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VACUUM POLARISATION IN SOME STATIC NONUNIFORM MAGNETIC FIELDS

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Abstract

Vacuum polarisation in QED in presence of some configurations of external magnetic fields is investigated. The configuration considered correspont to fields lying in a plane and without sources. The motion of a Dirac electron in this field configuration is studied and arguments are found to conclude that the lowest level gives the most important contribution. The result is that the main effect is not very different from the uniform case, the possibilities of calculating the corrections due to the nonuniformity is explicitly shown.

A typical effect of nonuniformity of the field shows out in the refractivity of the vacuum.

1.Introduction

Vacuum polarisation in presence of strong external electromagnetic fields is an example of problem where nonperturbative effects may be looked for in QED and also, more modestly, where techniques to deal with such kind of problems are tried and tested. There is also some more phenomenological interest in dealing with these questions: one may look for the possibility, sometimes claimed [1], that in similar circumstances a change of regime can be produced in QED, giving rise to a confining phase, a quantitative signal of this fact could be found in an increasig value of the fine stucture constant α. When we go from these broad statements down to actual calculations or estimates, the generality of the investigation is usually drastically reduced because of the difficulty of dealing with charged particles in given fields configurations: nonetheless different kind of actual field configurations have been studied, mainly of magnetic type both because they are easier to deal with and because the interpretation of the result in term of effective coupling constants is clearer (these two facts are certainly related). Even in the case of the magnetic field the research,which obviously started by the consideration of a constant and uniform field [2], has been carried out mainly in the frame of the fields that keep constant their direction [3]. There is an evident difference between the formal conditions we require for the field, *e.g.* to be constant and uniform everywhere, and the actual relevance of this condition in calculating the vacuum polarisation, which must be satisfied in the space where the virtual pair of charged particles actually lives, so the condition of uniform field or of constant direction is less severe than it could seem at first sight. It must be, however, remembered that a magnetic field constant in direction but not in strength implies $curl\mathbf{B} \neq 0$ and thus the presence of charged currents. From some points of view this fact can be welcome both because is may be real, think at the strong fields produced in heavy-ion collisions, and also because the presence of currents and charge density might drive α to higher values [4], but on the other side this amounts to the introduction of another physical element in the game: the charge density implies a chemical potential for the electrons, So trying to push on the investigation on the side of the $curl\mathbf{B} = 0$ field configuration can present some interest. Some exact or approximate solutions of the Dirac equation, in other specific kinds of magnetic and electric fields, have been presented, for the moment they have not been used to calculate the vacuum polarisation effect [5,6]. This paper deals with some investigation about the vacuum polarisation in a family of non uniform magnetic fields satisfying the $curl\mathbf{B} = 0$ conditions.

Although for the vacuum polarisation effects the relevant size of the region where the field behaviour matters is of the order of the Compton wave length of the electron, in the formal study of the Dirac equation boundary condition at infinity are given, this is a technical device and the only real requirement is that they are a reasonable extrapolation of the behaviour of the field in the above described small region, this statement applies, in particular, also to the constant field case. Keeping in mind this general and trivial observation a suggestion coming from the uniform field case will be used in an essential way; it was found that for strong uniform fields the main part of the effect is produced by the lowest lying Landau level of the charged particle in the given magnetic field so the first step is precisely a study of this level in some nonuniform field configuration, then, after some observations about the higher states, an estimate of the vacuum polarisation effects is presented.

As a sort of byproduct other kind of polarisation effects, related not to the variation of the effective coupling constant, but to the free photon propagation are put into evidence.

2. Determination of the lowest level in a non uniform magnetic field

In this subsection the evaluation of the wave function of the lowest level in some particular configurations of a non uniform magnetic field is presented.

The configurations are characterised by two requirements: the field is plane, i.e. it is always perpendicular to a definite direction that will be chosen as the z-axis; the field has no sources, at any finite point. The requirements $div\,\mathbf{B}=0$, $curl\,\mathbf{B}=0$, read in the plane (x,y), allow the definition of the dual, in the plane, ${}^*\mathbf{B}$ having also the properties $div\,{}^*\mathbf{B}=0$, $curl\,{}^*\mathbf{B}=0$, so that ${}^*\mathbf{B}=-grad\,\chi$ and by defining $A_z=\chi, A_x=A_y=0$ the usual relation , $\mathbf{B}=curl\,\mathbf{A}$ is obtained.

