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REGGE CALCULUS AND ASHTEKAR VARIABLES

Regge calculus and Ashtekar variables

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Spacetime discretized in simplexes, as proposed in the pioneer work of Regge, is described in terms of selfdual variables. In particular, we elucidate the "kinematic" structure of the initial value problem, in which 3–space is divided into flat tetrahedra, paying particular attention to the role played by the reality condition for the Ashtekar variables. An attempt is made to write down the vector and scalar constraints of the theory in a simple and potentially useful way.

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1. Introduction.

Discussing the (1971) future of Regge calculus [1], Misner, Thorne and Wheeler [2] expressed the hope that "Regge's truly geometric way of formulating general relativity will someday make the content of the Einstein field equations ... stand out sharp and clear,...". Perhaps this hope has not been fulfilled (though not for want of trying), but one may still turn to Regge calculus for an intuitive geometric interpretation of formal developments, or for a clean and geometrically meaningful regularization of the theory.

In the latest and very promising attempt to quantize gravity, this time in loop space [3], it is the lack of a reliable regularization that hampers the progress towards a consistent theory. A regularization is needed to deduce the form of the constraints, and inevitably this breaks the backround independence that is central to the whole program. Worse still, when the cutoff is removed, backround independence is not recovered: the action of the Hamiltonian constraint on the wave function can be expressed in terms of genuine loop operations, weighted with the angles between the branches of the loop at an intersection [4]. With such an obscure expression it is impossible to decide whether the constraint algebra closes or not, quite apart from the technical difficulty of the calculation.

This situation could perhaps be remedied if one could describe Regge's discretized space in terms of the new canonical variables for general relativity introduced by A.Ashtekar [5] [6], on which the loop representation is based. This program was begun by V. Khatsymovsky [7], and there have been other attempts to formulate a discretized canonical theory along these lines [8]. In this work we complete the "kinematical" part of such a program, but do not attempt to formulate a canonical theory, nor to investigate in detail the structure of the constraints.

Quite apart from the aims stated above, a formulation of this type might have an interest of its own because it is to some extent modeled on recent work by G. 't Hooft [9] in $2+1$ dimensions. It appears that, in this admittedly simpler case, on can go surprisingly far in the construction of a quantum theory, in particular on the crucial problem of the connection with the Euclidean formulation, on the formulation of a canonical theory and on the analysis of the constraints [10].

We recall in §2 the basic ideas of Regge calculus, in the tetrad form developed by various authors [11] in the last few years, and their translation in terms of selfdual variables.

In §3 we analyze the initial value picture one obtains separating 3–space from time, emphasizing in particular the definition and the properties of the Ashtekar variables. In a

3–manifold divided in tetrahedra, there is a set of variables $\bar{E}^{a\mu}$ for each tetrahedron, while the transport matrices $L(AB)_{ab}$, related to the Ashtekar connection A^a_μ, are associated to the links of the dual lattice. We use the reality conditions [12] [6], to derive the form of the curvature one finds associated to the polygons in the dual lattice, and find that it is consistent with one would expect from the continuum case [13].

We attempt in §4 to express the scalar and the vector constraints of the Ashtekar formulation in a way that is suitable to be applied to Regge calculus, using arguments based on the weak coupling case and on an intuitive picture borrowed from [14]; the form we obtain is perhaps sufficiently attractive to deserve further investigation. It is also encouraging that it agrees with what one would expect from the work of [15] [16].

A more sophisticated treatment, along the lines of [17] is perhaps possible, but has not been attempted here.

2. Generalities about Regge calculus.

I shall here describe the main elements of Regge calculus that are needed in the following, in the case of $3 + 1$ dimension, referring to ref. [18] for a proper review and a complete bibliography.

