ISTITUTO NAZIONALE DI FISICA NUCLEARE

Sezione di Milano

M. Pavšič, E. Recami, W.A. nRodrigues Jr.:

ELECTRON STRUCTURE, ZITTERBEWEGUNG AND A NEW NON–LINEAR DIRAC–LIKE EQUATION

PACS: 03.70.+ k ;   11.10.Qr   ;   14.60.Cd

1

ELECTRON STRUCTURE, ZITTERBEWEGUNG AND A NEW NON-LINEAR DIRAC–LIKE EQUATION[†]

Matej PAVŠIČ[*], Erasmo RECAMI[**], Waldyr A. RODRIGUES Jr.[***]

I.N.F.N., Sezione di Milano, 16 via Celoria, 20133 Milano, Italy.

ABSTRACT: The recent literature shows a renewed interest, with various independent approaches, in the classical models for spin. Considering the possible interest of those results, at least for the electron case, we purpose in this paper to explore their physical and mathematical meaning, by the natural and powerful language of Clifford algebras (which, incidentally, will allow us to unify those different approaches). In such models, the ordinary electron is in general associated to the mean motion of a point–like "constituent" Q, whose trajectory is a cylindrical helix. We find, in particular, that the object Q obeys a new, non-linear Dirac–like equation, such that —when averaging over an internal cycle (which corresponds to linearization)— it transforms into the ordinary Dirac equation (valid for the electron as a whole).

3

1. INTRODUCTION

The possibility of constructing formal classical models of spin was already realized at least 60 years ago, from different points of view.[1] In particular, Schrödinger's suggestion[2] that the electron spin was related to its Zitterbewegung (zbw) motion has been investigated by several authors[3].

In ref.[4], for instance, one meets even the proposal of models with clockwise and anti-clockwise "inner motions" as classical analogues of quantum relativistic spinning particles and antiparticles, respectively. The use of grassmannian variables in a classical lagrangian formulation for spinning particles was proposed, on the contrary, by Berezin and Marinov[5] and by Casalbuoni[6]. Pauri,[7] moreover, showed how a relativistic (or non-relativistic) spin can enter a classical formulation, in the ordinary phase space [without resource now to grassmannian quantities]', just as a consequence of the algebraic structure of the Poincaré (or Galilei) group. In such an interesting approach, it was found[7] that a relativistic classical zbw motion can directly follow as a "spin effect" from the requirements of existence of a lagrangian and of a covariant position in Dirac's instant form of relativistic dynamics. The quantum analogue of those developments had been studied in ref.[8].

The number of papers appeared on the subject of classical models for spin, starting from the fifties, is so large that it would be difficult to try quoting them here. A recent approach, based on a generalization of Dirac non-linear electrodynamics, where the spin of the electron is identified with the momentum of the Poynting vector in a soliton solution of that theory, can be found in ref.[9].

In this paper we choose, by making recourse to the natural and powerful language of Clifford algebras, to refer ourselves mainly to the model by Barut and Zanghi[10,11] (BZ), which relates the spin (at least in the case of the electron) to a helical motion. Namely, we first recast the BZ model into the Clifford formalism, in the meantime clarifying its physical and mathematical meanings; then, we quantize that model for the electron case. In particular, we derive from the BZ model a *new* non-linear equation for the "spinorial variable" of the model, which when linearized reduces to the ordinary Dirac equation. Solutions of this non-linear equation will be discussed in another paper[12].

2. SPIN AND ELICAL MOTION

In the BZ model,[10] a classical electron is characterized, besides by the usual pair of conjugate variables (x^μ, p_μ), by a second pair of conjugate classical *spinor* variables $(z, i\bar{z})$, representing internal degrees of freedom, which are functions of an invariant time parameter τ, that when convenient will be identified with the proper time of the center of mass (CM). Quantity z was a Dirac spinor, while $\bar{z} \equiv z^\dagger \gamma^0$. Barut and Zanghi, then, introduced for a spinning particle the classical lagrangian [$c = 1$]

4

$$\mathcal{L} = \tfrac{1}{2}\lambda i(\dot{\bar{z}}z - \bar{z}\dot{z}) + p_\mu(\dot{x}^\mu - \bar{z}\gamma^\mu z) + eA_\mu(x)\bar{z}\gamma^\mu z \ , \tag{1}$$

where λ has the dimension of an action, γ^μ are the Dirac matrices, and the particle velocity is

$$v_\mu \equiv \bar{z}\gamma_\mu z \ . \tag{1'}$$

We are not writing down explicitly the spinorial indices of $\bar{z}$ and z. Let us consider the simple case of a *free* electron ($A_\mu = 0$). Then a possible solution of the equations of motion corresponding to the lagrangian (1) is:[#1]

$$z(\tau) = \left[\cos m\tau - i\gamma^\mu \frac{p_\mu}{m} \sin m\tau\right] z(0) \ , \tag{2a}$$

$$\bar{z}(\tau) = \bar{z}(0)\left[\cos m\tau + i\gamma^\mu \frac{p_\mu}{m} \sin m\tau\right] \ , \tag{2b}$$

and $p_\mu = \text{constant}$; $p^2 = m^2$; $H = p_\mu\bar{z}\gamma^\mu z \equiv p_\mu v^\mu$; and finally:

$$\dot{x}_\mu = v_\mu = \frac{p_\mu}{m^2}H + [\dot{x}_\mu(0) - \frac{p_\mu}{m^2}H]\cos 2m\tau + \frac{\ddot{x}_\mu(0)}{2m}\sin 2m\tau \tag{2c}$$

(which in ref.[10] appeared with two misprints). In connection with this "free" general solution, let us remark that H is a constant of motion so that we can set $H = m$. Solution (2) exhibits the classical analog of the phenomenon known as "zitterbewegung" (zbw): in fact, the velocity $v_\mu \equiv \dot{x}_\mu$ contains the (expected) term p_μ/m plus a term describing an oscillatory motion with the characteristic frequency $\omega = 2m$. The velocity of the *center of mass* will be given by $W_\mu = p_\mu/m$. Notice incidentally that, instead of adopting the variables z and $\bar{z}$, one can work in terms of the spin variables, *i.e.* in terms of the dynamical variables $(x_\mu, v_\mu, \pi_\mu, S_{\mu\nu})$, where:

$$v_\mu = \dot{x}_\mu; \quad \pi_\mu = p_\mu - eA_\mu; \quad S_{\mu\nu} = \frac{1}{4}i\bar{z}[\gamma_\mu, \gamma_\nu]z \ ,$$

so that $\dot{S}_{\mu\nu} = \pi_\mu v_\nu - \pi_\nu v_\mu$ and $\dot{v}_\mu = 4S_{\mu\nu}\pi^\nu$. In the case of a free electron, by varying the action corresponding to $\mathcal{L}$ one finds as generator of space-time rotations the conserved quantity $J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$, where $S_{\mu\nu}$ is just the particle spin.[#2]

[#1] For other solutions, see ref.[12].