It is convenient to perform a change of coordinates in order that the lines of flux of $\mathbf{B}$ become one family of coordinates lines in the plane (x,y). A way of implementing this request is making complex the (x,y) plane, by defining $\mathcal{Z}=x+iy$ and then introducing the complex function $\mathcal{W}=u+iv$, with the requirement that $bu=A_z$, the parameter b characterises the overall strength of the magnetic field. It is convenient to express the derivatives of the new coordinates with respect to the old ones as $\partial_x u=\partial_y v=\rho\,cos\theta\,;\partial_y u=-\partial_x v=\rho\,sin\theta$. The auxiliary variable ρ is strictly related to the intensity of the magnetic field; from the previous definitions it results $b\partial_x u=-B_y\,, b\partial_y u=B_x$ so that $b\rho=B$. Since $b\,grad\,u=-{}^*\mathbf{B}$ if $\mathbf{n}$ is the unit vector in the direction of $\mathbf{B}$ it follows $\mathbf{n}\cdot\partial u=0$; in other words the lines $u=const$ define the lines of flux of $\mathbf{B}$. The area element is transformed accordingly as $dx\,dy=\rho^{-2}du\,dv$. Together with the coordinates also the spinorial function is transformed by defining $\psi=\Re\Psi$, with $\Re=\cos\frac{1}{2}\theta+i\sigma_z\sin\frac{1}{2}\theta\,.$

After these transformations the expression for the energy becomes:

$$E\int dxdydz\Psi^\dagger\Psi=E\int\frac{1}{\rho^2}dudvdz\psi^\dagger\psi= \tag{1}$$

$$\int\frac{1}{\rho^2}dudvdz\psi^\dagger[\alpha_z(-i\partial_z-ebu)-i\rho(\alpha_x\partial_u+\alpha_y\partial_v)+\frac{i}{\rho^3}(\alpha_x u+\alpha_y v)+\beta m]\psi\,.$$

The trivial dependence on z is separated and with the definition

$$\psi = \sqrt{\rho} e^{ipz} \phi \tag{2}$$

the expression of previous eq.(1) is brought to the form

$$E \int dudv (1/\rho) \phi^\dagger \phi = \tag{3}$$

$$\int dudv \phi^\dagger \{(1/\rho)[\alpha_z (p - ebu) + \beta m] - i(\alpha_x \partial_u + \alpha_y \partial_v)\} \phi .$$

The spinor ψ can be decomposed into the two eigenstates of σ_y , ϕ_+, ϕ_-, which are coupled by the matrices α_x and α_z; the state with eigenvalue -1 is, however, decoupled from the other one if its u-dependence is the one prescribed by the equation

$$(\partial_s + ebs/\rho)\phi_- = 0 \tag{4}$$

where, in order to simplify the notation the auxiliary variable $s = u - u_o = u - p/eb$ has been introduced. The s-dependence of ϕ_- is expressed by the factor $e^{-\gamma}$, with

$$\gamma(s, v) = \int_0^s ds' \ (ebs'/\rho) \tag{5}$$

where is must be noted that ρ depends on the integration variable s'. The whole s-dependence is expressed by means of the definitions:

$$\phi_-(s, v) = \chi(v) f_o(s, v) \tag{6a}$$

$$f_o(s, v) = \frac{1}{\sqrt{N(v)}} e^{-\gamma} , \tag{6b}$$

$$N(v) = \int ds \, e^{-2\gamma} \tag{6c}.$$

The function f_o is real and is normalised with respect to the variable s. In this way it results $\int ds f_o \partial_v f_o = 0$ and the expression for the energy simplifies very much becoming

$$E \int dv g(v) \chi^\dagger \chi = \int dv \chi^\dagger [g(v) \beta m - i\alpha_y \partial_v] \chi ; \tag{7}$$

having defined the function

$$g(v) = \int ds \, f_o(s, v)^2 / \rho .$$

This kind of procedure is an application of the Born-Oppenheimer approximation [5], in which the motion in s is the "fast" dynamics and the motion along v is the "slow" dynamics. In this a simple expression for χ is obtained in the form

$$\chi(v) = \zeta e^{i\kappa\lambda} \, , \; \lambda(v) = \int_0^v dv' \, g(v') \tag{8}$$

and the constant spinor $\zeta_\pm$ is common eigenstate of σ_y, with eigenvalue -1 and of $\alpha_y q + \beta m$, with eigenvalues $E = \pm\sqrt{\kappa^2 + m^2}$.