4-space is divided in 4-simplices σ^4, each with a flat metric determined by the assigned lengths of its links σ^1_α, $\alpha = 1, ..., 10$. A Lorentz frame can therefore be attached to each σ^4, through vierbeins $e^i_\mu(\sigma^4)$, constant within σ^4, which associate to each link σ^1_α a Lorentz vector $v^i(\sigma^1_\alpha|\sigma^4) = e^i_\mu(\sigma^n)\Delta x^\mu_\alpha$. Two neighbouring 4-simplices, say σ^4_A, σ^4_B share a tetrahedron σ^3; since the lengths of of all $\sigma^1_\alpha \in \sigma^3$ must be the same in the two frames, a Lorentz transformation will link them, such that:

$$v^i(\sigma^1_\alpha|\sigma^4_A) = \Lambda(A\,B)^i{}_j v^j(\sigma^1_\alpha|\sigma^4_B), \quad \forall\, \sigma^1_\alpha \in \sigma^3 = \sigma^4_A \cap \sigma^4_B \tag{1}$$

There are 3 independent such equations, which fix Λ_{AB} uniquely[1]. This is the "metricity" condition; in the continuum the condition that the torsion be zero fixes the Lorentz connection ω^{ij}_μ to be equal to the metric compatible Levi–Civita connection Ω^{ij}_μ, here distributional, with support on σ^3, and with $\Lambda_{AB} = P \exp \int_A^B \Omega$.

Curvature is found going around a "bone" $\sigma^2 = \sigma^4_1 \cap ... \cap \sigma^4_N$, where we find:

$$R(\sigma^2|\sigma^4_1) = \Lambda(1N)...\Lambda(21) \tag{2}$$

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[1] Assigning the action of an $O(n)$ (or $O(n-1,1)$) transformation on $1, 2, ..., n-1$ independent vectors fixes $(n-1)$, $(n-1) + (n-2), ...$, all its $n(n-1)/2$ parameters

a Lorentz tansformation, which for consistency must be such that:

$$R(\sigma^2|\sigma_1^4)^i{}_j v^j(\sigma_\alpha^1|\sigma_1^4) = v^i(\sigma_\alpha^1|\sigma_1^4), \quad \forall\, \sigma_\alpha^1 \in \sigma^2 \tag{3}$$

This gives us 2 independent equations, that by themselves fix R up to one parameter (the "deficit angle", which in fact may or may not be an angle). Explicitely:

$$R = \exp(F), \qquad F^i{}_j = \alpha\, \epsilon^i{}_{jkl} v_1^k v_2^l \tag{4}$$

In the Ashtekar approach the emphasis is on the use and the transport of self-dual vectors, which are projected from antisymmetric tensors by

$$v^a := C^a_{ij} V^{ij} = -\tfrac{1}{2}\epsilon_{abc} V^{bc} + i\, V^{0a} ; \quad a, b, \ldots = 1, 2, 3 \tag{5}$$

and transform under the $(1,0)$ representation of the Lorentz group:

$$\Lambda^i{}_j = \delta^i{}_j + \lambda^i{}_j + \ldots \iff L_{ab} = \delta_{ab} + \epsilon_{acb} C^c_{ij} \lambda^{ij} + \ldots \tag{6}$$

The 3×3 matrices L_{ab} are orthogonal and (in general) complex; for an ordinary rotations they are real with $L_{ab} = \Lambda_{ab}$; for a Lorentz boost (a pure Lorentz transformation) they are hermitian. Selfduality means that $\tfrac{1}{2}\epsilon_{ij}{}^{kl} C^a_{kl} = iC^a_{ij}$, therefore to $\Lambda^i{}_j = \delta^i{}_j + \epsilon^i{}_{jkl}\Delta^{kl} + \ldots$ corresponds $L_{ab} = \delta_{ab} + 2i\epsilon_{acb} C^c_{ij}\Delta^{ij} + \ldots$.

The arguments above can be repeated focusing on bones rather than links, associating to a bone σ^2 with given $S^{\mu\nu}$ a 3–vector by $v^a_{\text{bone}} = C^a_{ij} e^i_\mu e^j_\nu S^{\mu\nu}$, in each frame. Notice that there are 10 links, but also 10 triangles for each σ^4, so that assigning the areas of the triangles fixes the metric just like assigning the lengths of the links [19].

Going round a bone:

$$R = \exp(F) \qquad F_{ab} := \epsilon_{acb} F^c = 2i\alpha\epsilon_{acb} v^c_{\text{bone}} \quad \text{real } \alpha \tag{7}$$

which follows because R has to leave the bone invariant.

3. 3+1 separation.

