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FIELD THEORY OF THE SPINNING ELECTRON: I - INTERNAL MOTIONS. (*)

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Abstract – One of the most satisfactory picture of spinning particles is the Barut-Zanghi (BZ) classical theory for the relativistic electron, that relates the electron spin with the so-called *zitterbewegung* (zbw). The BZ theory has been recently studied in the lagrangian and hamiltonian simplectic formulations, both in flat and in curved space-time. The BZ motion equations constituted the starting point for two recent works about spin and electron structure, co-authored by us, which adopted the Clifford algebra formalism.

In this letter, by employing on the contrary the ordinary tensorial language, we first write down a meaningful (real) equation of motion, describing particle classical paths, quite different from the corresponding (complex) equation of the standard Dirac theory. As a consequence, we succeed in regarding the electron as an extended object with a classically intelligible structure (thus overcoming some long-standing, wellknown problems).

Second, we make explicit the kinematical properties of the 4-velocity field v^μ, which also result to be quite different from the ordinary ones, valid for scalar particles.

At last, we analyze the inner zbw motions, both time-like and light-like, as functions of the initial conditions (in particular, for the case of *classical* uniform motions, the z component of spin s is shown to be *quantized*). In so doing, we make explicit the strict correlation existing between electron polarization and zbw kinematics.

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1. – A new motion equation for the spinning (free) electron.

Attempts to put forth classical models for the spinning electron are known since more than seventy years [1]. In the BZ theory,[2] the classical electron was actually characterized, besides by the usual pair of conjugate variables (x^μ, p^μ), by a second pair of conjugate classical *spinorial* variables $(\psi, \overline{\psi})$, representing internal degrees of freedom, which were functions of the (proper) time τ measured in the electron global center-of-mass (CM) system; the CM frame (CMF) being the one in which $\boldsymbol{p} = 0$ at every instant of time. Barut and Zanghi, then, introduced a classical lagrangian that in the free case (i.e., when the *external* electromagnetic potential is $A^\mu = 0$) writes $[c = 1]$

$$\mathcal{L} = \frac{1}{2} i\lambda(\dot{\overline{\psi}}\psi - \overline{\psi}\dot{\psi}) + p_\mu(\dot{x}^\mu - \overline{\psi}\gamma^\mu\psi) \; , \tag{1}$$

where λ has the dimension of an action and ψ and $\overline{\psi} \equiv \psi^\dagger\gamma^0$ are ordinary $\mathbb{C}^4$–bispinors, the dot meaning derivation with respect to τ. The four Euler–Lagrange equations, with $-\lambda = \hbar = 1$, yield the following motion equations:

$$\begin{cases} \dot{\psi} + ip_\mu\gamma^\mu\psi = 0 & \text{(2a)} \\ \dot{x}^\mu = \overline{\psi}\gamma^\mu\psi & \text{(2b)} \\ \dot{p}^\mu = 0 \; , & \text{(2c)} \end{cases}$$

besides the hermitian adjoint of eq.(2a), holding for $\overline{\psi} = \psi^+\gamma^0$. From eq.(1) one can also see that

$$H \equiv p_\mu v^\mu = p_\mu\overline{\psi}\gamma^\mu\psi \tag{3}$$

is a contant of the motion [and precisely is the energy in the CMF].[2–4] Being H the BZ hamiltonian in the CMF, we can suitably set $H = m$, quantity m being the particle rest-mass. The general solution of the equations of motion (2) can be shown to be:

$$\psi(\tau) = [\cos(m\tau) - i\frac{p_\mu\gamma^\mu}{m}\sin(m\tau)]\psi(0) \; , \tag{4a}$$

$$\overline{\psi}(\tau) = \overline{\psi}(0)[\cos(m\tau) + i\frac{p_\mu\gamma^\mu}{m}\sin(m\tau)] \; , \tag{4b}$$

with $p^\mu =$ constant; $p^2 = m^2$; and finally:

$$\dot{x}^\mu \equiv v^\mu = \frac{p^\mu}{m} + [\dot{x}^\mu(0) - \frac{p^\mu}{m}]\cos(2m\tau) + \frac{\ddot{x}^\mu}{2m}(0)\sin(2m\tau) \; . \tag{4c}$$