[#2] Alternatively, Barut and Zanghi,[10] in order to study the internal dynamics of the considered (classical) particle, did split x_μ and $v_\mu \equiv \dot{x}_\mu$ as follows: $x_\mu \equiv X_\mu + Q_\mu$; $v_\mu \equiv W_\mu + U_\mu$ (where by definition $W_\mu = \dot{X}_\mu$ and $U_\mu = \dot{Q}_\mu$). In the particular case of a *free* particle, $\dot{W}_\mu = 0$; $W_\mu = p_\mu/m$. One can now interpret X_μ and p_μ as the CM coordinates, and Q_μ and $P_\mu \equiv mU_\mu$ as the *relative* position and momentum, respectively. For a free particle, then, one finds that the internal variables are coordinates oscillating with the zbw frequency $2m$; and that, again, the total angular momentum $J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$ is

Let us explicitly observe that solution (2c) is the equation of a space-time cylindrical helix, *i.e.* it represents in 3–space a helical motion. Let us also stress that this motion describes particle spin at a classical level. In fact, such a classical system has been shown to describe, after quantization, the Dirac electron. Namely, Barut and Pavšič[13] started from the classical hamiltonian corresponding to eq.(1):

$$\mathcal{H} = \bar{z}\gamma^{\mu}z(p_{\mu} - eA_{\mu}) ; \qquad (3)$$

passed to its quantum version, in which the above quantities are regarded as operators; and considered in the Schroedinger picture the equation

$$i\frac{\partial\varphi}{\partial\tau} = \mathcal{H}\varphi \qquad (4)$$

where $\varphi = \varphi(\tau, x, z)$ is the wave function, and the operators in eq.(3) are $p_{\mu} \longrightarrow -i\partial/\partial x^{\mu}$ and $i\bar{z} \longrightarrow -i\partial/\partial z$. The wave function φ can be expanded in the z variable, around $z = 0$, as follows:

$$\varphi(\tau, x, z) = \bar{\Phi}(\tau, x) + \bar{\psi}_{\alpha}(\tau, x)z^{\alpha} + \bar{\psi}_{\alpha\beta}(\tau, x)z^{\alpha}z^{\beta} + \dots$$

where α, β are spinorial indices. Quantity $\bar{\psi} \equiv \psi^{\dagger}\gamma^{0}$ is the Dirac adjoint spinor, which a charge $\bar{e}$ and a mass $\bar{m}$ are attributed to. By disregarding the spin–zero term $\bar{\Phi}(\tau, x)$, and retaining only the second (spin $\frac{1}{2}$) term, thus neglecting also the higher–spin terms, from eq.(4) they then derived the equation

$$i\frac{\partial\bar{\psi}}{\partial\tau} = (i\partial_{\mu} + \bar{e}A_{\mu})\bar{\psi}\gamma_{\mu} . \qquad (5)$$

Taking the Dirac adjoint of such an equation, with[14] $\bar{e} = -e$; $\bar{m} = -m$, we end up just with the Dirac equation! This confirms, by the way, that the term $\bar{\psi}_{\alpha}z^{\alpha}$ refers to the spin $\frac{1}{2}$ case, i.e. to the case of the electron.[#3]

An alternative approach leading to a classical description of particles with spin is the one by Pavšič,[16,17] who made recourse to the (extrinsic) curvature of the particle world-line in Minkowski space; so that his starting, classical lagrangian [α, β = constants]

$$\mathcal{L} = \sqrt{\dot{x}^2}\,(\alpha - \beta K^2); \quad K^{\mu} \equiv \frac{1}{\sqrt{\dot{x}^2}} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{x}^{\mu}}{\sqrt{\dot{x}^2}}\right) \qquad (6)$$

contained the extra (kinematical) term $\beta\sqrt{\dot{x}^2}\,K^2$. The conserved generator of rotations, belonging to lagrangian (6), is once more $J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$, where now, however,

---

a constant of motion, quantities $S_{\mu\nu}$ being the spin variables.

[#3] For the case of higher spins, cf. ref.[15].

φ

$S_{\mu\nu} = \dot{x}_\mu p_\nu^{(2)} - \dot{x}_\nu p_\mu^{(2)}$, while $p_\nu^{(2)} \equiv \partial\mathcal{L}/\partial\ddot{x}^\mu = 2\beta\ddot{x}_\mu/\sqrt{\dot{x}^2}$ is the second–order canonical momentum, conjugated to $\dot{x}^\mu$. The equations of motion, which correspond to eq.(6) for constant K^2, do again admit as solution[16,17] the *helical* motion with the zbw frequency $\omega = 2m$. For a suitable choice of the constant K^2, and when the affine parameter τ is the CM proper–time, the equations of motion result to be $[v_\mu \equiv \dot{x}_\mu]$

$$\dot{v}_\mu = 4S_{\mu\nu}p^\nu; \quad \dot{S}_{\mu\nu} = \pi_\mu v_\nu - \pi_\nu v_\mu \,,$$

i.e., they are the same (for the free case: $A_\mu = 0$) as in the Barut–Zanghi (BZ) model. Moreover, the constraint due to reparametrization invariance can be written as $p_\mu v^\mu - m = 0$, which reminds us of the Dirac equation; and Poisson brackets are obtained which obey the same algebra as the Dirac γ matrices.[#4] Actually, the velocity v_μ can be also expressed, in terms of the canonically conjugate variables $\dot{x}^\mu$ and $p_\mu^{(2)}$, as follows:

$$v^\mu = k\frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} \,; \qquad k \equiv \frac{4\beta m}{4\alpha\beta + p_\mu^{(2)}p^{(2)\mu}} \,.$$

Then, in ref.[16] there were obtained the Poisson brackets:

$$\{v^\mu, v^\nu\} = 4S^{\mu\nu} \,;$$

$$\{S^{\mu\nu}, v^\rho\} = g^{\mu\rho}v^\nu - g^{\nu\rho}v^\mu \,,$$

where k was chosen in such a way that $k^3/(2\beta m) = 4$.