The qualitative shape of the wave function deserves some word of comment, which could also make clearer the elements of the Born-Oppenheimer approximation in this case. One can expand the expression of γ, eq (5) in powers of s as

$$\gamma(s,v) = eb[A_o + A_1 s + A_2 s^2 + A_3 s^3 + A_4 s^4 + \cdots] \, , \tag{5'}$$

with the result, for the first parameters,

$$A_o = A_1 = 0$$

$$A_2 = 1/2\rho(u_o, v)$$

$$A_3 = -\dot{\rho}(u_o,v)/3\rho^2(u_o,v) \;\; \text{with} \;\; \dot{\rho} = \partial\rho/\partial s.$$

The expansion is meaningful if $|A_3 s^3| << |A_2 s^2|$, in other form

$$s^2 << (A_2/A_3)^2 = (3\rho/2\dot{\rho})^2 \, . \tag{9}$$

In these considerations it is always understood $\rho = \rho(u_o, v)$. When eq (9) holds it is possible to estimate the order of magnitude of s^2 with the result $< s^2 >= \rho/2eb$. Since ρ is directly connected with the magnetic field it is possible to get a condition involving precisely the magnetic field and its gradients, by using the relation $\partial/\partial u = -(b/B)\,{}^*\mathbf{n} \cdot \partial$ one obtains:

$${}^*\mathbf{n} \cdot \partial B << 3\sqrt{e/2} B^{3/2} \, . \tag{9'}$$

From this expression it appears that, at fixed geometrical shape of the field, the stronger the magnetic field becomes the better the condition is satisfied.

Until eq(9,9') hold and the motion along s is confined by a Gaussian wave function with a size depending on v, the function f_o has the explicite form

$$f_o = \left(\frac{eb}{\pi\rho}\right)^{1/4} \exp\left[-\frac{eb(u-u_o)^2}{2\rho}\right]. \tag{10}$$

Here also $\rho = \rho(u_o, v)$ is understood, but with the same degree of approximation one could use the same expression with $\rho = \rho(u, v)$, the differences will show out in the term A_3. More precisely, if one wish to show the first correction to the Gaussian approximation, it results

$$\gamma = \tilde{\gamma} + s^3 \dot{\rho}/6\rho^2 + \cdots, \quad \tilde{\gamma} = \tfrac{1}{2} s^2/\rho, \tag{5''}$$

with further corrections at the level of A_4. This kind of representation will be useful in the next section.

The motion in v takes places around the line of flux $u = p/eb$ and may be treated separately using eq(7), this motionon in v is not limited contrary to what one would expect in analogy with the classical case, where there is the magnetic-mirror effect, see *e.g.* [6]; the fact is that in the particular case under consideration the spin effect precisely compensates the orbital motion, so there is no diamagnetism, this property is in accordance with the result that of the energy of the lowest level does not depend on the magnetic field strength.

In order to have a better understanding of the motion along v it is useful to make more explicit the meaning of the auxiliary variable $\lambda(v)$. By definition it results $\partial_i(bv) = -B_i$ and one can introduce, on the curves $u = const$ the curvilinear abscissa ℓ such that $\partial(bv)/\partial\ell = B$. According to eq(8)

$$\lambda(v) = \int_0^v dv'\, g(v') = \int_0^\ell d\mu \frac{\partial v}{\partial \mu} g(v(\mu)) \tag{8'}$$

When in the definition of $g(v)$ it is possible to interpret ρ as $\rho(u_o, v)$ it results also $g(v) = b/B(v)$, so that,at the end, $\lambda(v) = \ell$

It straightforward, although lengthy, to verify the correct normalisation of the variational wave function. Starting from the original integral

$$\int d^3 r \Psi^\dagger_{p,q} \Psi_{p',q'}$$

the main steps are: the z-integration gives a $\delta(p' - p)$ term, so the auxiliary variable s is the same in both functions; the subsequent s integration yields, through the definitions in eq.s (1,2,3), a factor $1/\rho$, which reconstruct a factor $g(v)$, needed in the final integration over v which in turn yields the $\delta(q' - q)$ term*.

The connection of the calculated wave function and the wave function for constant $\mathbf{B}$ is very straightforward; by choosing $B_y = -b$, $B_x = B_z = 0$ it turns out that the coordinate transformation is the identical transformation $\mathcal{W} = \mathcal{Z}$ so, in particular $\rho = 1$.

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* Since there is, anyway, a preferred frame, a noncovariant normalisation is chosen; there could also be an integer factor to be put by hand depending on how many times the (x, y) plane covers the (u, v) plane

3. A glance to the excited states

The possibility of performing an approximate calculation of the polarisation effects in strong fields using only the lowest level arises from the existence of an energy gap which separates it from the higher levels, it is therefore necessary to know something about the excited states. To this end it is useful to display more in detail the content of eq.(3) by writing it after the decomposition of ϕ.

$$E\int dudv(1/\rho)[\phi_-^\dagger\phi_- + \phi_+^\dagger\phi_+] = \tag{4'}$$

$$\int dudv[\phi_-^\dagger L\phi_- + \phi_+^\dagger L\phi_+ + \phi_+^\dagger\alpha_x K\phi_- + \phi_-^\dagger\alpha_x K^\dagger\phi_+]\,.$$