This general solution exhibits a classical analogue of the phenomenon known as zitterbewegung: in fact, the velocity v^μ contains the (expected) term p^μ/m plus a term describing an oscillating motion with the characteristic zbw frequency $\omega = 2m$. The velocity of the CM will be given by p^μ/m. Let us explicitly observe that the general solution (4c) represents a helical motion in the ordinary 3-space: a result that has been met also by means of other, alternative approaches.[5,6]

Before studying the time evolution of our electron, we want to write down its motion equation in a "kinematical" form, suitable a priori for describing a point-like object; i.e., at variance with eqs.(2), expressed not in terms of ψ and $\overline{\psi}$, but on the contrary in terms of quantities related to the particle trajectory (such as p^μ and v^μ). To this aim, we can introduce the spin variables, and adopt the set of dynamical variables

$$x^\mu \, , \; p^\mu \, ; \; v^\mu \, , S^{\mu\nu} \, ,$$

where

$$S^{\mu\nu} \equiv \frac{i}{4} \overline{\psi} [\gamma^\mu, \gamma^\nu] \psi \; ; \tag{5a}$$

then, we get the following motion equations:

$$\dot{p}^\mu = 0 \; ; \quad v^\mu = \dot{x}^\mu \; ; \quad \dot{v}^\mu = 4 S^{\mu\nu} p_\nu \; ; \quad \dot{S}^{\mu\nu} = v^\nu p^\mu - v^\mu p^\nu \; . \tag{5b}$$

[By varying the action corresponding to $\mathcal{L}$, one finds as generator of space-time rotations the conserved quantity $J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$, where $L^{\mu\nu} \equiv x^\mu p^\nu - x^\nu p^\mu$ is the orbital angular momentum tensor, and $S^{\mu\nu}$ is just the particle spin tensor: so that $\dot{J}^{\mu\nu} = 0$ implies $\dot{L}^{\mu\nu} = -\dot{S}^{\mu\nu}$].

By deriving the third one, and using the first one, of eqs.(5b), we obtain

$$\ddot{v}^\mu = 4 \dot{S}^{\mu\nu} p_\nu \; ; \tag{6}$$

by substituting now the fourth one of eqs.(5b) into eq.(6), and imposing the previous constraints $p_\mu p^\mu = m^2$ and $p_\mu v^\mu = m$, we end with the time evolution[3] of the *field four-velocity*:

$$v^\mu = \frac{p^\mu}{m} - \frac{\ddot{v}^\mu}{4m^2} \, , \tag{7}$$

such a motion equation corresponding to the whole system of eqs.(2). Let us recall, for comparison, that the analogous equation *for the standard Dirac case:*[1]

$$v^\mu = \frac{p^\mu}{m} - \frac{i}{2m} \dot{v}^\mu \tag{7'}$$

was totally devoid of a classical, intuitive meaning, because of the known appearance of an imaginary unit i in front of the acceleration (connected with the well-known fact that the position operator is not hermitian therein).

Let us observe that, by differentiating the relation $p_\mu v^\mu = m =$ constant, one immediately finds that the (internal) acceleration $\dot{v}^\mu \equiv \ddot{x}^\mu$ is orthogonal to the electron impulse p^μ, since $p_\mu \dot{v}^\mu = 0$ at any instant. To conclude, let us stress that, while the Dirac electron has no classically meaningful internal structure, our electron on the contrary (an *extended–type* particle) does possess an internal structure, and internal motions, which are all endowed with a "realistic" meaning, from both the geometrical and kinematical points of view: as we are going to see in the next section.

2 – Spin and internal kinematics.