The above relations are analogous to the corresponding quantum equations, in which v^μ is replaced by the Dirac operators γ^μ, and the Poisson brackets by commutators. The classical frequency of the orbital motion is equal to the zbw frequency of the Dirac theory.

Let us finally mention that the *same* classical equations of motion (and the same Poisson-bracket algebra) have been found also in a third approach, which consists in

---

[#4] This conclusion, referring to a half–integer spin, has been criticized in refs.[18] by noticing a seeming analogy with the case of the ordinary orbital angular momentum (which admits integer spin values only). We believe such arguments to not apply to the present situation, where spin is the orbital momentum in the $q^\mu \equiv \dot{x}^\mu$ variables space; since the corresponding, canonically conjugate momentum $p_\mu^{(2)}$ is never a constant of motion. [Namely, when we consider a congruence of world-lines which are solutions of the equations of motion, one finds a non–zero curl of the field $p_\mu^{(2)}(q)$, so that the phase $\int p_\mu^{(2)}dq^\mu$ is not a single–valued function. Thus, a basis of single–valued wave functions does not exist, nor the operator representation $p_\mu^{(2)} \longrightarrow +i\partial/\partial q^\mu$; and the ordinary arguments about orbital angular momentum do no longer apply].

$\mathcal{G}$

adding to the ordinary lagrangian an extra term containing Grassmann variables.[19]

3. ABOUT THE ELECTRON STRUCTURE

Considering the interest of the previous results (which suggest in particular that the helical motion can have a role in the origin of spin), we purpose to explore their physical meaning more deeply, by the very natural —and powerful— language of the Clifford algebras:[20,21] in particular of the "space-time algebra (STA)" $\mathbb{R}_{1,3}$. First of all, let us preliminarily clarify *why* Barut and Zanghi had to introduce the Dirac spinors z in their lagrangian, by recalling that classically the motion of a spinning top has to be individuated by (i) the world–line σ of its center of mass (*e.g.*, by the ordinary coordinates x^μ and the conjugate momenta p_μ), and (ii) a Frenet tetrad[#5] attached[22] to the world–line σ. This continues to be true when wishing to describe the motion of a point-like spinning particle. For the Frenet tetrad[23] we have:

$$e_\mu = R\gamma_\mu\tilde{R} = \Lambda_\mu^\nu\gamma_\nu \; ; \quad \Lambda_\mu^\nu \in L_+^\uparrow \tag{7}$$

where e_0 is parallel to the particle velocity v (even more, $e_0 = v$ whenever one does use as parameter τ the CM system proper–time); the tilde represents the *reversion*[#6]; and $R = R(\tau)$ is a "Lorentz rotation" [more precisely, $R \in \mathrm{Spin}_+(1,3)$, and a Lorentz transform of quantity a is given by $a' = Ra\tilde{R}$]. Moreover $R\tilde{R} = \tilde{R}R = 1$. The Clifford STA fundamental unit–vectors γ_μ should not be confused with the Dirac *matrices* γ_μ. Let us also recall that, while the orthonormal vectors $\gamma_\mu \equiv \partial/\partial x^\mu$ constitute a *global* tetrad in Minkowski space-time (associated with a given inertial observer), on the contrary the Frenet tetrad e_μ is defined only along σ, in such a way that e_0 is tangent to σ. At last, it is: $\gamma^\mu = \eta^{\mu\nu}\gamma_\nu$, and $\gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3$.

Notice that $R(\tau)$ does contain all the essential information carried by a Dirac spinor. In fact, out of R, a "Dirac–Hestenes" (DH) spinor[24] ψ_{DH} can be constructed as follows:

$$\psi_{\mathrm{DH}} = \rho^{\frac{1}{2}}e^{\beta\gamma_5/2}R \tag{8}$$

where ρ is a normalization factor; and $e^{\beta\gamma_5} = +1$ for the electron (and -1 for the positron); while, if ε is a primitive idempotent of the STA, any Dirac spinor ψ_{D} can be represented in our STA as:[25]

$$\psi_{\mathrm{D}} = \psi_{\mathrm{DH}}\varepsilon . \tag{9}$$

[#5] The use of Frenet tetrads in connection with the Dirac formalism was first investigated in ref.[22].

[#6] The main anti-automorphism in $\mathbb{R}_{1,3}$ (called reversion), denoted by the tilde, is such that $\widetilde{AB} = \tilde{B}\tilde{A}$, and $\tilde{A} = A$ when A is a scalar or a vector, while $\tilde{F} = -F$ when F is a 2-vector.

8

For instance, the Dirac spinor z introduced by BZ is obtained from the DH spinor[#7] by the choice $\varepsilon = \frac{1}{2}(1+\gamma_0)$. Incidentally, the Frenet frame can also write $\rho e_\mu = \psi_{\text{DH}}\gamma_\mu\tilde{\psi}_{\text{DH}}$.

Let us stress that, to specify how does the Frenet tetrad rotate as τ varies, one has to single out a particular $R(\tau)$, and therefore a DH spinor ψ_{DH}, and eventually a Dirac spinor ψ_{D}. This makes intuitively clear why the BZ Dirac–spinor z provides a good description of the "spin motion" of a classical particle.

Let us now repeat what precedes on a more formal ground. In the following, unless differently stated, we shall indicate the DH spinors ψ_{DH} simply by ψ.

Let us translate the BZ lagrangian into the Clifford language.

In eq.(1) quantity $z^{\text{T}} = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \end{pmatrix}$ is a Dirac spinor $\tau \longrightarrow z(\tau)$, and $\bar{z} = z^\dagger\gamma^0$. To perform our translation, we need a matrix representation of the Clifford space-time algebra; this can be implemented by *representing* the fundamental Clifford vectors (γ_0, γ_1, γ_2, γ_3) by the *ordinary* Dirac matrices γ_μ. Choosing:

$$\gamma_0 \longrightarrow \gamma_0 \equiv \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} ; \quad \gamma_i \longrightarrow \gamma_i \equiv \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} ,$$

the representative in $\mathbb{R}_{1,3}$ of Barut–Zanghi's quantity z is

$$z \longrightarrow \Psi \equiv \psi\varepsilon ; \quad \varepsilon \equiv \tfrac{1}{2}(1+\gamma_0) \tag{10}$$

where ψ, and $\tilde{\psi}$, are represented (with $\psi\tilde{\psi} = \mathbb{1}$) by

$$\psi = \begin{pmatrix} z_1 & -\bar{z}_2 & z_3 & \bar{z}_4 \\ z_2 & \bar{z}_1 & z_4 & -\bar{z}_3 \\ z_3 & \bar{z}_4 & z_1 & -\bar{z}_2 \\ z_4 & -\bar{z}_3 & z_2 & \bar{z}_1 \end{pmatrix} ; \quad \tilde{\psi} = \begin{pmatrix} \bar{z}_1 & \bar{z}_2 & -\bar{z}_3 & -\bar{z}_4 \\ -z_2 & z_1 & -z_4 & z_3 \\ -\bar{z}_3 & -\bar{z}_4 & \bar{z}_1 & -\bar{z}_2 \\ -z_4 & z_3 & -z_2 & z_1 \end{pmatrix} . \tag{11}$$

