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## A NEW LOOK AT THE DIRAC QUANTIZATION CONDITIONS

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### ABSTRACT

In this paper we look at the Dirac quantization conditions as  $\hbar$ -dependent constraints on the tangent bundle to *classical* phase-space. Starting from the path-integral version of classical mechanics ( whose operatorial counterpart is the Liouville formulation ) and using the associated extended Poisson brackets structure in the tangent bundle to phase-space, we handle the above constraints using the standard Dirac theory for constrained systems. The hope is to obtain, as total Hamiltonian, the Moyal operator of time evolution and as Dirac brackets the Moyal ones. Unfortunately the program fails indicating that something is missing. We put forward at the end some ideas for future work which may overcome this failure.



In the last few years there has been a revival of interest in quantum mechanics (QM) especially in connection with the black hole evaporation problem<sup>[1]</sup>, with quantum gravity and strings<sup>[2]</sup>, with the wave function of the Universe<sup>[3]</sup> and last but not least with the well-known measurement problem<sup>[4]</sup>. People are slowly realizing that some of the "secrets" of QM<sup>[5]</sup> are still there and it may be worthwhile to investigate them further. The light shed on these "secrets" may turn out to be light shed also on quantum gravity.

In this paper we study the old Dirac quantization conditions<sup>[6]</sup> by analyzing it with modern tools borrowed from the theory of constrained systems and gauge theories<sup>[7]</sup>. Part of the inspiration for this work came from ref.[8], but the reader will soon realize that our work is totally different.

By the Dirac quantization conditions we mean the usual rules of quantization

$$\begin{aligned} p &\longrightarrow -i\hbar \frac{\partial}{\partial q} \\ &\text{or} \\ q &\longrightarrow i\hbar \frac{\partial}{\partial p} \end{aligned} \tag{1}$$

In technical language this condition can be interpreted as an  $\hbar$ -dependent constraint between elements  $\phi \equiv (q, p)$  of the phase-space  $\mathcal{M}$  of the system and elements  $(\frac{\partial}{\partial p}, \frac{\partial}{\partial q})$  of the tangent space  $T_\phi\mathcal{M}$  in  $\phi$  to  $\mathcal{M}$ . In order to really interpret this as a constraint one should work in a space which unifies the phase-space  $\mathcal{M}$  with all the tangent spaces  $T_\phi\mathcal{M}$  of  $\mathcal{M}$ . This space is what is known as the tangent bundle to phase space and it is indicated by  $T\mathcal{M}$ . A formulation of classical mechanics (CM) in  $T\mathcal{M}$  has been given in ref.[9] through a path-integral approach to the operatorial version of CM put forward in 1931 by Koopman and von Neumann<sup>[10]</sup>.

Let us now briefly review this path-integral formulation. In classical mechanics (CM) the propagator  $P(\phi_2, t_2 | \phi_1, t_1)$ , which gives the classical probability for a particle to be at the point  $\phi_2$  at time  $t_2$ , given that it was at the point  $\phi_1$  at time  $t_1$ , is just a delta function

$$P(\phi_2, t_2 | \phi_1, t_1) = \delta^{2n}(\phi_2 - \Phi_{cl}(t_2, \phi_1)) \tag{2}$$

where  $\Phi_{cl}(t, \phi_0)$  is a solution of Hamilton's equation

$$\dot{\phi}^a(t) = \omega^{ab} \partial_b H(\phi(t)) \tag{3}$$

subject to the initial conditions  $\phi^a(t_1) = \phi_1^a$  and where

$$\phi^a = \begin{cases} q^a, & \text{if } 1 \leq a \leq n \\ p^{a-n}, & \text{if } n+1 \leq a \leq 2n \end{cases}$$

Here  $H$  is the conventional Hamiltonian of a dynamical system defined on some phase-space  $\mathcal{M}_{2n}$  with local coordinates  $\phi^a, a = 1 \dots 2n$  and a constant symplectic structure  $\omega = \frac{1}{2} \omega_{ab} d\phi^a \wedge d\phi^b$ .