We wish first of all to make explicit the kinematical definition of v^μ, *rather different from the ordinary one* valid for scalar particles.(7). In fact, from the very definition of v^μ, we get

$$v^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}\tau^\mu \equiv (\mathrm{d}t/\mathrm{d}\tau; \mathrm{d}\boldsymbol{x}/\mathrm{d}\tau) \equiv (\frac{\mathrm{d}t}{\mathrm{d}\tau}; \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau})$$
$$= (1/\sqrt{1-\boldsymbol{w}^2};\ \boldsymbol{u}/\sqrt{1-\boldsymbol{w}^2}), \qquad [\boldsymbol{u} \equiv \mathrm{d}\boldsymbol{x}/\mathrm{d}t] \tag{8}$$

where $\boldsymbol{w} = \boldsymbol{p}/m$ is the velocity of the CM in the chosen reference frame (i.e., in the frame in which quantities x^μ are measured). Below, it will be convenient to choose as reference frame the CMF (even if quantities as $v^2 \equiv v_\mu v^\mu$ are frame invariant); so that

$$v^\mu_{\mathrm{CM}} = V^\mu \equiv (1; \boldsymbol{V}) , \tag{9}$$

wherefrom one deduces for the speed $|\boldsymbol{V}|$ of the internal motion (i.e., for the zbw speed) the new conditions:

$$\begin{aligned} 0 < V^2(\tau) < 1 \quad &\Leftrightarrow \quad 0 < \boldsymbol{V}^2(\tau) < 1 \qquad &\text{(time-like)} \\ V^2(\tau) = 0 \quad &\Leftrightarrow \quad \boldsymbol{V}^2(\tau) = 1 \qquad &\text{(light-like)} \\ V^2(\tau) < 0 \quad &\Leftrightarrow \quad \boldsymbol{V}^2(\tau) > 1 \qquad &\text{(space-like)} , \end{aligned} \tag{10}$$

where $V^2 = v^2$. Notice that, in general, the value of V^2 does vary with τ; except in special cases (e.g., the case of polarized particles: as we shall see). Coming back to the expression of the 4-velocity, eq.(4c), it is possible after some algebra to recast this equation in a "spinorial" form, i.e., to write it as a function of the initial spinor $\psi(0)$:

$$v^\mu = p^\mu/m + E^\mu \cos(2m\tau) + H^\mu \sin(2m\tau) , \tag{11}$$

where $[\alpha^\mu \equiv \gamma^0\gamma^\mu]$

$$E^\mu = \frac{1}{2}\overline{\psi}(0)[\frac{\not p}{m}\, , \ \alpha^\mu]\psi(0) \tag{12a}$$

$$H^\mu = \frac{i}{2}\overline{\psi}(0)(\alpha^\mu - \frac{\not p}{m}\alpha^\mu\frac{\not p}{m})\psi(0)\ . \tag{12b}$$

In the chosen CM frame, eqs.(12) read:

$$E^\mu = \overline{\psi}(0)\gamma^\mu\psi(0) - \frac{p^\mu}{m} \tag{13a}$$

$$H^\mu = i\overline{\psi}(0)(\alpha^\mu - g^{0\mu})\psi(0)\ , \tag{13b}$$

where $g^{\mu\nu}$ is the metric tensor. Bearing in mind that (in the CMF) it holds $v^0 = 1$ [cf. eq.(9)], and therefore $\overline{\psi}\gamma^0\psi = 1$ (which, incidentally, implies the normalization $\psi^\dagger\psi = 1$ in the CMF), one obtains

$$E^\mu = (0;\ \overline{\psi}(0)\vec{\gamma}\psi(0)) \tag{14a}$$

$$H^\mu = (0;\ i\overline{\psi}(0)\vec{\alpha}\psi(0))\ . \tag{14b}$$

By eq.(4), for V^2 we can write:

$$V^2 = 1 + E^2\cos^2(2m\tau) + H^2\sin^2(2m\tau) + 2E_\mu H^\mu \sin(2m\tau)\cos(2m\tau)\ . \tag{15}$$

Now, let us single out the solutions ψ of eq.(2) corresponding to *constant* V^2 and A^2, where $A^\mu \equiv \mathrm{d}V^\mu/\mathrm{d}\tau \equiv (0; \boldsymbol{A})$, quantity $V^\mu \equiv (1; \boldsymbol{V})$ being the zbw velocity. In the present frame, therefore, we shall suppose quantities

$$V^2 = 1 - \boldsymbol{V}^2 \quad ; \quad A^2 = -\boldsymbol{A}^2$$

to be constant in time:

$$V^2 \ = \ \text{constant}\ ; \qquad A^2 \ = \ \text{constant}\ , \tag{16}$$

so that $\boldsymbol{V}^2$ and $\boldsymbol{A}^2$ are constant in time too. (Let us recall that we are dealing with the internal motion only, in the CMF; thus, our results are independent of the global 3-impulse $\boldsymbol{p}$ and hold both in the relativistic and in the non-relativistic case). Requirements (16), inserted into eq.(15), yield the following interesting constraints:[(7)]

$$\begin{cases} E^2 = H^2 & \text{(17a)} \\ E_\mu H^\mu = 0\ . & \text{(17b)} \end{cases}$$