The translation of the various terms in eq.(1) is then:

$$\begin{aligned} \tfrac{1}{2}i(\dot{\bar{z}}z - z\dot{z}) &\longrightarrow \langle\tilde{\dot{\psi}}\psi\gamma_1\gamma_2\rangle_0 \\ p_\mu(\dot{x}^\mu - \bar{z}\gamma^\mu z) &\longrightarrow \langle p(\dot{x} - \psi\gamma_0\tilde{\psi})\rangle_0 \\ eA_\mu\bar{z}\gamma^\mu z &\longrightarrow e\langle A\psi\gamma_0\tilde{\psi}\rangle_0 , \end{aligned}$$

where $\langle\ \rangle_0$ means "the scalar part" of the Clifford product. Thus, the lagrangian $\mathcal{L}$ in the Clifford formalism is

$$\mathcal{L} = \langle\tilde{\dot{\psi}}\psi\gamma_1\gamma_2 + p(\dot{x} - \psi\gamma_0\tilde{\psi}) + eA\psi\gamma_0\tilde{\psi}\rangle_0 , \tag{12}$$

[#7] The DH spinors can be regarded as the *parent* spinors, since all the other spinors of common use among physicists are got from them by operating as in eq.(9). We might call them "the *fundamental spinors*".

which is analogous, incidentally, to Krüger's lagrangian[26] (apart from a misprint).

As we are going to see, by "quantizing" it, also in the present formalism it is possible (and, actually, quite easy) to derive from $\mathcal{L}$ the Dirac–Hestenes equation:[#8]

$$\partial \psi(x)\,\gamma_1\gamma_2 \;+\; m\,\psi(x)\,\gamma_0 \;+\; e\,A(x)\,\psi(x) \;=\; 0\,, \tag{13}$$

which is nothing but the ordinary Dirac equation written down in the Clifford formalism.[20,24] Quantity $\partial \;=\; \gamma^\mu \partial_\mu$ is the Dirac operator. Let us notice that p in eq.(12) can be regarded as a Lagrange multiplier, when the velocity $v = \dot{x}$ is represented by $\psi\gamma_0\tilde{\psi}$. The BZ model is indeed a hamiltonian system, as proved by using Clifford algebras in ref.[27] (cf. also ref.[28]). The dynamical variables are then $(\psi, \tilde{\psi}, x, p)$, and the Euler–Lagrange equations yield a system of three independent equations:

$$\dot{\psi}\gamma_1\gamma_2 + \pi\psi\gamma_0 \;=\; 0 \tag{14a}$$
$$\dot{x} \;=\; \psi\gamma_0\tilde{\psi} \tag{14b}$$
$$\dot{\pi} \;=\; eF \cdot \dot{x} \tag{14c}$$

where $F \equiv \partial \wedge A$ is the electromagnetic field (a bivector, in Hestenes' language) and $\pi \equiv p - eA$ is the kinetic momentum. [Notice incidentally, from eq.(14b), that $\dot{x}^2 = \rho(\tau)$].

At this point, let us consider a velocity vector *field* $V(x)$ together with its integral lines (or stream–lines). Be σ the stream–line along which a particle moves (*i.e.*, the particle world–line). Then, the velocity distribution V is required to be such that its restriction $V(x)_{|\sigma}$ to the world–line σ is the ordinary velocity $v = v(\tau)$ of the considered particle.

If we moreover recall[29,30] that any Lorentz "rotation" R can be written $R = e^{\mathcal{F}}$, where $\mathcal{F}$ is a *bivector*, then along any stream–line σ we shall have:[19]

$$\dot{R} \equiv \frac{dR}{d\tau} \;=\; \frac{1}{2}v^\mu\Omega_\mu R \;=\; \frac{1}{2}\Omega R\,, \tag{15}$$

with $\partial_\mu R = \Omega_\mu R/2$, where $\Omega_\mu \equiv 2\partial_\mu\mathcal{F}$, and where $\Omega \equiv v^\mu\Omega_\mu$ is the angular-velocity bivector (also known, in differential geometry, as the "Darboux bivector"). Therefore, for the tangent vector *along any line* σ we obtain the relevant relation:

$$\frac{d}{d\tau} \;=\; v^\mu\partial_\mu \;=\; v \cdot \partial$$

The [total derivative] equation (14a) thus becomes:[#8]

$$v \cdot \partial\psi\gamma_1\gamma_2 + \pi\psi\gamma_0 \;=\; 0\,, \tag{16}$$

[#8] Observe that in eq.(12) it is $\psi = \psi(\tau)$, while in eq.(13) we have $\psi = \psi(x)$ with $\psi(x)$ such that its *restriction* $\psi(x)_{|\sigma}$ to the world–line σ coincides with $\psi(\tau)$. Below, we shall meet the same situation, for instance, when passing from eq.(14a) to eqs.(16)–(16').

which is a non-linear [partial derivative] equation, as it is easily seen by using eq.(14b) and rewriting it in the noticeable form

$$(\psi\gamma_0\tilde{\psi}) \cdot \partial\psi\gamma_1\gamma_2 + \pi\psi\gamma_0 = 0 \,. \tag{16'}$$

Equation (16') constitutes a new *non-linear* Dirac–like equation. The solutions of this new equation will be explicitly discussed elsewhere.

Let us here observe only that the probability current $J \equiv V$ is conserved: $\partial \cdot V = 0$, as we shall show elsewhere.