Having set

$$L = -\mathrm{i}\alpha_y\partial_v + \beta m/\rho$$

$$K = -\mathrm{i}\partial_s - \mathrm{i}ebs/\rho\ ,\ K^\dagger = -\mathrm{i}\partial_s + \mathrm{i}ebs/\rho\,.$$

The problem in the s-variable, where v is treated as an external parameter is solved again with the Born-Oppenheimer procedure generalising eq(6a) to

$$\phi_\pm(s,v) = \chi(v)_\pm f_\pm(s,v)\,. \tag{11}$$

In this way the matrix equation in the s variable is produced

$$\mathcal{K}F = \Omega F, \tag{12}$$

with the definitions:

$$\mathcal{K} = \begin{pmatrix} 0 & K^\dagger \\ K & 0 \end{pmatrix}$$

$$F = \begin{pmatrix} f_- \\ f_+ \end{pmatrix}$$

and by squaring eq.(12) it results

$$\mathcal{K}^2 F = \Omega^2 F \tag{12'}$$

$$\mathcal{K}^2 = \begin{pmatrix} K^\dagger K & 0 \\ 0 & KK^\dagger \end{pmatrix}$$

which suggests also the position $f_+ \propto Kf_-$.

In this way eq (12') becomes

$$K^\dagger K f_- \equiv \{-\partial_s^2 + (ebs/\rho)^2 - [\partial_s\,,\,ebs/\rho]\}f_- = \Omega^2 f_- \tag{12''}$$

The commutator gives as result $eb/\rho - ebs\dot{\rho}/\rho^2$ so that, at least for wave functions for which small values of s are more important, eq.(12') can be interpreted as representing a harmonic oscillator with a v depending frequency

$$\Omega_n^2 = 2neb/\rho\,; \tag{13}$$

the integer n may assume also the value 0 reproducing thus the ground state. The approximation requires the condition $s\dot{\rho}/\rho^2 << 1$, which is the same as eq (9),within a factor $\frac{3}{2}$, it amounts neglecting the terms with higher negative powers of ρ; in this situation the first excited state will have the dependence in s dictated respectively by the real factors $f_+ = f_o$ and $f_- = \Omega_n^{-1}\mathrm{i}Kf_o$. It is possible to get the equivalent of eq.(7), which for the first excited state is:

$$E\int dv g(v)[\chi_-^\dagger\chi_- + \chi_+^\dagger\chi_+] = \tag{14}$$

$$\int dv\{\chi_-^\dagger[g(v)\beta m - \mathrm{i}\alpha_y\partial_v]\chi_- + \chi_+^\dagger[g(v)\beta m - \mathrm{i}\alpha_y\partial_v]\chi_+ + \mathrm{i}[\chi_+^\dagger\alpha_x\chi_- - \chi_-^\dagger\alpha_x\chi_+]h(v)\}$$

having defined the function $h(v) = \int ds f_o(s,v)^2\Omega_1$.

With the definition of ℓ used in eq(8'), the same equation is transcribed as

$$E\int d\ell[\chi_-^\dagger\chi_- + \chi_+^\dagger\chi_+] = \tag{14'}$$

$$\int d\ell\{\chi_-^\dagger[\beta m - \mathrm{i}\alpha_y\partial_\ell]\chi_- + \chi_+^\dagger[\beta m - \mathrm{i}\alpha_y\partial_\ell]\chi_+ + \mathrm{i}[\chi_+^\dagger\alpha_x\chi_- - \chi_-^\dagger\alpha_x\chi_+]C(\ell)\}\,,$$

having now defined $C(\ell) = h(v)/g(v)$. The term in C is the new feature of the excited states, it is not present in the ground state. The qualitative behaviour of the solution of eq.(14') depends on the behaviour of the function $C(\ell)$ and so on the behaviour of the functions $g(v)$ and $h(v)$ for large v. It may be noticed that with the same approximation used to derive the relation $\ell = \lambda$, *i.e.* narrow wave function in s, it results also $\Omega_1 = b\sqrt{2e/B}$ and finally $C(\ell) = \sqrt{2eB(v(\ell))}$