Constraints (17) are necessary and sufficient (initial) conditions to get a circular *uniform* motion (the only finite, uniform motion possible in the CMF). Since both E and H do not depend on τ, also eqs.(17) hold at any time. In the euclidean 3-dimensional space, and at any time, constraints (17) may read:

$$\begin{cases} \boldsymbol{A}^2 = 4m^2\boldsymbol{V}^2 & (18a) \\ \boldsymbol{V}\cdot\boldsymbol{A} = 0 & (18b) \end{cases}$$

which explicitely correspond to a uniform circular motion with radius

$$R = |\boldsymbol{V}|/2m \ . \tag{19}$$

Quantity R is the "zitterbewegung radius"; the zbw frequency was already found to be $\Omega = 2m$. By means of eqs.(14), conditions (17) or (18) can be written in spinorial form (still for any time instant τ) as follows:

$$\begin{cases} (\overline{\psi}\vec{\gamma}\psi)^2 = -(\overline{\psi}\vec{\alpha}\psi)^2 & (20a) \\ (\overline{\psi}\vec{\gamma}\psi)\cdot(\overline{\psi}\vec{\alpha}\psi) = 0 \ . & (20b) \end{cases}$$

At this point, let us show that this classical uniform circular motion, around the z-axis (which in the CMF can be chosen arbitrarily, while in a generic frame is parallel to the global three-impulse $\boldsymbol{p}$, as we shall see below), does just correspond to the case of *polarized* particles with $s_z = \pm\frac{1}{2}$. It may be interesting to notice that in this case the *classical* requirements (17) or (18) —namely, the uniform motion conditions— play the role of the ordinary *quantization* conditions $s_z = \pm\frac{1}{2}$.

It is straightforward to realize also that the most general spinors $\psi(0)$ satisfying the conditions

$$s_x = s_y = 0 \tag{21a}$$

$$s_z = \frac{1}{2}\overline{\psi}(0)\Sigma_z\psi(0) = \pm\frac{1}{2} \tag{21b}$$

($\vec{\Sigma}$ being the spin operator) possess in the standard representation the form

$$\psi^{\mathrm{T}}_{(+)}(0) = (\ a\ 0\ |\ 0\ d\ ) \tag{22a}$$

$$\psi^{\mathrm{T}}_{(-)}(0) = (\ 0\ b\ |\ c\ 0\ )\ , \tag{22b}$$

and obey in the CMF the normalization constraint $\psi^\dagger\psi = 1$. [It could be easily shown that, for generic initial conditions, it is always $-\frac{1}{2} \le s_z \le \frac{1}{2}$]. In eqs.(22) we separated the first two from the second two components, bearing in mind that in the standard

Dirac theory (and in the CMF) they correspond to the positive and negative frequencies, respectively. With regard to this point, let us observe that the "negative-frequency" components c and d do *not* vanish at the non-relativistic limit (since, let us repeat, in the CMF it is $\boldsymbol{p} = 0$); but the field hamiltonian H is *nevertheless* positive and equal to m, as already stressed. Now, from relation (22a) we are able to deduce that (with $* \equiv$ complex conjugation):

$$\begin{aligned} <\vec{\gamma}> &\equiv \overline{\psi}\vec{\gamma}\psi = 2(\mathrm{Re}[a^*d], +\mathrm{Im}[a^*d], 0) \\ <\vec{\alpha}> &\equiv \overline{\psi}\vec{\alpha}\psi = 2i(\mathrm{Im}[a^*d], -\mathrm{Re}[a^*d], 0) \end{aligned}$$

and analogously, from eq.(22b), that

$$\begin{aligned} <\vec{\gamma}> &\equiv \overline{\psi}\vec{\gamma}\psi = 2(\mathrm{Re}[b^*c], -\mathrm{Im}[b^*c], 0) \\ <\vec{\alpha}> &\equiv \overline{\psi}\vec{\alpha}\psi = 2i(\mathrm{Im}[b^*c], +\mathrm{Re}[b^*c], 0) \; , \end{aligned}$$

which just imply relations (20):

$$\begin{cases} <\vec{\gamma}>^2 = -<\vec{\alpha}>^2 \\ <\vec{\gamma}> \cdot <\vec{\alpha}> = 0 \; . \end{cases}$$