To separate space from time an appropriate set of σ^3-s has to be selected as a Cauchy surface of constant time. Each inherits a Lorentz frame from (say) its future σ^4; but while the lengths of the $\sigma^1_\alpha \in \sigma^3$ fix the (flat) 3–metric, to determine the Lorentz vectors v^i_α, hence the e^i_μ, $i = 0, 1, 2, 3$, $\mu = 1, 2, 3$ we need also informations about the movement of those links in time. However, since the lengths are positive, the 3–metric $q_{\mu\nu} = \eta_{ij} e^i_\mu e^j_\nu$ must be positive definite; therefore for each σ^3 there is a Lorentz transformation Λ such that $(\Lambda e_\mu)^0 = -n_\mu$, the unit normal the the 3–space hypersurface, i.e. such that $(\Lambda v(\sigma^1|\sigma^3))^0 = 0$. This choice of frames we shall call "time gauge", and will be used a lot in the following.

A Lorentz transformation links the frames of a pair σ^3_A, σ^3_B sharing a σ^2, the same that links the corresponding σ^4-s. But if we only have access to the spacelike links, we can only get 2 independent equations from eq.(1), and the Lorentz transformation is fixed only up to one parameter: $\Lambda(AB)$ depends on the 3–d Levi–Civita connection, but also on the extrinsic curvature at σ^2. Suppose we choose the time gauge for both σ^3_A, σ^3_B, and write $\Lambda(AB)$ as a rotation times a boost. Since $v^0_\alpha(A) = v^0_\alpha(B) = 0$, the boost must be *transversal* to all $v_\alpha(B) \in \sigma^2$, which fixes it up to a constant (related to the extrinsic curvature), and the 3–rotation is uniquely determined. This transversality of the Lorentz boost is an essential ingredient in G. 't Hooft's treatment of the $2 + 1$ dimensions theory [9].

In the continuum we have the analogous statement that pulling back the 4–d Levi-Civita connection one has:

$$q_\mu{}^\nu \Omega^{ab}_\nu = {}^3\Omega^{ab}_\mu \; ; \quad q_\mu{}^\nu \Omega^{0a}_\nu = e^{a\nu} K_{\mu\nu} \qquad a, b = 1, 2, 3 \; ; \quad \text{(time gauge)} \qquad (8)$$

Where ${}^3\Omega^{ab}_\mu$ is the 3–d Levi–Civita connection, $K_{\mu\nu} = K_{\nu\mu}$ the extrinsic curvature.

Going round a $\sigma^1 = \sigma^3_1 \cap ... \cap \sigma^3_N$ we get the curvature associated by eq.(2) to the corresponding timelike bone: a Lorentz transformation, determined up to 3 parameters by the condition that it leaves invariant the vector associated to σ^1.

The canonical formalism of A. Ashtekar [6] is obtained separating time from space in the selfdual picture: one defines "new variables" from the pull backs of the connection and the vierbein forms by:

$$A^a_\mu := q_\mu{}^\nu C^a_{ij} \omega^{ij}_\mu \; ; \qquad \tilde{E}^{a\mu} := -C^a_{ij} e^i_\nu e^j_\lambda \tilde{\epsilon}^{\mu\nu\lambda} = -2i C^a_{ij} e\, n_\nu e^{i\nu} e^{j\mu} \qquad (9)$$

where $e = \det(q_{\mu\nu})^{1/2}$. In time gauge and for the Levi-Civita connection:

$$A^a_\mu = -\tfrac{1}{2}\epsilon_{abc}\,^3\Omega^{bc}_\mu + ie^{a\nu}K_{\mu\nu}\ ; \qquad \tilde{E}^{a\mu} = ee^{a\mu} \qquad \text{(time gauge)} \tag{10}$$

The $\tilde{E}^{a\mu}$ associate quite naturally (complex) 3–vectors to surfaces in 3-space [20], by

$$S^a = \tilde{E}^{a\mu}\underset{\sim}{\epsilon}_{\mu\nu\lambda}S^{\nu\lambda} = -2\,C^a_{ij}e^i_\mu e^j_\nu S^{\mu\nu} = \ \text{(in time gauge)}\ \epsilon_{abc}e^b_\mu e^c_\nu S^{\mu\nu} \tag{11}$$

