, 0, m_3, m_4 + 1] = 0, \end{aligned} \quad (\text{A.5})$$

which are relevant for the second step. Note that the corresponding recursion relations for the fusion transformation of the DF integrals are recovered from (A.2) and (A.4) simply dropping the last terms.

Next define coefficients  $A$ ,  $B$ , and  $C$  by

$$\begin{aligned} I[m_1, m_2, 0, m_4] &= \sum A_{k_1, k_4}^{(2)} I[m_1 + k_1, m_2 - k_1 - k_4, 0, m_4 + k_4] \\ &+ \sum B_{k_1, k_4}^{(2)} Z[m_1 + k_1, m_2 - k_1 - k_4, 0, m_4 + k_4], \\ Z[m_1, m_2, 0, m_4] &= \sum C_{k_1, k_4}^{(2)} Z[m_1 + k_1, m_2 - k_1 - k_4, 0, m_4 + k_4], \end{aligned}$$

relevant for the first step (the sums are over  $k_1$  and  $k_4$  with  $k_1 + k_4$  fixed) and

$$\begin{aligned} I[m_1, 0, m_3, m_4] &= \sum A_{k_1, k_3}^{(4)} I[m_1 + k_1, 0, m_3 + k_3, m_4 - k_1 - k_3] \\ &+ \sum B_{k_1, k_3}^{(4)} Z[m_1 + k_1, 0, m_3 + k_3, m_4 - k_1 - k_3], \\ Z[m_1, 0, m_3, m_4] &= \sum C_{k_1, k_3}^{(4)} Z[m_1 + k_1, 0, m_3 + k_3, m_4 - k_1 - k_3], \end{aligned}$$

relevant for the second step. The relations (A.2), (A.3) and (A.4), (A.5) translate into recursion relations for the coefficients  $A$ ,  $B$ ,  $C$

$$\begin{aligned} -A_{k_1 k_4}^{(2)} &= \frac{\sin(\pi(-a - c + (k_1 + 2k_4 - m_1 - 2m_2 + 1)\kappa)) A_{k_1 - 1, k_4}^{(2)} + \sin(\pi(b + (m_4 + k_4 - 1)\kappa)) A_{k_1, k_4 - 1}^{(2)}}{\sin(\pi(-c + (k_1 + k_4 - m_2)\kappa))} \\ -B_{k_1 k_4}^{(2)} &= \frac{\sin((\pi\kappa)) A_{k_1 k_4}^{(2)}}{\sin(\pi(-c + (k_1 + k_4 - m_2 + 1)\kappa))} \\ &+ \frac{\sin(\pi(-a - c + (k_1 + 2k_4 - m_1 - 2m_2 + 1)\kappa)) B_{k_1 - 1, k_4}^{(2)} + \sin(\pi(b + (m_4 + k_4 - 1)\kappa)) B_{k_1, k_4 - 1}^{(2)}}{\sin(\pi(-c + (k_1 + k_4 - m_2 - 1)\kappa))} \end{aligned} \quad (\text{A.6})$$

and

$$\begin{aligned} A_{k_1, k_3}^{(4)} &= \frac{\sin(\pi(a + (m_1 + k_1 - 1)\kappa)) A_{k_1 - 1, k_3}^{(4)}}{\sin(\pi(b + c + (2m_3 + m_4 + k_3 - k_1)\kappa))} - \frac{\sin(\pi(c + (m_3 + k_3 - 1)\kappa)) A_{k_1, k_3 - 1}^{(4)}}{\sin(\pi(b + c + (2m_3 + m_4 + k_3 - k_1 - 2)\kappa))} \\ B_{k_1, k_3}^{(4)} &= \frac{\sin(\pi(a + (m_1 + k_1 - 1)\kappa)) B_{k_1 - 1, k_3}^{(4)}}{\sin(\pi(b + c + (2m_3 + m_4 + k_3 - k_1)\kappa))} \\ &+ \frac{-\sin(\pi(c + (m_3 + k_3)\kappa)) B_{k_1, k_3 - 1}^{(4)} + \sin((\pi\kappa)) A_{k_1, k_3 - 1}^{(4)}}{\sin(\pi(b + c + (2m_3 + m_4 + k_3 - k_1 - 2)\kappa))} \end{aligned} \quad (\text{A.7})$$