The delta function in (2) can be rewritten as

$$\delta^{2n}(\phi_2 - \Phi_{cl}(t_2, \phi_1)) = \left\{ \prod_{i=1}^{N-1} \int d\phi_{(i)} \delta^{2n}(\phi_{(i)} - \Phi_{cl}(t_i, \phi_1)) \right\} \delta^{2n}(\phi_2 - \Phi_{cl}(t_2, \phi_1)) \quad (4)$$

where we have sliced the interval  $[0, t]$  in  $N$  intervals and labelled the various instants as  $t_i$  and the fields at  $t_i$  as  $\phi_{(i)}$ . Each delta function contained in the product on the RHS of (4) can be written as:

$$\delta^{2n}(\phi_{(i)} - \Phi_{cl}(t_i, \phi_1)) = \prod_{a=1}^{2n} \delta(\dot{\phi}^a - \omega^{ab} \partial_b H) |_{t_i} \det [\delta_b^a \partial_t - \partial_b (\omega_{ac}(\phi) \partial_c H(\phi))] |_{t_i} \quad (5)$$

where the argument of the determinant is obtained from the functional derivative of the equation of motion (3) with respect to  $\phi_{(i)}$ . Introducing Grassmannian variables  $c^a$  and  $\bar{c}_a$  to exponentiate the determinant, and an auxiliary variable  $\lambda_a$  to exponentiate the delta functions, we can re-write the propagator above as a path-integral.

$$P(\phi_2, t_2 | \phi_1, t_1) = \int_{\phi_1}^{\phi_2} \mathcal{D}\phi \mathcal{D}\lambda \mathcal{D}c \mathcal{D}\bar{c} \exp i\tilde{S} \quad (6)$$

where  $\tilde{S} = \int_{t_1}^{t_2} dt \tilde{\mathcal{L}}$  with

$$\tilde{\mathcal{L}} \equiv \lambda_a [\dot{\phi}^a - \omega^{ab} \partial_b H(\phi)] + i\bar{c}_a (\delta_b^a \partial_t - \partial_b [\omega^{ac} \partial_c H(\phi)]) c^b \quad (7)$$

In the path-integral (6) we have used the slicing (4) and then taken the limit of  $N \rightarrow \infty$ . Holding  $\phi$  and  $c$  both fixed at the endpoints of the path-integral, we obtain the

kernel,  $K(\phi_2, c_2, t_2 | \phi_1, c_1, t_1)$ , which propagates distributions in the space  $(\phi, c)$

$$\tilde{\varrho}(\phi_2, c_2, t_2) = \int d^{2n} \phi_1 d^{2n} c_1 K(\phi_2, c_2, t_2 | \phi_1, c_1, t_1) \tilde{\varrho}(\phi_1, c_1, t_1) \quad (8)$$

The distributions  $\tilde{\varrho}(\phi, c)$  are finite sums of monomials of the type

$$\tilde{\varrho}(\phi, c) = \frac{1}{p!} \varrho_{a_1 \dots a_p}^{(p)}(\phi) c^{a_1} \dots c^{a_p} \quad (9)$$

The kernel  $K(\cdot | \cdot)$  is represented by the path-integral

$$K(\phi_2, c_2, t_2 | \phi_1, c_1, t_1) = \int \mathcal{D}\phi^a \mathcal{D}\lambda_a \mathcal{D}c^a \mathcal{D}\bar{c}_a \exp i \int_{t_1}^{t_2} dt \tilde{\mathcal{L}} \quad (10)$$

with the boundary conditions  $\phi^a(t_{1,2}) = \phi_{1,2}^a$  and  $c^a(t_{1,2}) = c_{1,2}^a$ . The function  $\tilde{\varrho}$  of eq. (8) is the *classical* analogue of the Wigner function. It is also easy from here to build a classical generating functional  $Z_{cl}$  from which all correlation-functions can be derived. It is given by

$$Z_{cl} = \int \mathcal{D}\phi^a(t) \mathcal{D}\lambda_a(t) \mathcal{D}c^a(t) \mathcal{D}\bar{c}_a \exp i \int dt \{ \tilde{\mathcal{L}} + \text{source terms} \} \quad (11)$$

where the Lagrangian can be written as

$$\tilde{\mathcal{L}} \equiv \lambda_a \dot{\phi}^a + i \bar{c}_a \dot{c}^a - \tilde{\mathcal{H}} \quad (12)$$

with the "Hamiltonian" given by

$$\tilde{\mathcal{H}} = \lambda_a h^a + i \bar{c}_a \partial_b h^a c^b \quad (13)$$