For example for a triangle $(\Delta x_1, \Delta x_2, \Delta x_1 - \Delta x_2)$, $S^{\mu\nu} = \tfrac{1}{2}\Delta x_1^{[\mu}\Delta x_2^{\nu]}$, so we find $S^a S^a = \tfrac{1}{4}(\Delta x_1^2 \Delta x_2^2 - (\Delta x_1 \Delta x_2)^2) = $ area square. We therefore have the necessary consistency condition:

$$S^a = \tilde{E}^{a\mu}s_\mu \ \Rightarrow\ S^a S^a = \text{real, positive}\quad \forall\, s_\mu\ \text{real} \tag{12}$$

a first set of "reality conditions", from which the existence of a time gauge follows. In time gauge the $\tilde{E}^{a\mu}$-s, and therefore any S^a, are real. In a different frame S^a will be complex, with (orthogonal) real and imaginary parts that carry the informations about the surface and its orientation, and about the components of its velocity perpendicular to this orientation.

In a discretized world, if tetrahedra σ^3_A and σ^3_B share a triangle to which S^a_A, S^a_B are associated, one passes from from one to the other by a Lorentz transformation:

$$L(A\,B) = P\exp\int_A^B A\ :\qquad S^a_A = L(A\,B)_{ab}S^b_B \tag{13}$$

This metricity condition corresponds to the continuum "Gauss law": $D_\mu \tilde{E}^{a\mu} = 0$. However it is not really enough, because it fixes L_{AB} only up to one *complex* parameter. In the continuum the missing information is supplied by a second set of reality conditions:

$$\tilde{P}^{\mu\nu} := i\epsilon_{abc}\tilde{E}^{c\rho}\big(\tilde{E}^{a\mu}D_\rho\tilde{E}^{b\nu} + \tilde{E}^{a\nu}D_\rho\tilde{E}^{b\mu}\big) = \ \text{real} \tag{14}$$

If the connection is the pull back of the selfdual part of the 4-d Levi–Civita connection, one finds $\tilde{P}^{\mu\nu} = 2e^3(K^{\mu\nu} - q^{\mu\nu}K)$.

To be definite, suppose that the tetrahedra σ^3_B, σ^3_A, σ^3_C, all sharing a common link σ^1, are separated by the triangles σ^2_1, σ^2_2, σ^2_3, so that σ^3_A has sides σ^2_1, σ^2_2 etc., and $S^{\mu\nu}_I = \tfrac{1}{2}\Delta x^{[\mu}\Delta x^{\nu]}_I$, $S_{I\mu} := \tfrac{1}{2}\underset{\sim}{\epsilon}_{\mu\nu\lambda}S^{\nu\lambda}_I$, with $I = 1, 2, 3$.

At the boundary between σ^3_B and σ^3_A, the covariant derivative of $\tilde{E}^{a\mu}$ is non zero only in the direction normal to σ^2_1, so a plausible translation of eq. (14) is:

$$i\,\epsilon_{abc}\tilde{E}^{c\lambda}_A S_{1\lambda}\big(\tilde{E}^{a\mu}_A L_{bd}\tilde{E}^{d\nu}_B + \tilde{E}^{a\nu}_A L_{bd}\tilde{E}^{d\mu}_B\big) = \text{real} \tag{15}$$

To interpret this condition, let us associate a (densitized) 3-vector to a link by:

$$\tilde{l}^a := \tfrac{1}{2}\epsilon_{abc}\tilde{E}^{b\mu}\tilde{E}^{c\nu}\underset{\sim}{\epsilon}_{\mu\nu\lambda}\Delta x^\lambda = -2ieC_{ij}^a e_\mu^i e_\nu^j n^\mu \Delta x^\nu = \text{(in time gauge)} \; ee_\mu^a \Delta x^\mu \qquad (16)$$

Notice that if (in time gauge) we organize the frames in two neighbouring tetrahedra so that the Lorentz transformation from one to the other is just a boost transversal to the common triangle, the $\tilde{l}^a$ associated to the sides of the triangle will *not* be invariant.