Let us pass now to the *free* case ($A_\mu = 0$), when eq.(14a) may be written

$$\dot{\psi}\gamma_1\gamma_2 + p\psi\gamma_0 = 0 \,, \tag{14'a}$$

and admits some simple solutions.[12] Actually, in this case p is constant [cf. eq.(14c)] and one can choose the γ_μ frame so that $p = m\gamma_0$ is a constant vector in the direction γ_0. Since $\dot{x} = \psi\gamma_0\tilde{\psi}$, it follows that

$$v = \frac{1}{m}\psi p\tilde{\psi} \,; \quad \frac{p}{m} = \psi^{-1}v\tilde{\psi}^{-1} \,. \tag{17}$$

The mean value of v over a zitterbewegung period is then given by the relation

$$\langle v \rangle_{\text{zbw}} = p/m = \psi^{-1}v\tilde{\psi}^{-1} \tag{18}$$

which resembles the ordinary quantum–mechanical mean value for the wave–function ψ^{-1} (recall that in Clifford algebra for any ψ it exists its inverse). Let us recall, by comparison, that the time average of v_μ given in eq.(2c) over a zbw period is evidently equal to p_μ/m.

Let us explicitly stress that, due to the first one [$z \longrightarrow \psi\varepsilon$] of eqs.(10), the results found by BZ for z are valid as well for ψ in our formalism. For instance, for BZ [cf. eq.(1')] it was $v^\mu = \bar{z}\gamma^\mu z \equiv v_{\text{BZ}}^\mu$, while in the Clifford formalism [cf. eq.(11)] it is $v = \psi\gamma^0\psi = \langle\gamma^0\tilde{\psi}\gamma^\mu\psi\rangle_0\gamma_\mu = v_{\text{BZ}}^\mu\gamma_\mu$. As a consequence, σ refers in general to a cylindrical helix (for the free case) also in our formalism.

Going back to eq.(14'a), by the second one of eqs.(17) we finally obtain our non-linear (free) Dirac–like equation in the following form:

$$v \cdot \partial\psi\gamma_1\gamma_2 + m\psi^{-1}v\tilde{\psi}^{-1}\psi\gamma_0 = 0 \,, \qquad . \tag{19}$$

which in the ordinary, tensorial language would write: $i(\bar{\Psi}\gamma^\mu\Psi)\partial_\mu\Psi = \gamma^\mu p_\mu\Psi$, where Ψ and γ^μ are now an ordinary Dirac spinor and the ordinary Dirac matrices, respectively, and $\hat{p}_\mu \equiv i\partial_\mu$.

In connection with this fundamental equation of motion (19), let us explicitly notice the following. At a classical level, the equation of motion in the BZ model was eq.(14a), which held for the world–line σ. In other words, eq.(14a) was valid for *one* world–line; on the contrary, eq.(19) is a *field* equation, satisfied by quantities $\psi(x)$ such that $\psi(x)_{|\sigma} = \psi(\tau)$. A change in interpretation is of course necessary when passing from the classical to the "quantum" level: and therefore eq.(19) is now to be regarded as valid for *a congruence* of world lines, that is to say, for a congruence of stream–lines of the velocity field $V = V(x)$. In the quantum case, the "particle" can follow any of those integral lines, with probability amplitude ρ. In this context, it must be recalled that a tentative interpretation of the Dirac equation within the Clifford algebra approach has been suggested in ref.[20], and later in ref.[31]. However, in the present paper we shall not put forth, nor discuss, any interpretation of our formalism.

As we have seen, eq.(19) will hold for our helical motions. Notice moreover that, since [cf. eq.(8)] it is $\psi = \rho^{\frac{1}{2}}e^{\beta\gamma_5/2}R$, in the case of the helical path solution the Lorentz "rotation" R will be the product of a pure space rotation and a boost.

It is important to observe that, if we replace v in eq.(19) by its mean value over a zbw period, eq.(18), then we end up with the Dirac–Hestenes equation [*i.e.*, the ordinary Dirac equation!], valid now for the center-of-mass world–line. In fact, since $\langle v \rangle_{\text{zbw}} = p/m$, eq.(19) yields

$$p \cdot \partial\psi\gamma_1\gamma_2 + mp\psi\gamma_0 = 0 \,, \tag{19'}$$

and therefore —if we recall that for the eigenfunctions of p in the DH approach[20] it holds $\partial\psi\gamma_1\gamma_2 = p\psi$, so that $(p \cdot \partial)\psi\gamma_1\gamma_2 = p\partial\psi\gamma_1\gamma_2$— one obtains:

$$p(\partial\psi\gamma_1\gamma_2 + m\psi\gamma_0) = 0 \,, \tag{19''}$$

which is satisfied once it holds the *ordinary* Dirac equation (in its Dirac–Hestenes form):

$$\partial\psi\gamma_1\gamma_2 + m\psi\gamma_0 = 0 \,. \tag{20}$$

Let us observe that all the eigenfunctions of p are solutions both of eq.(19) and of the Dirac equation.

In conclusion, our non-linear Dirac–like equation (19) is a quantum–relativistic equation, that can be regarded as "sub-microscopic" in the sense that it refers to the internal motion of a point-like "constituent" $\mathcal{Q}$. In fact, the density current $\psi\gamma^0\tilde{\psi}$, relative to the solutions ψ of eq.(19), does oscillates in time in a helical fashion, in complete analogy to the initial equation (2c); so that x and $\dot{x}$ refer to $\mathcal{Q}$. On the contrary, the ordinary (linear) Dirac equation can be regarded as the equation describing the global motion

of the geometrical centre of the system (*i.e.*, of the whole "electron"); actually, it has been obtained from eq.(19) by linearization, that is to say, replacing the density current $v \equiv \psi \gamma^0 \tilde{\psi}$ by its time average p/m over the zbw period [cf. eqs.(17)–(18)].

At last, let us underline that, in the free case, eq.(14'a) admits also a trivial solution σ_0, corresponding to rectilinear motion.

4. A VERY SIMPLE SOLUTION

In the free case, the equation

$$\dot{\psi}\gamma_1\gamma_2 + p\psi\gamma_0 = 0 \tag{14'a}$$

admits also a very simple solution (the limit of the ordinary helical paths when their radius r tends to zero), which —incidentally— escaped BZ's attention. In fact, let us recall that in this case eq.(14c) implies p to be constant, and we were thus able to choose $p = m\gamma_0$. As a consequence, quantities R entering any solution ψ of eq.(14'a) become pure *space* rotations. Actually, if solution ψ is essentially a pure space rotation, it holds $\psi\gamma_0\tilde{\psi} = \gamma_0\rho$, and eq.(14'a) becomes $\psi^{-1}\dot{\psi} = m\gamma_1\gamma_2$, so that one verifies that

$$\psi = \rho^{\frac{1}{2}} \exp[-\gamma_2\gamma_1 m\tau] = \psi(0) \exp[-\gamma_2\gamma_1 m\tau] \tag{21}$$

is a (very simple) solution of its. Moreover, from eq.(14b) it follows

$$v \equiv \dot{x} = \rho\gamma_0$$

and we can set $\rho = 1$, which confirms that our trivial solution (21) corresponds to rectilinear uniform motion, and that τ in this case is just the proper–time along the particle world–line σ_0. (In this case, of course, the zbw disappears).