and where  $h^a$  are the components of the Hamiltonian vector field<sup>[11]</sup>

$$h^a(\phi) \equiv \omega^{ab} \partial_b H(\phi) \quad (14)$$

From the path-integral (10) and (11) we can see<sup>[9]</sup> that the variables  $(\phi, \lambda)$  and  $(c, \bar{c})$  form conjugate pairs satisfying the ( $Z_2$ -graded) commutation relations<sup>\*</sup>

$$\begin{aligned} [\phi^a, \lambda_b] &= i \delta_b^a \\ [c^a, \bar{c}_b] &= \delta_b^a \end{aligned} \quad (15)$$

The commutators above are defined in precise terms in ref.[9]. Because of these commutators,<sup>[9]</sup> the variables  $\lambda_a$  and  $\bar{c}_a$  can be represented, in a sort of "Schroedinger-like" picture,

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\* We can define commutators because we have a path-integral.

as

$$\lambda_a = -i \frac{\partial}{\partial \phi^a} \equiv -i \partial_a \quad ; \quad \bar{c}_a = \frac{\partial}{\partial c^a} \quad (16)$$

So one sees immediately that the  $\lambda_a$  represent a basis in the tangent space  $T_\phi \mathcal{M}$ . Inserting (16) in (13) the Hamiltonian becomes an operator

$$\widehat{\mathcal{H}} = -i l_h \quad (17)$$

where

$$l_h = h^a \partial_a + c^b (\partial_b h^a) \frac{\partial}{\partial c^a} \quad (18)$$

is the Lie derivative operator<sup>[11]</sup> associated to the Hamiltonian flow (for details see ref.[9]). Its bosonic part coincides with the Liouvillian  $\hat{L} = h^a \partial_a$ , which gives the evolution of standard distributions  $\varrho^{(0)}(\phi)$  in phase-space:

$$\partial_t \varrho^{(0)}(\phi, t) = -l_h \varrho^{(0)}(\phi, t) = -\hat{L} \varrho^{(0)}(\phi, t) \quad (19)$$

This is the standard operatorial version of CM of Liouville, Koopmann and von Neumann, so this proves that our path-integral is really what is behind this operatorial formulation.

Let us now ask ourselves how we would quantize CM starting from the Liouville operator  $\hat{L}$  of eq.[19]. If we had started from the Hamiltonian,  $H = \frac{p^2}{2} + V(q)$ , we would have applied the *Dirac quantization conditions* of eq. (1):

$$H \longrightarrow \hat{H} = -\frac{\hbar^2}{2} \frac{\partial^2}{\partial q^2} + V(q)$$

but if we start with the Liouville operator  $\hat{L} = \frac{\partial H}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial}{\partial p}$ , we see immediately that we cannot apply the correspondence rules of Dirac (1), because we have to say what is  $\frac{\partial}{\partial p}$  in  $\hat{L}$  at the quantum level. As  $p$  becomes an operator in QM,  $\frac{\partial}{\partial p}$  becomes *the derivative of an operator* and this is a strange concept which find its natural place in the realm of non-commutative geometry. Trying to answer this question is the aim of this work without anyhow embarking in the mathematics of non-commutative geometry.

The strategy that we shall adopt is the following : both QM and CM are now on similar grounds because both are formulated via path-integrals so it may be easier to go from one to the other. In particular we will try to look at the Dirac quantization condition as a constraint in the tangent bundle and impose that constraint in the path-integral for CM

in order to obtain QM. The reader may object that constraints usually act on a classical phase-space and not in an operatorial space as our CM path-integral seems to be. Well, in ref. [9] we showed that this path-integral ,besides providing an operatorial version of CM, is also naturally endowed with a classical Poisson brackets which we will use to handle the constraint mentioned above. In fact we can view  $(\phi^a, \lambda_a)$  as coordinates on the tangent bundle  $T\mathcal{M}$  and introduce an "enlarged" Poisson brackets (epb) structure  $\{\cdot, \cdot\}_{epb}$  which reproduces the classical equations of motion from the Hamiltonian  $\tilde{\mathcal{H}}$  of eq. (13). The enlarged Poisson brackets must satisfy the relation