The two first order equtions for χ_- and χ_+ arising from eq(14') can be reduced to a second order equation

$$(E^2 - m^2 + \partial_\ell^2)\chi_- = \left(C^2 + \frac{\dot{C}}{C}[\partial_\ell - \mathrm{i}\alpha_y(E - \beta m)]\right)\chi_-\,;$$

here the dot indicates the derivative with respect to ℓ, defining the function $\Phi = C^{-1/2}\chi_-$ and remembering the relation between C and the absolute value of the magnetic field B, the alternative form is produced:

$$(E^2 - m^2 + \partial_\ell^2 - 2eB)\Phi + \left(\frac{\ddot{B}}{4B} - \frac{3\dot{B}^2}{16B^2} + \mathrm{i}\alpha_y(E - \beta m)\frac{\dot{B}}{2B}\right)\Phi = 0\,. \tag{15}$$

In order to get more definite results about the wave function of the excited states some specification on the behaviour of the field B are needed: it has already pointed out that all the treatment implies large values for B, so in particular lines of flux on which the field can become zero at some finite point must be avoided. If the correspondence between $\mathcal{W}$ and $\mathcal{Z}$ is polynomial B grows at infinity and the positive term $2eB$ certainly dominates over the second parenthesis; eq(15) describes a confined system, the bound-state energy levels depend on how B goes to infinity but in any case there will be a gap in the energy of order,at least, of B_{min}; if the correspondence is different *e.g.* $\mathcal{W} = e^{a\mathcal{Z}}$, then it may happen that the system is unbound. One must remember,however, that the behaviour at infinity of B is, in the real case, a sort of conventional statement because, as already remarked in the introduction, one is interested only in small region of space.

4. Estimate of the vacuum polarisation

According to standard QED the expression to be calculated is:

$$\mathrm{i}D(x-y) + e^2 \int D(x-\xi)\Pi(\xi,\eta)D(\eta-y)\, d\xi\, d\eta. \tag{16}$$

The factor $\Pi(\xi,\eta)$ is evidently a tensor explicitly given by

$$\Pi_{\mu\nu}(\xi,\eta) = tr[\gamma_\mu S_F(\xi,\eta)\gamma_\nu S_F(\eta,\xi)]$$

and the Feynman propagators S_F are expressed through the solutions Ψ. As already stated the magnetic field is static but inhomogeneous in space, it is convenient to consider a "partial" Fourier transform, where the mean position, here given explicitly by $\mathbf{R} = \frac{1}{2}(\mathbf{x}+\mathbf{y})$, is not transformed: in this way, eq (16) gives rise to:

$$\mathrm{i}\tilde{D}(\mathbf{q}) + \frac{e^2}{(2\pi)^3} \int \tilde{D}(\mathbf{q}-\tfrac{1}{2}\mathbf{Q})\tilde{D}(\mathbf{q}+\tfrac{1}{2}\mathbf{Q})e^{\mathrm{i}\mathbf{a}\cdot\mathbf{Q}}\tilde{\Pi}(\mathbf{R}-\mathbf{a};\mathbf{q})\, d^3a\, d^3Q\,; \tag{17}$$

the normalisation is such that $\tilde{D}(\mathbf{q}) = 1/\mathbf{q}^2$. For the particular component of $\Pi_{\mu\nu}$ we are mainly interested *i.e.* Π_{oo}, one can take advantage from the fact that the spin-rotation matrices $\Re$ commute with γ_o, so they disappear from the trace and one is allowed to work with the wave functions ψ. The system shows, moreover, a transationally invariance along the z-axis due to the restrictions that have been put on the variations of $\mathbf{B}$, so the relevant component of the trace can be expressed as:

$$\tilde{\Pi}_{oo} = -\mathrm{i}\int e^{\mathrm{i}q_\perp\cdot(\xi-\eta)_\perp} d(\xi-\eta)_\perp \int \frac{dk\, dl}{E_k+E_l}\times \tag{18}$$

$$[\delta(q_z - k_z + l_z)\psi_k^\dagger(\xi)\psi_l(\xi)\psi_l^\dagger(\eta)\psi_k(\eta) + \delta(q_z + k_z - l_z)\psi_l^\dagger(\xi)\psi_k(\xi)\psi_k^\dagger(\eta)\psi_l(\eta)]$$

Here the functions ψ_k and ψ_l depend only on the transverse variables, they refer respectively to electrons with momentum components k and the to positons with momentum components $-l$; the energies E_k and E_l are both positive. The integrations in $dk\,dl$ indicates the integations over the momenta along z, i.e. k_z, l_z, which do not contribute to the energy and on the momenta along the curvilinear abscissa λ eq(8,8'), which will be indicated as κ_k, κ_l; since among the oscillatory modes only the ground state is selected out no summation over discrete quantum number is implied, in the same way also for the spin only one state is taken into account, eq(4). Within these limitations the various terms appearing in eq(18') are separately calculated. The first factor comes from the spinorial structure: going back to the definition given in eq(8) it results