But, recalling eq.(16), equation (14'a) in the free case may read

$$v \cdot \partial\psi\gamma_1\gamma_2 + mv\psi\gamma_0 = 0$$

and finally (using an argument analogous to the one leading to eq.(19"))

$$v(\partial\psi\gamma_1\gamma_2 + m\psi\gamma_0) = 0 \,,$$

which is satisfied once it holds the equation

$$\partial\psi\gamma_1\gamma_2 + m\psi\gamma_0 = 0 \tag{20'}$$

which, as expected, is just the Dirac equation in the Clifford formalism.

Before going on, we want to explicitly put forth the following observation. Let us first recall that in our formalism the (Lorenz force) equation of motion for a charged particle moving with velocity w in an electromagnetic field F is

$$\dot{w} \;=\; \frac{e}{m} F \cdot w \,. \tag{22}$$

Now, for *all* the free–particle solution of the BZ model in the Clifford language, it holds the "Darboux relation":

$$\dot{e}_\mu \;=\; \Omega \cdot e_\mu \,, \tag{23}$$

so that the "sub–microscopic" point–like object $\mathcal{Q}$, moving along the helical path σ, is endowed [cf. eq.(15)] with the angular–velocity bivector

$$\Omega \;=\; \tfrac{1}{2}\, \dot{e}_\mu \wedge e^\mu \;=\; \tfrac{1}{2}\, \dot{e}_\mu e^\mu \,, \tag{24}$$

as it follows by recalling that e_μ can always be written, like in eq.(7), as $e_\mu = R\gamma_\mu \tilde{R}$. Finally, let us observe that eq.(23) yields in particular $\dot{e}_0 = \Omega \cdot e_0$, which is formally identical to eq.(22). Thus, the formal, algebraic way we chose [see eq.(22)] for describing that the system as a whole possesses a non-vanishing magnetic dipole structure suggests that the bivector field Ω may be regarded as a kind of *internal* electromagnetic–like field, which keeps the "sub–microscopic" object $\mathcal{Q}$ moving along the helix.[32] In other words, $\mathcal{Q}$ may be considered as confining itself along σ [*i.e.*, along a circular orbit, in the electron CM], via the generation of the internal, electromagnetic–like field

$$F_{\text{int}} \;\equiv\; \frac{m}{2e}\, \dot{e}_\mu \wedge e^\mu \,.$$

We shall further discuss this point elsewhere.

5. ABOUT HESTENES' INTERPRETATION

In connection with our new equation (16'), or rather with its (free) form (19), we met solutions corresponding —in the free case— to helical motions with constant radius r; as well as a limiting solution, eq.(21), for $r \to 0$. We have seen above that the latter is a solution also of the ordinary (free) Dirac equation.

Actually, the solution of the Dirac equation for a free electron in its rest frame can be written in the present formalism as:[20]

$$\psi(x) = \psi(0) \exp[-\gamma_2\gamma_1 m\tau] \tag{25}$$

which coincides with our eq.(21) along the world–line σ_0.

It is interesting to examine how in refs.[20], even if confronting themselves only with the usual (linear) Dirac equation and with eq.(25), those authors were led to propose for the electron the existence of internal helical motions. It was first noticed that, in correspondence with solution (25), it is $\dot{e}_0 = 0$; $\dot{e}_3 = 0$, so that e_0, e_3 are constants; while e_2, e_3 are *rotating*[9] in the e_2e_1 plane (*i.e.*, in the spin plane[20]) with the zbw frequency $\omega = 2m$ $[\hbar = c = 1]$:

$$e_1(\tau) = e_1(0) \cos 2m\tau + e_2(0) \sin 2m\tau$$

$$\tag{26}$$

$$e_2(\tau) = e_2(0) \cos 2m\tau - e_1(0) \sin 2m\tau$$

as follows from eq.(7) with $R = \exp[-\gamma_2\gamma_1 m\tau]$. Incidentally, by recalling that in the Clifford formalism the spin bivector S is given by

$$S = R\gamma_2\gamma_1\tilde{R}\frac{\hbar}{2} = e_2e_1\frac{\hbar}{2}, \tag{27}$$

whilst the angular–velocity bivector Ω is given by eq.(24), one then gets

$$p \cdot v = \Omega \cdot S = m, \tag{28}$$

which seems to suggest[20] the electron rest–mass to have an (internal) kinetic origin! Hestenes could do nothing but asking himself (following Lorentz[33,34]): what is rotating?

If something was rotating inside the electron, since $v = e_0$ refers *in this case* to the electron mean motion, *i.e.*, is the velocity of the whole electron, in refs.[20] it was *assumed* for the velocity of the internal "constituent" Q the value

$$u = e_0 - e_2; \quad e_0 \equiv v \tag{29}$$

which is a light–like quantity. Eq.(29) represents a null vector since $e_0{}^2 = +1$; $e_2{}^2 = -1$ (quite analogously, they could have chosen $u = e_0 - e_1$). Notice that the ordinary Dirac current will correspond to the average $\bar{u}$ of velocity u over a zbw period; *i.e.*, due to eqs.(26), to: $\bar{u} = e_0 = v$, as expected. However, if we set $u \equiv \dot{\zeta}$, then one gets[20]

$$\zeta(\tau) = (e^{\Omega\tau} - 1)R_0 + \zeta_0, \tag{30}$$

which is just the parametric equation of a light–like helix $\zeta(\tau) = x(\tau) + R(\tau)$ centered on the stream–line σ_0 with radius R_H given by $[\omega = 2m]$:

$$R_H(\tau) = e^{\Omega\tau}R_0 = \frac{-e_1}{\omega} = \frac{-\dot{u}}{\omega^2} . \tag{31}$$

The parameters in eqs.(30)-(31) were chosen by Hestenes[20] in *such a way* that the helix diameter equals the Compton wavelength of the electron, and the angular momentum of the zbw motion yields the correct electron spin. It is possible, incidentally, that such a motion be also at the origin of the electric charge; in any case, we saw that, if the electron is associated with a clock-wise rotation, then the positron will be associated with an anti–clock-wise rotation, with respect to the motion direction.