$$\{\phi^a, H\}_{pb} = \{\phi^a, \tilde{\mathcal{H}}\}_{epb} \quad (20)$$

in order to reproduce the classical equations of motion. The  $\{\cdot, \cdot\}_{pb}$  are the old standard Poisson Brackets on  $\mathcal{M}$  while  $\{\cdot, \cdot\}_{epb}$  have the structure<sup>[9]</sup>

$$\begin{aligned} \{\phi^a, \lambda_b\}_{epb} &= \delta_b^a, \quad \{\phi^a, \phi^b\}_{epb} = 0 = \{\lambda_a, \lambda_b\} \\ \{c^a, \bar{c}_b\}_{epb} &= -i\delta_b^a, \quad \text{all others} = 0 \end{aligned} \quad (21)$$

Basically this formulation of CM provides a manner to generate the dynamics on the full  $T\mathcal{M}$ . Actually, as the  $c^a$  are a basis<sup>[9]</sup> of the cotangent bundle  $T^*\mathcal{M}$ , our  $\tilde{\mathcal{H}}$  generate the dynamics on the full tensor bundle over  $\mathcal{M}$  (for details see ref.[9]).

In this space now, remembering eq.(16), the first Dirac quantization condition in (1) can be written as the following constraint:

$$\Phi_{(0)}^a \equiv \theta(a - n)(\phi^a + \hbar\omega^{ab}\lambda_b) = 0 \quad (22)$$

where  $\theta(a - n)$  is the standard step function

$$\theta(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases}$$

and "n" is the number of degrees of freedom (half the dimension of phase-space). The second Dirac quantization condition in (1) can be represented as

$$\Psi_{(0)}^a \equiv \theta(n + 1 - a)(\phi^a - \hbar\omega^{ab}\lambda_b) = 0 \quad (23)$$

Note that the role of the  $\theta(n + 1 - a)$  or  $\theta(a - n)$  in the equation above is to choose either one of the two "polarization": "q" as Schrodinger picture or "p" as Fourier transform picture. One could also envision a way to get the complex polarization (creation and annihilation operators).

Now that we have a CM in this enlarged space with the Hamiltonian given by (13) and its "epb" structure given by (21), we should treat the constraint (22) as it is usually done in constrained systems<sup>[12]</sup>. The first thing to do is to calculate the *secondary* constraints which are those obtained by the evolution of the primary ones  $\Phi_0^a$ . The sub-index "(0)" is to indicate that it is primary. We will use the sub-index "(1),(2),(3),..." to indicate the *secondary* and *tertiary*, etc. constraints generated this way. This procedure starts by using what is called<sup>[7]</sup> the total Hamiltonian  $\tilde{\mathcal{H}}_T$  defined as

$$\tilde{\mathcal{H}}_T \equiv \tilde{\mathcal{H}} + \sum_a u_a \Phi_{(0)}^a \quad (24)$$

where  $u_a$  are Lagrange multipliers. The secondary constraint  $\Phi_{(1)}^a$  are then generated by imposing that the primary ones are left invariant by the time evolution under  $\tilde{\mathcal{H}}_T$ :

$$\begin{aligned} \dot{\Phi}_{(0)}^a &= \{\Phi_{(0)}^a, \tilde{\mathcal{H}}_T\}_{epb} \\ &= \theta(a-n) \left( \hbar^a - \hbar \omega^{ab} \lambda_e \omega^{ef} \frac{\partial^2 H}{\partial \phi^b \partial \phi^f} \right) \end{aligned} \quad (25)$$

So the secondary constraints  $\Phi_{(1)}^a$  are

$$\Phi_{(1)}^a = \theta(a-n) \left( \omega^{ab} \frac{\partial H}{\partial \phi^b} - \hbar \omega^{ab} \lambda_e \omega^{ef} \frac{\partial^2 H}{\partial \phi^b \partial \phi^f} \right) \quad (26)$$

The result (26) is obtained because the primary constraints  $\Phi_{(0)}^a$  have zero "epb" among themselves as it is easy to verify:

$$\begin{aligned} \{\Phi_{(0)}^a, \Phi_{(0)}^c\}_{epb} &= \{(\phi^a + \omega^{ab} \lambda_b) \theta(a-n), (\phi^c + \omega^{cd} \lambda_d) \theta(c-n)\}_{epb} \\ &= \{\phi^a + \omega^{ab} \lambda_b, \phi^c + \omega^{cd} \lambda_d\} \theta(a-n) \theta(c-n) \\ &= 2\omega^{ac} \theta(a-n) \theta(c-n) = 0 \end{aligned} \quad (27)$$