From the definitions one can derive the identities:

$$S_{IA}^a \tilde{l}_A^b = S_{IA}^a \tilde{l}_{IA}^b = 0 \; ; \quad \epsilon_{abc}S_{1A}^b S_{2A}^c = \tfrac{1}{4}\,\tilde{l}_A^a \underset{\sim}{\epsilon}_{\lambda\mu\nu}\Delta x^\lambda \Delta x_1^\mu \Delta x_2^\nu \; ; \quad \epsilon_{abc}\tilde{l}_A^b \tilde{l}_{IA}^c = e_A^2 S_{IA}^a \qquad (17)$$

(for $I = 1, 2$), that we use to prove that:

$$\epsilon_{abc}\tilde{l}_A^b L_{cd}\tilde{l}_B^d = k\epsilon_{abc}\epsilon_{bdc}\epsilon_{cfg}S_{1A}^d S_{2A}^e L_{ff'}S_{1B}^{f'}L_{gg'}S_{2B}^{g'} = $$
$$= kS_{1\lambda}\tilde{E}_A^{a\lambda}\,\epsilon_{bcd}\tilde{E}_A^{b\rho}\tilde{E}_A^{c\mu}L_{dc}\tilde{E}_B^{c\nu}S_{1\rho}S_{2\mu}S_{2\nu} \qquad (18)$$

with k real; by eq.(15) then:

$$\epsilon_{abc}\tilde{l}_A^b L_{cd}\tilde{l}_B^d = ih\,S_{1A}^a \qquad (19)$$

with h real. We may take this relation, which must hold for any of the sides of the triangle we cross going from one tethrahedron to another, as "reality condition" for L_{AB}, and we shall presently see that together with eq.(13), it gives us an adequate characterization of L_{AB}.

It follows from eqs.(13) and (19) that that the action of L_{AB} on $\tilde{l}_b$ must be of the form:

$$L_{ab}\tilde{l}_B^b = a\,\tilde{l}_A^a + ib\,\epsilon_{abc}\tilde{l}_A^b S_{1A}^c \qquad (20)$$

with a and b real constants. Furthermore, since $\tilde{l}_B^a$, S_{1B}^a, $\epsilon_{abc}\tilde{l}_B^b S_{1B}^c$ are orthogonal, we can align the orthonormal basis in B along their directions, do the same in A, then use eqs.(13), (19) to write the transformation in the form:

$$L_{AB} = \begin{pmatrix} \cosh\eta_1 & 0 & -i\sinh\eta_1 \\ 0 & 1 & 0 \\ i\sinh\eta_1 & 0 & \cosh\eta_1 \end{pmatrix} = e^{-i\eta_1 A_2} \; ; \quad (A_2)_{ab} = \epsilon_{a2b} \qquad (21)$$

Let us complete the picture, going all the way round the link σ^1. After the transformation (21) we change again the basis aligning it with $\tilde{l}_A^a$, S_{2A}^a, $\epsilon_{abc}\tilde{l}_A^b S_{2A}^c$ by a rotation

$$R_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_A & \sin\theta_A \\ 0 & -\sin\theta_A & \cos\theta_A \end{pmatrix} = e^{-\theta_A A_1} \; ; \quad (A_1)_{ab} = \epsilon_{a1b} \qquad (22)$$

In this way we go from B to A, to C and back to B round the link, rotating vector components in B by:

$$R = e^{-\theta_B A_1} e^{-i\eta_3 A_2} e^{-\theta_C A_1} e^{-i\eta_2 A_2} e^{-\theta_A A_1} e^{-i\eta_1 A_2} = e^{\alpha A_1 + i\beta A_2 + i\gamma A_3} := e^F \qquad (23)$$

More explicitely, we find that $F^c = \frac{1}{2}\epsilon_{acb}F_{ab}$, in the frame of B, must have the form

$$F_B^c = a\,\tilde{l}_B^c + ib\,S_{1B}^c + ic\,\epsilon_{cde}\tilde{l}_B^d S_{1B}^e \qquad (24)$$

with a, b, c real constants. This structure of F agrees with what we should have expected from eq.(7): viewed in 4-space, we have gone round a bone with sides Δx^μ and a time–like link that we may represent by $(Mn^\mu + M^\mu)$, hence from eqs.(7), (16) F^c must be of the form :

$$F^c = 2i\alpha C_{ij}^c e_\mu^i e_\nu^j (Mn^\mu + M^\mu)\Delta x^\nu = -\alpha M \tilde{l}_B^c + i\,S^{\prime c}, \quad \tilde{l}_B^a S^{\prime a} = 0 \qquad (25)$$

From this expression we also see that, since one link has to be time–like:

$$F^a F^a > 0 \qquad (26)$$

This concludes our "kinematic" analysis of Regge calculus in the selfdual picture.