The solutions of (A.6) are

$$A_{k_1, k_4}^{(2)} = (-1)^{k_1+k_4} \frac{[k_1+k_4]!}{[k_1]![k_4]!} \quad (\text{A.8})$$

$$\frac{\prod_{i=0}^{k_1-1} \sin(\pi(-a-c+(2-m_1-2m_2+k_4+i)\kappa)) \prod_{i=0}^{k_4-1} \sin(\pi(b+(m_4+i)\kappa))}{\prod_{i=0}^{k_1+k_4-1} \sin(\pi(-c+(1-m_2+i)\kappa))},$$

$$B_{k_1, k_4}^{(2)} = \frac{-[k_1+k_4+1] \sin((\pi\kappa)) \sin(\pi(-c+(k_1+k_4-m_2)\kappa))}{\sin(\pi(-c-m_2\kappa)) \sin(\pi(-c+(k_1+k_4-m_2+1)\kappa))} A_{k_1, k_4}^{(2)}, \quad (\text{A.9})$$

while the solutions of (A.7) are

$$A_{k_1, k_3}^{(4)} = \frac{(-1)^{k_3} [k_1+k_3]!}{[k_1]![k_3]!} \frac{\prod_{i=0}^{k_1-1} \sin(\pi(a+(m_1+i)\kappa))}{\prod_{i=0}^{k_1-1} \sin(\pi(b+c+(2m_3+m_4+k_3-k_1+i)\kappa))} \quad (\text{A.10})$$

$$\frac{\prod_{i=0}^{k_3-1} \sin(\pi(c+(m_3+i)\kappa))}{\prod_{i=0}^{k_3-1} \sin(\pi(b+c+(2m_3+m_4-k_1-1+i)\kappa))},$$

$$B_{k_1, k_3}^{(4)} = \frac{-[k_3] \sin((\pi\kappa))}{\sin(\pi(c+m_3\kappa))} A_{k_1, k_3}^{(4)}. \quad (\text{A.11})$$

The recursion relations for  $C$  are the same as for  $A$  except for a shift of the parameters and their solutions are

$$C_{k_1, k_4}^{(2)} = \frac{\sin(\pi(c+(m_2-k_1-k_4)\kappa))}{\sin(\pi(c+m_2\kappa))} A_{k_1, k_4}^{(2)}, \quad (\text{A.12})$$

$$C_{k_1, k_3}^{(4)} = \frac{\sin(\pi(c+(m_3+k_3)\kappa))}{\sin(\pi(c+m_3\kappa))} A_{k_1, k_3}^{(4)}. \quad (\text{A.13})$$

Putting the coefficients  $A, B, C$  from the two steps together it is immediate to write the crossing matrices in a mixed basis consisting of  $I$ 's in the even part and  ${}^{(1)}Y$ 's in the odd part. In this basis the fusion transformation takes a block triangular form  $I_{mm'} = \sum \alpha_{mm', nn'} I^{nn'} + \beta_{mm', nn'} {}^{(1)}Y^{nn'}$ ,  ${}^{(1)}Y_{mm'} = \sum \gamma_{mm', nn'} {}^{(1)}Y^{nn'}$ . Schematically we have  $\alpha = A^{(2)} * A^{(4)}$ ,  $\gamma = C^{(2)} * \theta C^{(4)}$ ,  $\beta = A^{(2)} * B^{(4)} + B^{(2)} * \theta C^{(4)}$ . For example for  $\alpha_{m,n}$  using the above solutions for  $A^{(2)}$  and  $A^{(4)}$  and remembering that we have to include the factorials from (4.6) we obtain

$$\alpha_{mn} = \frac{[s-n]!}{[s-m]!} \frac{1}{\prod_{i=0}^{m-1} \sin(\pi(-c-i\kappa))} \prod_{i=0}^{n-1} \frac{\sin(\pi(c+i\kappa))}{\sin(\pi(b+c+(n-1+i)\kappa))}$$

$$\sum_k \frac{[k]!}{[s-k]![k-n]![k+m-s]!} \prod_{i=0}^{s-k-1} \sin(\pi(-a-c+(1-m-i)\kappa)) \quad (\text{A.14})$$

$$\prod_{i=0}^{k+m-s-1} \sin(\pi(b+(s-m+i)\kappa)) \prod_{i=0}^{k-n-1} \frac{\sin(\pi(a+(s-k+i)\kappa))}{\sin(\pi(b+c+(2n+i)\kappa))}.$$

This expression, properly normalised by the square root of the ratio  $X_{m+1}^{s+1}(a, b, c)/X_{n+1}^{s+1}(b, a, c)$ , cf. (5.9), can be brought (modulo signs) into the standard Racah form of the  $6j$ -symbols, see [30] for the classical counterparts of the various representations of these symbols.

The  $\alpha$ 's and  $\gamma$ 's here are the same as the ones in the  $\{Y, {}^{(1)}Y\}$  basis. It is a simple algebra to obtain the crossing matrices in the  $\{I, {}^{(1)}I\}$  basis.

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