The last step gives zero because of the form  $\omega^{ac}$ , which has elements  $\neq 0$  for  $a > n$  and  $c < n+1$  or for  $a < n+1$  and  $c > n$ . In both cases the product of the two  $\theta(\cdot)$  in the expression above give zero. Because of this in (26) we got a relation which does not involve the lagrange multipliers  $u_a$ , that means we got a truly new constraint  $\Phi_{(1)}^a$ . We have now to apply the same procedure for the secondary constraint and see if it generates a tertiary one. This procedure has to be done with the same total Hamiltonian  $\tilde{\mathcal{H}}_T$  defined in (24), that means we do not add the secondary constraints to  $\tilde{\mathcal{H}}$ . The reasons for this are explained in

detail in ref.[7] and we will not repeat them here. So let us calculate  $\Phi_{(1)}^a$ , using the simple Hamiltonian with one degree of freedom  $H = \frac{1}{2}p^2 + V(q)$ . The result is

$$\begin{aligned} \{\Phi_{(1)}, \tilde{\mathcal{H}}_T\}_{epb} &= -pV'' + V''\hbar\lambda_q - V''' \hbar p \lambda_p + u(2\hbar V'' + \hbar^2 V''' \lambda_p) \\ &\approx \frac{V'V'''}{V''}p + \frac{\hbar u}{V''}(2(V'')^2 - V'V''') \end{aligned} \quad (28)$$

where in the second step we have used  $\Phi_{(1)}$  and so the equality  $\approx$  indicates that the relation holds only on the constraint. The  $V'$  indicates derivatives with respect to  $q$ . From the relation (28) we see that we do not generate a new constraint, but instead we determine the lagrange multiplier  $u$ , which comes out to be

$$u = \frac{-V'''V'p}{[2(V'')^2 - V'V''']\hbar} \quad (29)$$

The lagrange multiplier is undetermined only for those potentials  $V$  which makes the denominator in (29) equal to zero. They are potentials of the form  $V = -C_2 \ln |C_1 - q| + C_3$  where the  $C_i$  are arbitrary constants.

We can say at this point that the chain of constraints (primary, secondary, tertiary, etc) stops at the secondary level. So we started with  $4n$  variables  $(\phi^a, \lambda_a)$  and we developed  $2n$  constraints,  $n$  primary  $\Phi_{(0)}^a$  and  $n$  secondary  $\Phi_{(1)}^a$ , so the number of left over independent variables is only  $2n$ , half of the one we started with. This somehow is "similar" to what happens in the geometric quantization formulation of  $QM^{[13]}$  in which one start from  $2n$  variables  $(p, q)$  in the pre-quantization and ends up, via the polarization, with only  $n$ . The wave-function in fact depends only on either the  $n$  momenta  $p$  or the  $n$  position  $q$ .

The next thing is to check if the  $\Phi_{(0)}^a$  and  $\Phi_{(1)}^a$  are first or second class<sup>[7]</sup>. We can check it for a simple system like the harmonic oscillator :  $H = \frac{1}{2}(P^2 + q^2)$ . The constraints are:

$$\begin{aligned} \Phi_{(0)} &= p - \hbar\lambda_q = 0 \\ \Phi_{(1)} &= -q - \hbar\lambda_p = 0 \end{aligned} \quad (30)$$

and their bracket is

$$\{\Phi_{(0)}, \Phi_{(1)}\}_{epb} = -2\hbar$$

So this indicates that they are second class. It is easy to see that this feature holds for any system. The lagrange multipliers  $u$ , using formula (29) turns out to be zero in this case, so  $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_T$ . Having established this, what is left to do is the calculation of the Dirac-brackets