$$|\zeta_k^\dagger \zeta_l|^2 = \tfrac{1}{2}[1 - (m^2 + \kappa_k\kappa_l)/E_k E_l]\,, \tag{19}$$

which is invariant for the exchange $k \leftrightarrow l$. This expression is inserted into the integration over κ_k, κ_l with the result

$$H_\ell = \int \frac{d\kappa_k\, d\kappa_l}{E_k + E_l} |\zeta_k\, \zeta_l|^2 \exp[\mathrm{i}(\kappa_k - \kappa_l)(\lambda_\xi - \lambda_\eta)] =$$

$$4\pi[\delta(\lambda_\xi - \lambda_\eta) - m\, Ki_1(2m|\lambda_\xi - \lambda_\eta|) \tag{20}$$

The following definition has been used[9]:

$$Ki_1(x) = \int_x^\infty K_o(t)dt\,.$$

Now the integration over the momenta along the z-axis has to be performed the nontrivial way in which this variable enter in ψ is through the definition of γ, see eq(5,5'). It is convenient to go back to the exponential representation of the δ- function so that the integrations are partially disentangled. Following eq(5') the expression to be integrated in dk is

$$\sqrt{eb/\pi}[\rho(\xi)\rho(\eta)]^{-1/4} \exp[-\tfrac{1}{2}eb(u_\xi - u_o)^2/\rho(\xi) - \tfrac{1}{2}eb(u_\eta - u_o)^2/\rho(\eta)]\mathrm{e}^{\mathrm{i}k_z z}\,, \tag{21}$$

with $u_o = k_z/eb$.

The integration is easily performed with the result

$$\mathcal{J} = \sqrt{2}eb\frac{[\rho(\xi)\rho(\eta)]^{3/4}}{\sqrt{\rho(\xi) + \rho(\eta)}}\times \tag{22}$$

$$\exp\left[-\tfrac{1}{2}eb\frac{(u_\xi - u_\eta)^2 + z^2\rho(\xi)\rho(\eta)}{\rho(\xi) + \rho(\eta)} + \mathrm{i}ebz\frac{u_\xi\rho(\eta) + u_\eta\rho(\xi)}{\rho(\xi) + \rho(\eta)}\right].$$

The integration in dl gives the same result, but for a change of the sign of z and therefore the product of the two integrals is simply $|\mathcal{J}|^2$.

The successive integration in dz yields

$$H_z = \frac{1}{2\pi}\int dz|\mathcal{J}|^2\, e^{iq_z z} = \tag{23}$$

$$\frac{(eb)^{3/2}\rho(\xi)\rho(\eta)}{\sqrt{\pi}[\rho(\xi)+\rho(\eta)]}\exp\left[-eb\frac{(u_\xi-u_\eta)^2}{\rho(\xi)+\rho(\eta)}\right]\exp\left[-q_z^2\frac{\rho(\xi)+\rho(\eta)}{4eb\rho(\xi)\rho(\eta)}\right].$$

It has been previously shown which is the expression of the first correction to the Gaussian approximation for the transverse wave function, eq(5'), now it is necessary to show that the expression previously obtained is stable with respect to this correction. To the first order in $\dot\rho/\rho^2$ one can take care of this correction by multiplying the expression in eq(21) by a factor

$$U = 1 + (u_\xi - u_o)^3\dot\rho(\xi)/\rho^2(\xi) + (u_\eta - u_o)^3\dot\rho(\eta)/\rho^2(\eta)$$

Using then the representation $u_o = (-i/eb)\partial_z$ it is easy, although lengthy, to find out the modifications to the result expressed in eq(22). Neglecting terms which are depressed by factors $1/eb$, what is consistent with the formal limit $b \to \infty$, the correction to H_z is expressible by the factor

$$C_z = 1 + (u_\xi - u_\eta)^3\frac{\dot\rho(\xi)\rho(\xi) - \dot\rho(\eta)\rho(\eta)}{[\rho(\xi)+\rho(\eta)]^3}$$

As anticipated in eq(17,18') one has now to perform the Fourier transform with respect to the relative transverse variables, these variables are contained not only in u and λ, but also in $\rho(\xi)$ and $\rho(\eta)$, which are proportional to the external magnetic field, so the definite field configuration enter unavoidably at this point. If a definite field configuration is not chosen, one may perform an expansion around the central position of the loop in the transverse plane,*i.e.* around the point $\sigma = \frac{1}{2}(\xi+\eta)$, with the justification given by eq(10). With this expansion it results

$$H_z = (2\pi)^{-1/2}[eb\rho(\sigma)]^{3/2}\exp[-\tfrac{1}{2}eb\rho(\sigma)(\xi-\eta)_t^2]\exp[-\tfrac{1}{2}q_z^2/eb\rho(\sigma)]$$

without any correction linear in $\dot\rho$. The index t means transverse with respect to the field $\mathbf{B}$, in the plane (x,y). The expression arising from C_z contains at least either $\dot\rho^2$ or $\ddot\rho$, so at this order it is consistent to set $C_z = 1$.