Choice (29), of course, suits perfectly well with the standard discussions about the velocity operator[35] for the Dirac equation, and as a consequence does naturally allow considering that helical motion as the classical analog of zbw.

However, such approach by Hestenes, even if inspiring and rich of physical intuition, seems rather *ad hoc* and in need of a few assumptions [particularly with regard to eqs.(29) and (31)]. In our opinion, to get a sounder theoretical ground, it is to be linked with our eqs.(16'), (19) and the related discussion above.

In the original BZ model, there exist particular initial conditions yielding *as a limiting case* a light–like velocity $v^\mu = \bar{z}\gamma^\mu z$; that is, such that $v^\mu v_\mu = 0$, as it can be checked from eq.(2b). [For instance, in ref.[24] light–like helical paths have been obtained in correspondence with Majorana (singular) DH spinors ψ]. However, in our Clifford formalism, which makes recourse to the Frenet tetrad, quantity v was bound to be time–like. We may, of course, allow for a light–like v; but, in this case, we have to change the explicit representation of the velocity vector in the BZ lagrangian: for instance, to comply with Hestenes' assumption (29), we have just to set: $v = \psi\gamma_0\tilde{\psi} - \psi\gamma_2\tilde{\psi}$, so that eq.(12) has to be rewritten accordingly as follows:

$$\mathcal{L} = \langle\dot{\tilde{\psi}}\dot{\psi}\gamma_1\gamma_2 + p(\dot{x} - \psi\gamma_0\tilde{\psi} + \psi\gamma_2\tilde{\psi}) + eA(\psi\gamma_0\tilde{\psi} - \psi\gamma_2\tilde{\psi})\rangle_0 . \tag{12'}$$

But choice (29) of ref.[20] is just one possibility; for example, one might choose

$$u = e_0 - e_1 - e_2 . \tag{32}$$

and in this case we would get for the rotating point–object $\mathcal{Q}$ a *space–like* velocity u, whose mean value $\bar{u}$ would still be

$$\bar{u} = e_0 .$$

This would correspond in our Clifford formalism to representing the velocity vector as $v = \psi\gamma_0\tilde{\psi} - \psi\gamma_1\tilde{\psi} - \psi\gamma_2\tilde{\psi}$, and modifying the lagrangian (12') accordingly.

In any case, one may observe that also with the choice (29) the light–like u results from the composition of a time–like velocity e_0 with a *space–like* velocity e_2. Some interesting work in this direction did already appear in refs.[36], by Campolattaro.

6. FURTHER REMARKS

We mentioned, at the beginning, about further methods for introducing an helical motion as the classical limit of the "spin motion". We want here to show, at last, how to represent in the Clifford formalism the *extrinsic curvature* approach,[16,17] corresponding to lagrangian (6), due to its interest for the development of the present work.

Let us first recall that in classical differential geometry one defines[23] the Frenet frame $\{e_\mu\}$ of a non-null curve σ by the so–called Frenet equations, which with respect to proper time τ write [besides $\dot{x} = e_0 = v$]:

$$\ddot{x} = \dot{e}_0 = K_1 e^1 ; \quad \dot{e}_1 = -K_1 e^0 + K_2 e^2 ; \quad \dot{e}_2 = -K_2 e^1 + K_3 e^3 ; \quad \dot{e}_3 = -K_3 e^2 \qquad (33)$$

where the i-th curvatures K_i (i=1,2,3) are scalar functions chosen in such a way that $e_j^2 = -1$, with j=1,2,3. Quantity K_1 is often called curvature, and K_2, K_3 torsions (recall that in the 3–dimensional space one meets only K_1 and K_2, called curvature and torsion, respectively). Inserting eqs.(33) into eq.(24), we get for the Darboux (angular-velocity) bivector:

$$\Omega = K_1 e^1 e^0 + K_2 e^2 e^1 + K_3 e^3 e^2 , \qquad (34)$$

so that one can build the following scalar function

$$\Omega \cdot \Omega = K_1^2 - K_2^2 - K_3^2 = (\dot{e}_\mu \wedge e^\mu) \cdot (\dot{e}_\nu \wedge e^\nu) . \qquad (34')$$

At this point, one may notice that the square, K^2, of the "extrinsic curvature" entering eq.(6) is equal to $-K_1^2$, so that the lagrangian adopted in refs.[16] results —after the present analysis— to take advantage only of the first part,

$$\ddot{x}^2 = \dot{e}_0^2 = -K_1^2 ,$$

of the Lorentz invariant (34'). On the contrary, in our formalism the whole invariant $\Omega \cdot \Omega$ suggests itself as the suitable, complete lagrangian for the problem at issue; and in future work we shall exploit it, in particular comparing the expected results with Plyushchay's.[37]

$\Lambda\dagger$

For the moment, let us stress here only the possibly important result that the lagrangian $\mathcal{L} = \Omega \cdot \Omega$ does coincide (factors apart) along the particle world–line σ with the auto–interaction term[38]

$$\theta^5 \, (\mathrm{d}\theta^\mu \wedge \theta_\mu) \cdot (\mathrm{d}\theta^\nu \wedge \theta_\nu)$$

of the *Einstein–Hilbert lagrangian density* written (in the Clifford bundle formalism) in terms of tetrads of 1-form fields θ^μ. Quantity $\theta^5 \equiv \theta^0 \theta^1 \theta^2 \theta^3$ is the volume element.

Finally, we can examine within our formalism the third approach: that one utilizing Grassmann variables.[19] For instance, if we recall that the Grassmann product is nothing but the external part $A_r \wedge B_s = \langle A_r B_s \rangle_{|r-s|}$ of the Clifford product (where A_r, B_s are a r-vector and a s-vector, respectively), then the Ikemori lagrangian[19] can be immediately translated into the Clifford language and shown to be equivalent to the BZ lagrangian, apart from the constraint $p^2 = m^2$.

Further considerations about the solutions of our non-linear, Dirac–like, new equation and the interesting consequences of the present formalism are in preparation and will appear elsewhere.

7. ACKNOWLEDGEMENTS

References

[1] See, *e.g.*, J. Frenkel: Z. Phys. **37** (1926) 243; M. Mathisson: Acta Phys. Pol. **6** (1937); L.H. Kramers: *Quantentheorie des Electron und der Strahlung* (1938); H. Hönl and A. Papapetrou: Z. Phys. **112** (1939) 512; H.J. Bhabha and A.C. Corben: Proc. Roy. Soc. (London) **A178** (1941) 273.