4. The constraints.

What makes gravity in $3 + 1$ dimensions so much more interesting than in $2 + 1$ dimensions is that F^a does not vanish, but has to satisfy first class constraints [6]. To be able to do "dynamics" we have to express these constraints in the Regge case.

To get an idea of what might be their form, we shall consider the situation for a smooth manifold with weak field. In this case we succeed in expressing the constraints in a form which uses the notions introduced in the previous section; this might be a useful starting point.

To begin with, consider a triangle $(1, 2, 3)$, and set $x_{\alpha\beta}^\mu := \frac{1}{2}(x_\alpha + x_\beta)^\mu$, $\Delta_{\alpha\beta}^\mu := (x_\alpha - x_\beta)^\mu$; then we find:

$$x_{12}^\mu \Delta_{21}^\nu + x_{23}^\mu \Delta_{32}^\nu + x_{31}^\mu \Delta_{13}^\nu = x_1^{[\mu} x_2^{\nu]} + x_2^{[\mu} x_3^{\nu]} + x_3^{[\mu} x_1^{\nu]} = 2S^{\mu\nu} \qquad (27)$$

from which for example we may derive, for small A, the well known equation:

$$\exp(A(x_{12})_\nu \Delta_{21}^\nu) \exp(A(x_{23})_\nu \Delta_{32}^\nu) \exp(A(x_{31})_\nu \Delta_{13}^\nu) = 1 + F_{\mu\nu} S^{\mu\nu} + \dots \qquad (28)$$

with:

$$A_\mu^{ab} = \epsilon_{acb} A_\mu^c ; \quad F_{\mu\nu}^{ab} = \epsilon_{acb} F_{\mu\nu}^c = \epsilon_{acb}(\partial_\mu A_\nu^c - \partial_\nu A_\mu^c + \epsilon_{cde} A_\mu^d A_\nu^e) \tag{29}$$

and $F_{\mu\nu}$ calculated somewhere in the middle of the triangle.

In a similar way, for a tetrahedron $(1, 2, 3, 4)$, setting:

$$
\begin{aligned}
& x_{\alpha\beta\gamma}^\mu := \tfrac{1}{3}(x_\alpha^\mu + x_\beta^\mu + x_\gamma^\mu) ; \quad S_{\alpha\beta\gamma}^{\mu\nu} := \tfrac{1}{2}(x_\alpha^{[\mu} x_\beta^{\nu]} + x_\beta^{[\mu} x_\gamma^{\nu]} + x_\gamma^{[\mu} x_\alpha^{\nu]}) \\
& \underline{V} := \tfrac{1}{6}\underline{\epsilon}_{\mu\nu\lambda}(x_1^\mu x_2^\nu x_3^\lambda + x_2^\mu x_1^\nu x_4^\lambda + x_3^\mu x_4^\nu x_1^\lambda + x_4^\mu x_3^\nu x_2^\lambda) \\
& \tilde{l}_I^a := \tfrac{1}{2}\epsilon_{abc}\tilde{E}^{b\nu}\tilde{E}^{c\lambda}\underline{\epsilon}_{\mu\nu\lambda} x_I^\mu \quad \text{for} \quad I := (123),\ (214),\ (341),\ (432)
\end{aligned}
\tag{30}
$$

one finds:

$$\sum_I x_I^\mu S_I^{\nu\lambda} = \tfrac{1}{2}\tilde{\epsilon}^{\mu\nu\lambda}\underline{V} \tag{31}$$

To use this identity in a Regge-like situation, suppose we take the origine where four links meet, in such a way that the x_I^μ are links of a Regge lattice; then the tetrahedron itself belongs to the *dual* lattice, the triangles S_I giving paths that encircle the links. With this picture in mind we can derive:

$$
\begin{aligned}
& \sum_I \epsilon_{acb} F_{\nu\lambda}^c S_I^{\nu\lambda} \tilde{l}_I^b = \tfrac{1}{2}\epsilon_{acb} F_{\mu\nu}^c \epsilon_{bdc}\tilde{E}^{d\rho}\tilde{E}^{c\sigma}\underline{\epsilon}_{\rho\sigma\lambda}\tilde{\epsilon}^{\lambda\mu\nu}\underline{V} = 2\tilde{E}^{a\mu}\tilde{E}^{b\nu} F_{\mu\nu}^b \underline{V} \\
& \sum_I F_{\mu\nu}^a S_I^{\mu\nu} \tilde{l}_I^a = \tfrac{1}{2} F_{\mu\nu}^a \epsilon_{abc}\tilde{E}^{b\rho}\tilde{E}^{c\sigma}\underline{\epsilon}_{\rho\sigma\lambda}\tilde{\epsilon}^{\lambda\mu\nu}\underline{V} = \epsilon_{abc}\tilde{E}^{a\mu}\tilde{E}^{b\nu} F_{\mu\nu}^c \underline{V}
\end{aligned}
\tag{32}
$$

The r.h.s. of these identities has the form of the vector and scalar constraints in the Ashtekar formulation, and should therefore be set to zero. These identities then provide what may be an appealing interpretation of the constraints, very similar to the interpretation given by K. Kuchar[14] to the properties of the curvature in triad dynamics. Notice that the real-imaginary structure of F^c we found in the previous section makes the first line of eqs.(32) pure imaginary, the second real, as they should be [13].

However to apply these identities to Regge calculus one should assume that the F^c-s associated to the different links that meet at some vertex can be obtained from a single $F_{\mu\nu}^c$, contracted with the different surface elements, which seems unlikely. It would appear more likely, if anything, to conjecture that a single $F_{\mu\nu}^c$ is associated to each σ^3, and that the different F_l^c-s one finds going round its six links $l = (\alpha, \beta)$ can be obtained contracting this $F_{\mu\nu}^c$ with the surfaces of six polygons belonging to the dual lattice.

Tentatively, we can try and simulate the actual dual lattice taking an origine 0 at the barycenter of σ^3, and fixing points ^{*1}, ^{*2}, ^{*3}, ^{*4}, , with coordinates $^*x_1^\mu = k(x_2^\mu + x_3^\mu + x_4^\mu)$,

..., k large enough so that the triangles $(0,\ ^*1,\ ^*2),\ ...$ encircle the links $(3,4),\ ...;$ define then:

$$\Delta^{\mu}_{\alpha\beta} := x^{\mu}_{\alpha} - x^{\mu}_{\beta}\ ; \quad {}^*S^{\mu\nu}_{\alpha\beta} := \sum_{\gamma\delta} \tfrac{1}{4} \epsilon_{\alpha\beta\gamma\delta}\ {}^*x^{[\mu}_{\gamma}\ {}^*x^{\nu]}_{\delta}\ , \quad \epsilon_{\alpha\beta\gamma\delta} = \pm 1\ ,\ \epsilon_{1234} = +1$$

$$\tilde{l}^a_l := \tfrac{1}{2}\epsilon_{abc}\tilde{E}^{b\nu}\tilde{E}^{c\lambda}\underline{\epsilon}_{\mu\nu\lambda}\Delta^{\mu}_l \quad \text{for} \quad l := (1,2),\ (2,3),\ (3,4),\ (1,4),\ (4,2),\ (3,1) \tag{33}$$

What we are supposing is that:

$$F^c_l = F^c_{\mu\nu}\ {}^*S^{\mu\nu}_l \tag{34}$$

To write the constraints we need another elementary geometric identity, similar to eq.(31), that again follows from the definitions:

$$\sum_l \Delta^{\mu}_l\ {}^*S^{\nu\lambda}_l = \frac{3}{2}k^2\ \underline{V}\ \tilde{\epsilon}^{\mu\nu\lambda} \tag{35}$$

The Ashtekar constraint would then be, for each tetrahedron:

$$\sum_l \epsilon_{acb}F^c_l \tilde{l}^b_l = \frac{3}{4}k^2 \epsilon_{acb}F^c_{\mu\nu}\epsilon_{bdc}\tilde{E}^{d\rho}\tilde{E}^{c\sigma}\underline{\epsilon}_{\rho\sigma\lambda}\tilde{\epsilon}^{\lambda\mu\nu}\ \underline{V} = 3k^2\ \tilde{E}^{a\mu}\tilde{E}^{b\nu}F^b_{\mu\nu}\ \underline{V} = 0$$

$$\sum_l F^a_l \tilde{l}^a_l = \frac{3}{4}k^2 F^a_{\mu\nu}\epsilon_{abc}\tilde{E}^{b\rho}\tilde{E}^{c\sigma}\underline{\epsilon}_{\rho\sigma\lambda}\tilde{\epsilon}^{\lambda\mu\nu}\ \underline{V} = \frac{3}{2}k^2\epsilon_{abc}\tilde{E}^{a\mu}\tilde{E}^{b\nu}F^c_{\mu\nu}\ \underline{V} = 0 \tag{36}$$