(in our case of the extended Dirac brackets (edb)). On two observables  $F$  and  $G$  they are defined<sup>[12]</sup> as

$$\{F, G\}_{edb} \equiv \{F, G\}_{epb} - \{F, \Psi_\alpha\}_{epb} C^{\alpha\beta} \{\Psi_\beta, G\}_{epb} \quad (31)$$

where we have indicated collectively with  $\Psi_\alpha$  the set of primary  $\Phi_{(0)}^a$  and secondary  $\Phi_{(1)}^a$  constraints and  $C^{\alpha\beta}$  is the inverse of the matrix  $C_{\alpha\beta}$  defined as

$$C_{\alpha\beta} \equiv \{\Psi_\alpha, \Psi_\beta\}_{epb} \quad (32)$$

For the harmonic oscillator above the matrix  $C_{\alpha\beta}$  is, for example,

$$[C^{\alpha\beta}] = \frac{1}{2\hbar} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (33)$$

Using the constraints in (30), we get as Dirac brackets for the harmonic oscillator:

$$\{F, G\}_{edb} = \{F, G\}_{epb} + \frac{1}{2\hbar} \left[ \left( -\frac{\partial F}{\partial \lambda_p} - \hbar \frac{\partial F}{\partial q} \right) \left( \frac{\partial G}{\partial \lambda_q} - \hbar \frac{\partial G}{\partial p} \right) - \left( -\frac{\partial F}{\partial \lambda_q} + \hbar \frac{\partial F}{\partial p} \right) \left( \frac{\partial G}{\partial \lambda_p} + \hbar \frac{\partial G}{\partial q} \right) \right] \quad (34)$$

Applying this bracket for the evolution of of an observable  $F$  we obtain:

$$\begin{aligned} \{F, \tilde{\mathcal{H}}_T\}_{edb} &= \frac{1}{2}(p + \hbar \lambda_q) \frac{\partial F}{\partial q} - \frac{1}{2}(q - \hbar \lambda_p) \frac{\partial F}{\partial p} \\ &\approx p \frac{\partial F}{\partial q} - q \frac{\partial F}{\partial p} \end{aligned} \quad (35)$$

Now that we have this formulation, which we could call an  $\hbar$ -constrained CM ( $\hbar$ -CCM), we want to compare it with real QM. Of course the formulation of QM, against which we want to compare our  $\hbar$ -CCM, must be a formulation in phase-space (as our  $\hbar$ -CCM is) and moreover it must handle everything not with operators but with c-numbers as our  $\hbar$ -CCM does. This formulation of QM exists and it was provided by Weyl-Wigner and Moyal<sup>[14]</sup>. We will not review here all the Moyal formalism, let us just say that it is a procedure which associates to each operator  $\hat{O}$  a function  $O(\phi)$  in phase space, called the *symbol* of the operator. This correspondence is what is known also as the *symbol* map. Indicating with  $\varrho(\phi)$  the symbols of the density matrix and with  $H(\phi)$  the symbols of the Hamiltonian operator  $\hat{H}$ , from the the Heisenberg evolution equation  $i\hbar \partial_t \hat{\varrho} = -[\hat{\varrho}, \hat{H}]$  one goes<sup>[14]</sup>, via

the symbol map, into

$$\partial_t \varrho(\phi^a, t) = -\{\varrho, H\}_{mb} \quad (36)$$

where the  $\{\cdot, \cdot\}_{mb}$  are known as the Moyal brackets and are defined as:

$$\begin{aligned} \{A, B\}_{mb} &= A(\phi) \frac{2}{\hbar} \sin\left[\frac{\hbar}{2} \overleftarrow{\partial}_a \omega^{ab} \overrightarrow{\partial}_b\right] B(\phi) \\ &= \{A, B\}_{pb} + O(\hbar^2) \end{aligned} \quad (37)$$

In the classical limit ( $\hbar \rightarrow 0$ ) the Moyal bracket reduces to the classical Poisson bracket.

