ISTITUTO NAZIONALE DI FISICA NUCLEARE

Sezione di Catania

INFN/AE–93/06
10 Febbraio 1993

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PACS.: 03.70.+k ; 11.10.Qr ; 14.60.Cd

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INFN – Istituto Nazionale di Fisica Nucleare
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ABOUT ZITTERBEWEGUNG AND ELECTRON STRUCTURE†

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ABSTRACT: We start from the spinning electron model by Barut and Zanghi, which has been recently translated into the Clifford algebra language. We "complete" such a translation, first of all, by expressing in the Clifford formalism the original Barut–Zanghi (BZ) solution, which referred (at the classical limit) to an "internal" helical motion with a *time-like* speed, and is here shown to originate from the superposition of positive and negative energy solutions of the Dirac equation (when *restricted* to the helical path). Moreover, in this paper we yield a general method for constructing solutions of the Dirac equation which represent at the classical limit time-like motions along a cylindrical helix.

Then, we show how to construct solutions of the Dirac equation describing helical motions with *light-like* speed, which meet very well the standard interpretation of the velocity operator in the Dirac equation theory (and agree with Hestenes' solution, even if that author reached it via ad-hoc assumptions that are unnecessary in the present approach). All solutions are analysed also in their quantum version.

The above results appear to support the conjecture that the zitterbewegung motion (a helical motion, at the classical limit) is really responsible for the electron spin.

† Work partially supported by INFN, CNR, MURST and by CNPq, FAPESP.

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1. *Introduction* – The mysterious zitterbewegung motion, associated since long with the electron structure, seems to be responsible for the electron spin. Indeed, Schroedinger[1] proposed the electron spin to be a consequence of a local circulatory motion, constituting the zbw and resulting from the interference between positive and negative energy solutions of the Dirac equation. Such an issue turned out to be of renewed interest, following recent work, *e.g.*, by Barut *et al.*,[2–4] Hestenes,[5,6] and Pavšič *et al.*[7]. A key point appeared in refs.[2,3] is the recognition that the pair of conjugate variables (x^μ, p_μ) is *not enough* to characterize the spinning particle. Actually, after introducing the additional classical spinor variables $(z, i\bar{z})$ [where z is a Dirac spinor; and $\bar{z} \equiv z^\dagger\gamma^0$], Barut and Zanghi[2] associated the electron spin and zitterbewegung (zbw) —at the classical limit— with a canonical system [a point $\mathcal{Q}$ moving along a cylindrical helix] which after quantization describes the Dirac electron.[4] In ref.[7], Pavšič *et al.* presented a thoughtful study of the above results by the Clifford algebra formalism. However, they[7] left some questions still open, and aim of the present article is analysing part of them.

First of all, let us recall that Hestenes' analysis[5] was based on his reformulation[5,6,8,9] of Dirac theory in terms of the so-called "Clifford space–time algebra (STA)" $\mathbb{R}_{1,3}$. For details about the Dirac–Hestenes (DH) spinors and the Clifford bundle formalism (used also below), see *e.g.* refs.[10–14]. In Hestenes' papers[5,6] on his "zbw interpretation of quantum mechanics", an ad-hoc assumption appeared, when he identified the electron velocity with the light–like vector $u = e_0 - e_2$, with $e_i = \psi\gamma_i\tilde{\psi}$, $[i = 0, 2]$, where ψ is a (plane–wave) DH spinor field[5,6,10–12] satisfying the Dirac–Hestenes equation (*i.e.*, the equation representing the ordinary Dirac equation in the Clifford formalism). Then, he represented the electron internal structure by a light–like helical *motion* of a sub-microscopic "constituent" $\mathcal{Q}$, such that (for a suitable choice of the helix parameters) the helix diameter equals the electron Compton wavelength and the angular momentum of the zbw yields the correct electron spin. At last, he directly associated the complex phase factor of the electron wave–function with the zbw motion.

We are going to show, among other things, that Hestenes' ad-hoc assumption, namely that $u = e_0 - e_2$, is not necessary. More in general, below we shall show:
(i) *how* to construct solutions of the DH equation which correspond (at the classical limit) to a helical motion with time–like velocity, as in the Barut–Zanghi (BZ) model.[2] They will result to be superpositions[(**)] of positive and negative energy solutions of the DH equation; that is to say (as thoroughly explained in refs.[15], only on the basis of relativistic classical physics), superpositions of particle and antiparticle solutions of the Dirac equation. This suggests, as already pointed out in ref.[7], the BZ model to be indeed equivalent

(**) Not everybody may like the appearance of such a superposition. It should be clearly noticed, however, that our "helical" wave-functions have to be expressed as superpositions of positive and negative energy solutions (of the Dirac equation) only when we want to regard themselves —as done in this paper— as solutions of the *Dirac* equation (17). On the contrary, when we regard those "helical" wave-functions as solutions of our new, non-linear, Dirac-like equation (6'), no superposition of that kind is needed: cf. ref.[7]; this can be considered a further point in favour of eq.(6').

equivalent to Dirac's theory;
(ii) that there exist also solutions of the DH equation that correspond to a helical path with the light speed c, in which case the velocity operator[16] (as expected in ref.[5]) *can* actually be identified with $u = e_0 - e_2$.