In conclusion, the (circular) polarization conditions, eqs.(21), do imply the internal zbw motion to be uniform and circular ($V^2 =$ constant; $A^2 =$ constant); equations (21), in other words, imply simultaneosly that s_z be conserved and quantized.(7).

When passing from the CMF to a generic frame, eqs.(21) transform into

$$\lambda \equiv \frac{1}{2}\overline{\psi}(x)\frac{\vec{\Sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}\psi(x) \;=\; \pm\frac{1}{2} \;=\; \text{constant} \; . \tag{23}$$

Therefore, to get a uniform motion around the $\boldsymbol{p}$-direction [cf. eq.(4c)], we have to require that the field helicity λ be constant (in space and in time), and quantized in the ordinary way: $\lambda = \frac{1}{2}$.

It may be interesting also to calculate $|\boldsymbol{V}|$ as a function of the spinor components a and d. With reference to eq.(22a), since $\psi^\dagger\psi \equiv |a|^2 + |d|^2 = 1$, we obtain (for the $s_z = +\frac{1}{2}$ case):

$$\boldsymbol{V}^2 \equiv <\vec{\gamma}>^2 = 4|a^*d|^2 = 4|a|^2\,(1 - |a|^2) \tag{24a}$$

$$\boldsymbol{A}^2 \equiv (2im <\vec{\alpha}>)^2 = 4m^2\boldsymbol{V}^2 = 16m^2|a|^2\,(1 - |a|^2) \; , \tag{24b}$$

and therefore the normalization value (valid now in any frame, at any time):

$$\overline{\psi}\psi = \sqrt{1 - \boldsymbol{V}^2} \ , \tag{24c}$$

showing that to the same speed and acceleration there correspond two spinors $\psi(0)$, related by an interchange of a and d. From eq.(24a) we derive also that, as $0 \leq |a| \leq 1$, it is:

$$0 \leq \boldsymbol{V}^2 \leq 1 \ ; \quad 0 \leq \overline{\psi}\psi \leq 1 \ . \tag{24d}$$

Correspondingly, from eq.(19c) we would obtain for the zbw radius that $0 \leq R \leq \frac{1}{2}m$.

The second of eqs.(24d) is a rather interesting (normalization) boundary condition. From eq.(24c) one can easily see that: (i) for $\boldsymbol{V}^2 = 0$ (no zbw) we have $\overline{\psi}\psi = 1$ and ψ is a "Dirac spinor"; (ii) for $\boldsymbol{V}^2 = 1$ (light-like zbw) we have $\overline{\psi}\psi = 0$ and ψ is a "Majorana" spinor"; (iii) for $0 < \boldsymbol{V}^2 < 1$ we meet, instead, spinors with characteristics "intermediate" between the Dirac and Majorana ones.

The "Dirac" case, corresponding to $\boldsymbol{V}^2 = \boldsymbol{A}^2 = 0$, i.e., to *no* zbw internal motion, is trivially represented (apart from phase factors) by the spinors:

$$\psi^{\mathrm{T}}(0) \equiv (1 \ \ 0 \mid 0 \ \ 0) \tag{25}$$

and (interchanging a and d):

$$\psi^{\mathrm{T}}(0) \equiv (0 \ \ 0 \mid 0 \ \ 1) \ . \tag{25'}$$

This is the unique case (together with the analogous one for $s_z = -\frac{1}{2}$) in which the zbw disappears, while the field spin is still present! In fact, even in terms of eqs.(25)–(25') one still gets that $\frac{1}{2}\overline{\psi}\Sigma_z\psi = +\frac{1}{2}$.