In the same way as for the classical case<sup>[9]</sup> where we went from the Poisson brackets to the extended Poisson brackets (see eq.(20)), we can ask if it is possible to go from the the Moyal brackets to the extended Moyal brackets (emb), *i.e.*, if it is possible to find a set of  $\{\cdot, \cdot\}_{emb}$  and a new Hamiltonian  $\tilde{\mathcal{H}}^{\hbar}$  such that

$$\{\phi^a, H\}_{mb} = \{\phi^a, \tilde{\mathcal{H}}^{\hbar}\}_{emb} \quad (38)$$

The solution has been found in ref.[15] and it is given by

$$\{A, B\}_{emb} \equiv A \ 2\sin\left[\frac{1}{2} \left(\frac{\overleftarrow{\partial}}{\partial\phi^a} \frac{\overrightarrow{\partial}}{\partial\lambda_a} - \frac{\overleftarrow{\partial}}{\partial\lambda_a} \frac{\overrightarrow{\partial}}{\partial\phi^a}\right)\right] B \quad (39)$$

and

$$\tilde{\mathcal{H}}^{\hbar}(\lambda, \phi) = \frac{1}{\hbar} \sinh[\hbar \lambda_a \omega^{ab} \partial_b] H(\phi) \quad (40)$$

where the  $\partial_b$  acts only on  $H(\phi)$ . In the classical limit we get from  $\tilde{\mathcal{H}}^{\hbar}(\lambda, \phi)$

$$\lim_{\hbar \rightarrow 0} \tilde{\mathcal{H}}^{\hbar}(\lambda, \phi) = \lambda_a \omega^{ab} \partial_b H = \lambda_a h^a = \tilde{\mathcal{H}}(\lambda, \phi) \quad (41)$$

The full expansion in  $\hbar$  of (40) is

$$\tilde{\mathcal{H}}^{\hbar} = \tilde{\mathcal{H}} + \frac{(\hbar)^2}{3!} \mathcal{M}_{(1)} + \frac{(\hbar)^4}{5!} \mathcal{M}_{(2)} + \dots \quad (42)$$

where

$$\begin{aligned} \mathcal{M}_{(1)} &= [\lambda_a \omega^{ab}] [\lambda_c \omega^{cd}] [\lambda_e \omega^{ef}] [\partial_b \partial_d \partial_f H] \\ \mathcal{M}_{(2)} &= [\lambda_a \omega^{ab}] [\lambda_c \omega^{cd}] [\lambda_e \omega^{ef}] [\lambda_g \omega^{gh}] [\lambda_l \omega^{lm}] [\partial_b \partial_d \partial_f \partial_h \partial_m H] \\ \mathcal{M}_{(3)} &= \dots \end{aligned} \quad (43)$$

The idea is to compare the quantum time evolution provided by the  $\tilde{\mathcal{H}}^{\hbar}$  under the extended Moyal brackets of eq. (39), with the  $\hbar$ -CCM time evolution provided by  $\tilde{\mathcal{H}}_T$  of

eq. (24) under the extended Dirac brackets of eq. (31). For the harmonic oscillator, using (35) and applying eq. (37) and (40) to the Hamiltonian of the harmonic oscillator :  $H = \frac{1}{2}(p^2 + q^2)$ , it is easy to see that

$$\{F, \tilde{\mathcal{H}}_T\}_{edb} = \{F, \tilde{\mathcal{H}}^{\hbar}\}_{emb} \approx p \frac{\partial F}{\partial q} - q \frac{\partial F}{\partial p} \quad (44)$$

*So for the harmonic oscillator our idea works!*

The careful reader, anyhow, will have already realized that the identification of extended Dirac brackets and extended Moyal brackets works only if one of the two observables is the Hamiltonian  $\tilde{\mathcal{H}}$ , but it does not work for generic observables:

$$\{F, G\}_{edb} \neq \{F, G\}_{emb} \quad (45)$$

In fact the number of terms on the two sides of eq. (45) is different: for the harmonic oscillator the RHS of (45) has an infinite number of terms while the LHS has just one. Even the fact that the correspondence above works if one of the two observables is the Hamiltonian, is just limited to the harmonic oscillator. In fact let us take the Hamiltonian  $H = \frac{1}{2}p^2 + \frac{1}{4}q^4$ , and let us build up all the formalism: the secondary constraints, the Lagrange multiplier and the  $\tilde{\mathcal{H}}_T$  are respectively:

$$\begin{aligned} \Phi_{(1)} &= -q^3 - 3\hbar q^2 \lambda_p \\ u &= -\frac{p}{2\hbar} \\ \tilde{\mathcal{H}}_T &= \frac{3}{2}p\lambda_q - q^3\lambda_p - \frac{p^2}{2\hbar} \end{aligned} \quad (46)$$