For what concerns H_λ, at this order of approximation the term $(\lambda_\xi - \lambda_\eta)$ corresponds simply to $(\xi-\eta)_\ell$ *i.e.* to the distance in the direction of the field $\mathbf{B}$. The Fourier transform indicated in eq(18') is therefore expressed in terms of the transverse and longitudinal momentum transfer q_t , q_ℓ and the results are

$$\int H_z\, e^{iq_t(\xi-\eta)_t} d(\xi-\eta)_t = eb\rho(\sigma)\exp[-q_t^2/2eb\rho(\sigma)]\exp[-q_z^2/2eb\rho(\sigma)] \tag{24}$$

$$\int H_\ell \, e^{iq_\ell(\xi-\eta)_\ell} d(\xi-\eta)_\ell = 2\pi\Big[1 - \frac{1}{w\sqrt{w^2+1}} \ln(w + \sqrt{w^2+1})\Big] \qquad w = q_\ell/2m\,. \tag{25}$$

in the limit of small $\mathbf{q}$ the two expressions reduce to:

$$\int H_z \, e^{iq_t(\xi-\eta)_t} d(\xi-\eta)_t \approx eb\rho(\sigma) - \tfrac{1}{2}(q_z^2 + q_t^2) + \cdots \tag{24'}$$

$$\int H_\ell \, e^{iq_\ell(\xi-\eta)_\ell} d(\xi-\eta)_\ell \approx 2\pi[q_\ell^2/3m^2 - q_\ell^4/15m^4 + \cdots] \tag{25'}$$

Since the limit $w \to 0$ is equivalent either to $\mathbf{q}^2 \to 0$ or to $m^2 \to \infty$ it is evident that a term independent of m, which seemed to be present in eq(25), really is not there

In this way the limit of very low $\mathbf{q}$ is obtained in the form

$$\tilde{\Pi}_{oo} \approx -2i\pi eb\rho(\sigma) q_\ell^2/3m^2 + \cdots$$

It represents the result that has to be inserted in eq(17) to get the expression of the vacuum polarisation, observing that $\sigma = (R-a)_\perp$. The result is somehow obvious when one remembers that $b\rho(\sigma)$ is nothing but the modulus of the field $\mathbf{B}$ at the centre of the loop, so the dependence is locally the same as for the constant field; the derivation presented states that this result is corect also at the first order in the gradients of the field. It is also clear how the succesive corrections could be calculated in terms of the coefficient A_4 of the expansion eq(5') and of analogous term that arise, *e.g.* from the expansion around σ.

One can deal with an example given by a particular choice of the magnetic field, which is the simplest non constant field and is generated by a quadratic correspondence between $\mathcal{Z}$ and $\mathcal{W}$. With a suitable choice of the coordinates the vector potential is $A_z = \frac{1}{2}b(y^2 - x^2)$ and so $B_x = by$, $B_y = bx$, the function ρ becomes simply $r_\perp = \sqrt{x^2+y^2}$, the expression for $q_\ell = (yq_x + xq_y)/r_\perp$ must be inserted in eq(18') and successively (17) in order to have a completely explicitly expression. In this situation one can calculate the Fourier transform in the small $\mathbf{q}$ limit of $\tilde{\Pi}_{oo}$ with respect to the variable $\sigma_\perp$. * It result that

$$\tilde{\tilde{\Pi}}_{oo}(P,\mathbf{q}) \propto 2\pi i \, \frac{q_\perp^2 P^2 - 3(q_x P_y + q_y P_x)^2}{P^5}. \tag{26}$$

so there is a sharp divergence for small P, due to the behaviour of $\mathbf{B}$; so one gets a transparent result; if one inserts the behaviour given by eq(26) into eq(17), with the explicit form of the static photon propagator, it appears evident that the

* The subscript $\perp$ will be dropped whenever possible; in particular P is by definition transverse.

main contribution comes from very small P, when one neglects them in the photon propagators, the final result is that the field is now calculated precisely where the polarisation effect is looked for. For the modification to the photon propagator one gets

$$i/\mathbf{q}^2 \to [i/\mathbf{q}^2](1-\Delta) \quad \text{with} \quad \Delta = \alpha|eB(R)|q_\ell^2/3\pi m^2\mathbf{q}^2.$$

With the particular configuration of magnetic field here proposed also the analysis of the excited stated may be pushed further, but in so doing nothing essentially new is gained.