To test whether this form of the constraints is correct would presumably require a careful derivation of the $3+1$ action from the Regge action, along the lines of [17]; and to see whether it is a useful form one would have to develop a proper canonical formalism, and test for the closure of the constraints under Poisson brackets, a feat so far achieved only in $2+1$ dimensions [10].

Notice that the idea that there should be an $F^a_{\mu\nu}$ for each tetrahedron is consistent with the solution of the constraints proposed in [15], and generalized in [16]to

$$F^a_{\mu\nu} = \Phi_{ab}\ \underline{\epsilon}_{\mu\nu\lambda}\tilde{E}^{b\lambda} \tag{37}$$

with Φ_{ab} a traceless symmetric tensor, since in Regge calculus there is certainly an $\tilde{E}^{a\mu}$ for each tetrahedron; it is unfortunate that I cannot suggest a geometric interpretation for Φ_{ab}.

References

[1] T. Regge, Il Nuovo Cim. **19** (1951) 558.

[2] C.W. Misner, K.S. Thorne and J.A. Wheeler, "Gravitation", Freeman, San Francisco, 1971.

[3] C. Rovelli and L. Smolin, Phys. Rev. Lett. **61** (1988) 1155 ;
     C. Rovelli, Class. Quantum Grav. **8** (1991) 1613.

[4] M. P. Blencowe, Nucl. Phys. **B341** (1990) 213 ;
     R. Gambini, Phys. Lett. **B255** (1991) 180 ;
     B. Brügmann and J. Pullin, Nucl. Phys. **B390** (1993) 399.

[5] A. Ashtekar, Phys. Rev. Lett. **57** (1986) 2244; Phys. Rev. **D36** (1986) 1587.

[6] A. Ashtekar, "Non–perturbative canonical quantum gravity", notes prepared in collaboration with R. Tate, World scientific, Singapore, 1991.

[7] V. Khatsymovsky, Class. Quantum Grav. **6** (1989) L249.

[8] P. Renteln and L. Smolin, Class. Quantum Grav. **6** (1989) 275 ;
     M. Miller and L. Smolin, Syracuse preprint, gr-qc 9304005.

[9] G. 't Hooft, Class. Quantum Grav. **10** (1993) 1653.

[10] H. Waelbroeck, Class. Quantum Grav. **7** (1990) 751.

[11] M. Bander, Phys. Rev. Lett. **57** (1986) 1825, Phys. Rev. **D36** (1987) 2297, **D38** (1988) ;
     V. Khatsymovsky, Class. Quantum Grav. **8** (1991) 1205 ;
     H. C. Ren, Nucl. Phys. **B301** (1988) 661 ;
     M. Caselle, A. D'Adda and L. Magnea, Phys. Rev. Lett. **232** (1989) 457.

[12] A. Ashtekar, J.D. Romano and R.S. Tate, Phys. Rev. **D40** (1989) 2572.

[13] G. Immirzi, Class. Quantum Grav. **10** (1993) 2347.

[14] K. Kuchar, in: General Relativity and Gravitation 1992, R.J. Gleiser, C.N. Kozameh and O.M. Moreschi eds., IOP pub., Bristol, 1993.

[15] J. Samuel, Class. Quantum Grav. **5** (1988) L123.

[16] R. Capovilla, J. Dell and T. Jacobson, Phys. Rev. Lett. **63** (1989) 2325.

[17] L. Brewin, Class. Quantum Grav. **5** (1988) 1193, **10** (1993) 947.

[18] R.M. Williams , P.A. Tuckey, Class. Quantum Grav. **9** (1992) 1406.

[19] C. Rovelli, Pittsburgh and Trento preprint, 1993.

[20] C. Rovelli, Phys. Rev. **D47** (1993) 1703.