[2] E. Schrödinger: Sitzunber. Preuss. Akad. Wiss. Phys.-Math. Kl. **24** (1930) 418. Cf. also A.H. Compton: Phys. Rev. **11** (1917) 330; **14** (1919) 20, 247.

[3] J. Weyssenhof and A. Raabe: Acta Phys. Pol. **9** (1947) 7; M.H.L. Pryce, Proc. Roy. Soc. (London) **A195** (1948) 6; C.N. Fleming: Phys. Rev. **B137** (1965) 188.

[4] H.C. Corben: Phys. Rev. **121** (1961) 1833.

[5] F.A. Berezin and M.S. Marinov: JETP Lett. **21** (1975) 320.

[6] R. Casalbuoni: Nuovo Cimento **33** (1976) 389.

[7] M. Pauri: in *Group Theoretical Methods in Physics*, Lectures Notes in Physics, vol. 135, p.615, ed. by J.Ehlers, K.Hepp, R.Kippenhahn, H.A.Weidenmüller and J.Zittartz (Springer-Verlag; Berlin, 1980). See also E.P. Wigner: Ann. of Phys. **40** (1939) 149.

[8] T.F. Jordan and N. Mukunda: Phys. Rev. **132** (1963) 1842.

[9] W.A. Rodrigues Jr., J. Vaz Jr. and E. Recami: Found. Phys. **23** (1993) 469.

[10] A.O. Barut and N. Zanghi: Phys. Rev. Lett. **52** (1984) 2009.

[11] F. Riewe: Lett. Nuovo Cimento **1** (1971) 807.

[12] W.A. Rodrigues Jr., J. Vaz Jr., E. Recami and G. Salesi: "About Zitterbewegung and Electron Structure", Report INFN/AE–93/06 (Frascati, 10.02.1993), to appear in Phys. Lett. B (1993).

[13] A.O. Barut and M. Pavšič: Class. Quant. Grav. **4** (1987) L131.

[14] Cf., e.g., E. Recami and G. Ziino: Nuovo Cim. **A33** (1976) 205.

[15] A.O. Barut: Phys. Lett. **B237** (1990) 436.

[16] M. Pavšič: Phys. Lett. **B205** (1988) 231; **B221** (1989) 264; Class. Quant. Grav. **7** (1990) L187.

[17] M. Pavšič: (in preparation). Cf. also H. Arodź, A. Sitarz and P. Wegrzyn: Acta Phys. Pol. **B20** (1989) 921; M.S. Plyushchay: Phys. Lett. **B236** (1990) 291.

[18] J. Isberg, U. Lindström, H. Nordström and J. Grundberg: Mod. Phys. Lett. **A5** (1990) 2491; M.S. Plyushchay: Phys. Lett. **B235** (1990) 50.

[19] A.O. Barut and M. Pavšič: Phys. Lett. **B216** (1989) 297; H. Ikemori: Phys. Lett. **B199** (1987) 239; F.A. Berezin and M.S. Marinov: Ann. of Phys. **104** (1977) 336.

[20] D. Hestenes: Found. Phys. **20** (1990) 1213. See also D. Hestenes: Found. Phys. **15** (1985) 63; J. Math. Phys. **14** (1973) 893; and R. Gurtler and D. Hestenes: J. Math. Phys. **16** (1975) 573.

[21] D. Hestenes: *Space–Time Algebra* (Gordon & Breach; New York, 1966); *New Foundations for Classical Mechanics* (Reidel; Dordrecht, 1989). See also A. Crumeyrolle: *Orthogonal and Symplectic Clifford Algebras: Spinor Structures* (Kluwer; Dordrecht, 1990).

[22] F. Gursey: Nuovo Cimento **5** (1957) 784.

[23] See D. Hestenes and G. Sobczyk: *Clifford Algebra to Geometric Calculus* (Reidel; Dordrecht, 1984), Chapt. **6**.

[24] D. Hestenes: J. Math. Phys. **14** (1973) 893; **16** (1975) 556. Cf. also W.A. Rodrigues jr. and E.C. Oliveira: Int. J. Theor. Phys. **29** (1990) 397.

[25] V.L. Figuereido, E.C. Oliveira and W.A. Rodrigues Jr.: Int. J. Theor. Phys. **29** (1990) 371.

[26] H. Krüger: in *The Electron*, ed. by D. Hestenes and A. Weingartshofer (Kluwer; Dordrecht, 1991), p.49. See also the contributions by R. Boudet and S. Gull into the same volume.

[27] J. Vaz, Jr. and W.A. Rodrigues Jr.: "Clifford Algebra Approach to the Barut-Zanghi Models as a Hamiltonian System", preprint RP 11/93 (Imecc-UNICAMP; Campinas, S.P., 1993).

[28] J. Rawnsley: Lett. Math. Phys. **24** 331 (1992).

[29] J.R.R. Zeni and W.A. Rodrigues Jr.: Int. J. Mod. Phys. **A7** (1992) 1793.

[30] J. Vaz Jr. and W.A. Rodrigues Jr.: Int. J. Theor. Phys. **6** 945 (1993).

[31] S. Gull, A. Lasenby and C. Doran: "Electron Paths, Tunnelling and Diffraction in the Spacetime Algebra", to appear in Found. Phys. (1993).

[32] Cf. also, *e.g.*, G. Szamosi and D. Trevisan: preprint (Univ. of Windsor, Ontario; 1978).

[33] See e.g. H.A. Lorentz: *Theory of Electrons*, 2nd edition (Teubner; Leipzig, 1916), p.2. Cf. also refs.[35].

[34] Cf. also W.A. Rodrigues Jr., J. Vaz Jr. and E. Recami: "Free Maxwell equations, Dirac equation, and Non-dispersive de Broglie Wave–Packets", to appear in *Courants, Amers, Écueils em Micro–Physique*, ed. by G. and P. Lochak.

[35] See e.g. S.S. Schweber: *An Introduction to Relativistic Quantum Field Theory* (Harper & Row; New York, 1961), p.91; J.D. Bjorken and S.D. Drell: *Relativistic Quantum Mechanics* (McGraw–Hill; New York, 1964).

[36] A. Campolattaro: Int. J. Theor. Phys. **29** (1990) 141, and refs. therein.

[37] M.S. Plyushchay: Phys. Lett. B**235** (1990) 47.

[38] W.A. Rodrigues Jr. and Q.A.G. de Souza: Found. Phys. **23** (1993) 1465.