Since we have been discussing a classical theory of the relativistic electron, let us finally notice that even the well-known change in sign of the fermion wave function, under 360°-rotations around the z-axis, gets in our theory a natural classical interpretation. In fact, a 360°-rotation of the coordinate frame around the z-axis (passive point of view) is indeed equivalent to a 360°-rotation of the constituent $\mathcal{Q}$ around the z-axis (active point of view). On the other hand, as a consequence of the latter transformation, the zbw angle $2m\tau$ does suffer a variation of 360°, the proper time τ does increase of a zbw period $T = \pi/m$, and the pointlike constituent does describe a complete circular orbit around the z-axis. It appears then obvious that, since the period $T = 2\pi/m$ of spinor $\psi(\tau)$ in eq.(4c) is *twice* as large as the zbw orbital period, the wave function of eq.(4c) does suffer a phase–variation of 180° only, and then does change its sign: as it occurs in the standard theory.

3. Special cases: light-like motions and linear motions.

Let us first fix our attention on the special case of *light-like* motions.[7,6] The spinor fields $\psi(0)$, corresponding to $V^2 = 0; \boldsymbol{V}^2 = 1$, are given by eqs.(22) with $|a| = |d|$ for the $s_z = +\frac{1}{2}$ case, or $|b| = |c|$ for the $s_z = -\frac{1}{2}$ case; as it follows from eqs.(24) for $s_z = +\frac{1}{2}$, as well as from the analogous equations

$$\boldsymbol{V}^2 = 4|b^*c| = 4|b|^2(1 - |b|^2) \tag{26a}$$

$$\boldsymbol{A}^2 = 4m^2\boldsymbol{V}^2 = 16m^2|b|^2(1 - |b|^2)\ , \tag{26b}$$

for the case $s_z = -\frac{1}{2}$. It can be easily seen that a difference in the phase factors of a and d (or of b and c, respectively) does *not* change the motion kinematics, nor its rotation direction; but it merely shifts the zbw phase angle at $\tau = 0$. Thus, one is entitled to choose *purely real* spinor components (as we did above). As a consequence, the *simplest* spinors may be written:

$$\psi^{\mathrm{T}}_{(+)} = \frac{1}{\sqrt{2}}(1 \ \ 0 \mid 0 \ \ 1) \tag{27a}$$

$$\psi^{\mathrm{T}}_{(-)} = \frac{1}{\sqrt{2}}(0 \ \ 1 \mid 1 \ \ 0)\ ; \tag{27b}$$

and then

$$< \vec{\gamma} >_{(+)} = (1,0,0)\ ;\ < \vec{\alpha} >_{(+)} = (0,-i,0)$$

$$< \vec{\gamma} >_{(-)} = (1,0,0)\ ;\ < \vec{\alpha} >_{(-)} = (0,i,0)$$

which, inserted into eqs.(14), yield

$$E^{\mu}{}_{(+)} = (0;1,0,0)\ ; \qquad H^{\mu}{}_{(+)} = (0;0,1,0)\ .$$

$$E^{\mu}{}_{(-)} = (0;1,0,0)\ ; \qquad H^{\mu}{}_{(-)} = (0;0,-1,0)\ .$$

Because of eq.(11), we meet now for $s_z = +\frac{1}{2}$ an *anti-clockwise* internal motion, with respect to the chosen z-axis:

$$V_x = \cos(2m\tau); \quad V_y = \sin(2m\tau); \quad V_z = 0\ ; \tag{28}$$

and a *chockwise* internal motion for $s_z = -\frac{1}{2}$:

$$V_x = \cos(2m\tau); \quad V_y = -\sin(2m\tau); \quad V_z = 0\ . \tag{29}$$

Let us explicitly observe that spinor (27a), associated with $s_z = +\frac{1}{2}$ (i.e., with an anti-clockwise internal rotation), gets contributions of equal magnitude from the positive–frequency spin-up component and from the negative–frequency spin-down component: in full agreement with our "reinterpretation" in terms of particles and antiparticles, given in refs.(8). Analogously, spinor (27b), associated with $s_z = -\frac{1}{2}$ (i.e., with a clockwise internal rotation), gets contributions of equal magnitude from the positive-frequency spin-down component and the negative-frequency spin-up component.(8)

As we have seen above [cf. eq.(23)], in a *generic* reference frame the polarized states are characterized by a helical uniform motion around the $\boldsymbol{p}$-direction; therefore, the $\lambda = +\frac{1}{2}$ $[\lambda = -\frac{1}{2}]$ spinor will correspond to an anti-clockwise [a clockwise] helical motion with respect to the $\boldsymbol{p}$-direction.