It is also easy to calculate  $\tilde{\mathcal{H}}^{\hbar}$  that is

$$\tilde{\mathcal{H}}^{\hbar} = p\lambda_q - q^3\lambda_p - \hbar^2(\lambda_p)^3 \quad (47)$$

Using eq. (46) and (47) we can then compare the Moyal time evolution and the extended Dirac evolution, the result is:

$$\begin{aligned} \{\tilde{\mathcal{H}}^{\hbar}, F\}_{emb} &= -p \frac{\partial F}{\partial q} + q^3 \frac{\partial F}{\partial q} - \frac{\hbar^2}{4} \frac{\partial^3 F}{\partial p^3} \\ \{\tilde{\mathcal{H}}_T, F\}_{edb} &= -\frac{3}{2}p \frac{\partial F}{\partial q} + q^3 \frac{\partial F}{\partial p} \end{aligned} \quad (48)$$

and from here one sees immediately, that the two evolutions are not the same.

There was actually a general reason why the two evolutions could not be the same in general. The reason lies in the fact that for the Moyal brackets every derivation<sup>[16]</sup>, is an internal one, while for the Poisson brackets\* not every derivation is an internal one. This property of the Dirac brackets may break down in case there are infinite constraints, and actually, in order to generate in  $\tilde{\mathcal{H}}_T$  all the terms which appear in (42), it seems we need infinite constraints.

The reader at this point may be a little bit puzzled at the idea of having infinite constraints. In fact, as the original degrees of freedoms were  $4n$ , subtracting from these the infinite constraints leaves us with  $-\infty$  effective degrees of freedoms!!!. This idea is not so crazy as it may sound. Let us be allowed to speculate a little bit. What may it mean that in QM there are effectively  $-\infty$  degrees of freedom?. The minus sign may indicate that these degrees of freedom are Grassmannian like forms<sup>[17]</sup>. So, like forms, they are non-local and this may explain where the *non-locality* (a la' EPR) of QM comes from. The infinity, instead, may tell us where the *statistical* character of QM comes from. So the two main secrets of QM (the statistical character and the non-locality) may be explained by these  $-\infty$  degrees of freedom. The reader may object that we do not need all this formalism to explain these two features. Already Bohm<sup>[17]</sup> understood that in QM, besides the  $p$  and  $q$ , one needs an object, the wave-function, which has a non-local character and it carries, being a function, an infinity of modes which could be understood as further degree of freedom. These two ideas (ours and Bohm's) could be tied together if one thinks that the space of wave-fronts makes what is called the Grassmannian Lagrangian that is an infinite dimensional space which can be parametrized by Grassmannian variables.

Anyhow, leaving these "wild" speculations aside, what do we need further in order to make the construction of the previous pages works, and really generate the Hamiltonian of eq.(43), ? It seems that injecting the uncertainty principle in CM, via the constraint (22), is not enough. What is missing? There are two ingredients that we fear may be missing:

- 1) we have somehow to inject in CM the complex structure because this is a crucial aspect of QM;
- 2) the relation (22) somehow gives us a connection for how to move in the tensor bundle while we move in the base-space, but it is a connection which tells us how to move only in the symmetric sub-bundle of the full tensor bundle (just because  $\lambda$  is a basis of the symmetric

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\* For the Dirac brackets it should be the same as for the Poisson brackets because the former are just Poisson brackets on a restricted space.

tensors<sup>[9]</sup>). We need a connection which tells us how to move also in the antisymmetric sub-bundle of the full tensor bundle, that means we need a further primary constraints besides (22). These two constraints may change the over-all structure of the secondary and tertiary etc. constraints and lead us to really infinite constraints. A further feature that we feel is necessary is that the constraints should be first class and not second class. The reason is that with first class we still have some freedom in determining the lagrange multipliers. This freedom is necessary because, in going from CM to QM we do not have a unique answer: from one CM model we could build various QM ones all different for the different orderings we can choose. This freedom, we think, may be provided in part by the freedom in determining the lagrange multipliers. Of course the freedom of determining the lagrange multipliers is much larger than the ordering freedom but still it may be related. Besides this, if the constraints are first class, it means that they are gauge generators and this is very important. In QM, in fact, one thing to understand, with respect to CM, is where the phases comes from. These phases may be related to this gauge invariance. All these issues are still somehow in a primitive stage and the reader may find them misleading or even contradictory but work is in progress on them.

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