5. Other possible effects of the static field

From what has been seen it would seem that the effects of the non unformity of the field $\mathbf{B}$ are always in the second order in the field gradients or, alternatively, they involve the second derivatives of the field, at least as long as one looks at static properties. An obvious question then arises, are there effects of the first order?

It is easy to produce an example that answer affirmatively to this question. The polarisation tensor $\Pi_{\mu\nu}$ influences also the propagation of the free photon: the relevant terms, in this case, are the spacial components Π_{ij}, as it appears particularly evident if one deals with the free photon in Coulomb gauge. In configuration space this tensor can be obtained, as already discussed, from the corresponding one built up with the nonrotated wave functions ψ letting the rotation matrices $\Re$ act on the γ matrices at the vertices of the loop. In the approximation always considered in the paper of including only the lowest discrete state the expression built by means of the ψ wave functions has a unique nonzero term, the one bearing the indices (y,y) because $\psi^\dagger\gamma_x\psi = \psi^\dagger\gamma_z\psi = 0$ when ψ is an eigenstate of σ_y, eq(4). Let us call it $\Pi^{(y)}$ this term , the rotated tensor is not symmetric; when the symmetric part is extracted it is easily seen that it contains a part which survives in the limit of constant field and a second order correction. The relevant antisymmetric term is

$$A_{x,y} = \tfrac{1}{2}\sin(\theta_2-\theta_1)\Pi^{(y)}\,,$$

geometrically it is proportional to the curvature of the line of force. If one expresses the sine in terms of $\mathbf{B}$ it results linear in the gradient in fact

$$\sin(\theta_2-\theta_1) = \epsilon_{zij}n_i(\xi)n_j(\eta) \approx \epsilon_{zij}[(\xi-\eta)_a\partial_a B_i]B_j/B^2\,,$$

where B and its gradients are evaluated at the mean position σ. Both the symmetric and the antisymmetric parts of $\Pi_{i,j}$ give rise to a rotation of the plane polarisations, but their effect is different: in the symmetric case there are two eigenvectors corresponding to the two polarisation that are not altered by the magnetic effect, it is a standard birefringence effect present also for constant magnetic fields, the antisymmetric piece, which is evidently absent for constant fields, couples always different direction of polarisation and induces therefore a rotation in every case and the amount of the rotation is proportional to the gradient of the magnetic field.

6. Conclusions

A sketchy version of the content of this paper had been presented at the IV Seminar on nonlinear phenomena in complex systems - Minsk, February 1995 [10]. Here the same problem has been elaborated more in detail with a more general field configuration.

It could be useful to remember once again which is the true physical problem: it is the vacuum polarisation in presence of an external magnetic field, so the real interest is to consider the variations of the fields for distances of the order of the Compton wave length of the electron; this limitation justifies the interest for the uniform, and quasi uniform, field situations, however if one think to some experimental situations envisaged to detect such kind of effects: i.e. the heavy-ion collisions it is clear that very strong field but also very large gradients and very large time variations can be encountered. The case here presented is intermediate: large gradients are present but the field is still static. The limit of uniform field can evidently be reobtained, but the effects of nonuniformity can be calculated since the expression for the corrections have been displayed, in a form with is consistent with the initial assumption of very strong field. The actual computation would require a choice of the field configuration and is in every case quite long.

It is possible to give a simple geometrical interpretation of the procedure: given the lines of flux of the magnetic field, with the constraints $div\mathbf{B} = 0, curl\mathbf{B} = 0$ at the first step one takes as the most significant element the tangent, at the second step the osculating plane, a further step would imply the variation in all three directions and so the absence of a quantum number trivially conserved, so the technical difficulty would increase again.

The solution of the Dirac problem in the given field, although non exact, show some relevant features: the presence of a lowest level with energy independent of the field strength is confirmed, this kind of configuration cannot be bound so that there is also a continuum quantum number associated with it, formally of the momentum type, but conjugated to a curvilinear coordinate. The excited states have energies separated by gaps growing with the strength of the field, this property allows a first study of the polarisation effects taking into account only the ground state together with the continuum quantum number associated with it. The excited states show, moreover, the possibility of being bound, for suitable field configurations, what is expected from classical analogy [8].

It has also been found that the inhomogeneity of the field affects the free-photon propagation, in these cases there is a first order effect. These phenomena are not directly related to the original aim of this investigation *i.e.* the modifications of the effective coupling constant in external fields, in small regions of space-time. There are, moreover, other kinds of problems for which the behaviour of a Dirac particle in a given nonuniform magnetic field is interesting [11].

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