Going back to the CMF, we have to remark that eq.(19) yields in this case for the zbw radius R the traditional result:

$$R = \frac{|\boldsymbol{V}|}{2m} \equiv \frac{1}{2m} \equiv \frac{\lambda}{2}\,, \tag{30}$$

where λ is the Compton wave-length. Of course, $R = \frac{1}{2}m$ represents the *maximum* size (in the CMF) of the electron, among all the uniform motion (A^2 =const; V^2 =const) solutions. The minimum, $R = 0$, corresponding to the limiting Dirac case with no zbw ($V = A = 0$), represented by eqs.(25), (25'): so that the Dirac free electron is a pointlike, extensionless object.

Before concluding this Section, let us shortly consider what happens when *releasing* the conditions (22)–(25) (and therefore abandoning the assumption of circular uniform motion), so to obtain an internal oscillating motion along a constant straight line. For instance, one may choose either

$$\psi^{\mathrm{T}}(0) \equiv \frac{1}{\sqrt{2}}(1 \;\; 0 \mid 1 \;\; 0)\,, \tag{31}$$

or $\psi^{\mathrm{T}}(0) \equiv \frac{1}{\sqrt{2}}(1 \;\; 0 \mid i \;\; 0)$, or $\psi^{\mathrm{T}}(0) \equiv \frac{1}{2}(1 \;\; -1 \mid -1 \;\; 1)$, or $\psi^{\mathrm{T}}(0) \equiv \frac{1}{\sqrt{2}}(0 \;\; 1 \mid 0 \;\; 1)$, and so on.

In case (31), for example, one actually gets

$$<\vec{\gamma}> \equiv (0,0,1)\;;\; <\vec{\alpha}> \equiv (0,0,0)$$

which, inserted into eqs.(14), yield

$$E^{\mu} = (0;0,0,1)\;;\; H^{\mu} = (0;0,0,0).$$

Therefore, because of eq.(21a), we have now a *linear, oscillating* motion [for which equations (22), (23), (24) and (25) do *not* hold: here $V^2(\tau)$ does vary from 0 to 1!] along the z-axis:

$$V_x(\tau) = 0; \quad V_y(\tau) = 0; \quad V_z(\tau) = \cos(2m\tau) \ .$$

All the spinor written above could describe an unpolarized, mixed state, since it is

$$\boldsymbol{s} \equiv \frac{1}{2}\overline{\psi}\vec{\Sigma}\psi = (0,0,0) \ ,$$

in agreement with the existence of a linear oscillating motion. Furthermore for such spinors it holds $\overline{\psi}\psi = \overline{\psi}\gamma^5\psi = 0$, but $\overline{\psi}\gamma^5\gamma^\mu\psi \neq 0$ and $\overline{\psi}S^{\mu\nu}\psi \neq 0$. This new class of spinors has been recently proposed and extensively studied by Lounesto,(9) by employing a new concept, called "boomerang", within the framework of Clifford algebras. A physical realization of those new spinors(9) seems now to be provided by our electron, in the present case.

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References

[1] A.H. Compton: Phys. Rev. **14** (1919) 20, 247; E. Schrödinger: Sitzunger. Preuss. Akad. Wiss. Phys. Math. Kl. **24** (1930) 418. See also P.A.M. Dirac: *The Principles of Quantum Mechanics*, 4th edition (Claredon; Oxford, 1958), p.262; J. Frenkel: Z. Phys. **37** (1926) 243; M. Mathisson: Acta Phys. Pol. **6** (1937) 163; H. Hönl and A. Papapetrou: Z. Phys. **112** (1939) 512; **116** (1940) 153; M.J. Bhabha and H.C. Corben: Proc. Roy. Soc. (London) A**178** (1941) 273; K. Huang: Am. J. Phys. **20** (1952) 479; H. Hönl: Ergeb. Exacten Naturwiss. **26** (1952) 29; A. Proca: J. Phys. Radium **15** (1954) 5; M. Bunge: Nuovo Cimento **1** (1955) 977; F. Gursey: Nuovo Cimento **5** (1957) 784; W.H. Bostick: "Hydromagnetic model of an elementary particle", in *Gravity Res. Found. Essay Contest* (1958 and 1961), and refs. therein; H.C. Corben: Phys. Rev. **121** (1961) 1833; T.F. Jordan and M. Mukunda: Phys. Rev. **132** (1963) 1842; B. Liebowitz: Nuovo Cimento A**63** (1969) 1235; H. Jehle: Phys. Rev. D**3** (1971) 306; F. Riewe: Lett. Nuovo Cim. **1** (1971) 807; G.A. Perkins: Found. Phys. **6** (1976) 237; D. Gutkowski, M. Moles and J.P. Vigier: Nuovo Cim. B**39** (1977) 193; A.O. Barut: Z. Naturforsch. A**33** (1978) 993; J.A. Lock: Am. J. Phys. **47** (1979) 797; M. Pauri: in *Lecture Notes in Physics, vol.135* (Springer; Berlin, 1980), p.615; J. Maddox: "Where Zitterbewegung may lead", Nature **325** (1987) 306; M. H. McGregor: *The enigmatic electron* (Kluwer; Dordrecht, 1992); W.A. Rodrigues, J. Vaz and E. Recami: Found. Phys. **23** (1993) 459.

[2] A.O. Barut and N. Zanghi: Phys. Rev. Lett. **52** (1984) 2009. See also A.O. Barut and A.J. Bracken: Phys. Rev. D**23** (1981) 2454; D**24** (1981) 3333; A.O. Barut and M. Pavsic: Class. Quantum Grav. **4** (1987) L131; A.O. Barut: Phys. Lett. B**237** (1990) 436.

[3] G. Salesi and E. Recami: "A Quantum Field Theory of the Extended Electron", preprint RP/50 (Imecc–Unicamp; Campinas, S.P, Brazil, 1993), submitted for pub.

[4] M. Pavsic, E. Recami, W.A. Rodrigues, G.D. Maccarrone, F. Raciti and G. Salesi: Phys. Lett. B**318** (1993) 481.

[5] M. Pavsic: Phys. Lett. B**205** (1988) 231; B**221** (1989) 264; Class. Quant. Grav. **7** (1990) L187.

[6] A.O. Barut and M. Pavsic: Phys. Lett. B**216** (1989) 297; F.A. Ikemori: Phys. Lett. B**199** (1987) 239. See also D. Hestenes: Found. Phys. **20** (1990) 1213; S. Gull, A.

Lasenby and C. Doran: "Electron paths, tunneling and diffraction in the space-time algebra", to appear in Found. Phys. (1993); D. Hestenes and A. Weingartshofer (eds.): *The electron* (Kluwer; Dordrecht, 1971), in particular the contributions by H. Krüger, by R. Boudet, and by S. Gull; A. Campolattaro: Int. J. Theor. Phys. **29** (1990) 141; D. Hestenes: Found. Phys. **15** (1985) 63.

[7] W.A. Rodrigues Jr., J. Vaz, E. Recami and G. Salesi: Phys. Lett. B**318** (1993) 623.

[8] For the physical interpretation of the negative frequency waves, without any recourse to a "Dirac sea", see e.g. E. Recami: Found. Phys. **8** (1978) 329; E. Recami and W.A. Rodrigues: Found. Phys. **12** (1982) 709; **13** (1983) 533; M. Pavsic and E. Recami: Lett. Nuovo Cim. **34** (1982) 357. See also R. Mignani and E. Recami: Lett. Nuovo Cim. **18** (1977) 5; A. Garuccio *et al.*: Lett. Nuovo Cim. **27** (1980) 60.

[9] P. Lounesto: "Clifford Algebras, Relativity and Quantum Mechanics", to appear in the Proceedings of *SILARG–VIII* (World Scientific; Singapore, 1994).