2. *Spin and electron structure* – Let us start by recalling that —as shown by us in ref.[7]— the dynamical behaviour of a spinning point-like particle, that follows a world-line $\sigma = \sigma(\tau)$, must be individuated —besides by the canonical variables (x^μ, p_μ)— also by the *Frenet tetrad*[7,17]

$$e_\mu \;=\; R\gamma_\mu\tilde{R} \;=\; \Lambda^\nu_\mu\gamma_\nu\,; \qquad \Lambda^\nu_\mu \in L^\uparrow_+ \tag{1}$$

where e_0 is parallel to the particle velocity v (even more, $e_0 = v$ when we use as parameter τ the particle proper–time). In eq.(1), the tilde denotes the reversion operation in the STA; namely: $\widetilde{AB} = \tilde{B}\tilde{A}$, and $\tilde{A} = A$ if A is a scalar or a vector, while $\tilde{F} = -F$ when F is a 2-vector. Quantity $R = R(\tau)$ is a "Lorentz rotation"[18] [more precisely, $R \in \mathrm{Spin}_+(1,3) \simeq \mathrm{SL}(2,\mathbb{C})$, and a Lorentz transform of quantity a is given by $a' = Ra\tilde{R}$]. Moreover $R\tilde{R} = \tilde{R}R = 1$. The Clifford STA fundamental unit–vectors γ_μ, incidentally, should not be confused with the Dirac *matrices* γ_μ. Let us also recall that, while the orthonormal vectors $\gamma_\mu \equiv \partial/\partial x^\mu$ constitute a *global* tetrad in Minkowski space-time (associated with a given inertial observer), on the contrary the Frenet tetrad e_μ is defined only along σ, in such a way that e_0 is tangent to σ. At last, it is: $\gamma^\mu = \eta^{\mu\nu}\gamma_\nu$, and $\gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3$ is the volume element of the STA.

If $\Psi_{\mathrm{D}} \in \mathbb{C}^4$ is an ordinary Dirac spinor, in the STA it will be represented by

$$\Psi_{\mathrm{D}} \longrightarrow \Psi \;=\; \psi\varepsilon \;\in\; \mathbb{R}_{1,3}\,, \tag{2}$$

where $\psi \in \mathbb{R}^+_{1,3}$ is called a Dirac–Hestenes spinor[8,14] and ε is an appropriate primitive idempotent of $\mathbb{R}_{1,3}$. It is noticeable that the spinor field ψ carries *all* the essential information contained in Ψ (and Ψ_{D}, and —when it is nonsingular, $\psi\tilde{\psi} \neq 0$— it admits[8,9] a remarkable canonical decomposition in terms of a Lorentz rotation R, a duality transformation[19] $\mathrm{e}^{\beta\gamma_5/2}$, and a dilation $\sqrt{\rho}$:

$$\psi \;=\; \rho^{\frac{1}{2}}\mathrm{e}^{\beta\gamma_5/2}R\,. \tag{3}$$

In eq.(3), the normalization factor ρ belongs to $\mathbb{R}^+$; quantity β is the Takabayasi angle;[20] and $\mathrm{e}^{\beta\gamma_5} = +1$ for the electron (and -1 for the positron). Then, the Frenet tetrad can also be written:

$$\rho e_\mu \;=\; \psi\gamma_\mu\tilde{\psi}\,. \tag{4}$$

Now, let us take as the lagrangian for a classical spinning particle, interacting with the electromagnetic potential A (a 1–vector) the expression[7]

$$\mathcal{L} = \langle \tilde{\psi}\dot{\psi}\gamma_1\gamma_2 + p(\dot{x} - \psi\gamma_0\tilde{\psi}) + eA\psi\gamma_0\tilde{\psi}\rangle_0 , \qquad (5)$$

which is the translation[7] of the BZ lagrangian[2] into the Clifford bundle formalism. In eq.(5), $\langle \quad \rangle_0$ means "the scalar part" of the Clifford product; the dot represents the derivation with respect to parameter τ; and p can be regarded as a Lagrange multiplier. Then, the Euler–Lagrange equations yield a system of three independent equations:

$$\dot{\psi}\gamma_1\gamma_2 + \pi\psi\gamma_0 = 0 \qquad (6)$$
$$\dot{x} = \psi\gamma_0\tilde{\psi} \qquad (7)$$
$$\dot{\pi} = eF \cdot \dot{x} , \qquad (8)$$

where $\pi \equiv p - eA$ is the kinetic momentum; $F \equiv \partial \wedge A$ is the electromagnetic field (a bivector, in Hestenes' language); $\partial = \gamma^\mu \partial_\mu$ is the Dirac operator; and symbols $\cdot$ and $\wedge$ denote the internal and external product, respectively, in the STA. The system (6)–(8) is just that one appeared in ref.[2], but written[7] in terms of the STA language.

Let us pass to the free case, $A = 0$, for which one gets

$$\dot{\psi}\gamma_1\gamma_2 + p\psi\gamma_0 = 0 \qquad (9)$$
$$\dot{x} = \psi\gamma_0\tilde{\psi} \qquad (10)$$
$$\dot{p} = 0 . \qquad (11)$$

If we choose the γ_μ frame in such a way that

$$p = m\gamma_0 \qquad (12)$$

is a constant vector in the γ_0 direction, with $p^2 = m^2$, then for the system (9)–(11) we find the *solution:*

$$\psi(\tau) = \cos(m\tau)\,\psi(0) + \sin(m\tau)\,\gamma_0\psi(0)\,\gamma_0\gamma_1\gamma_2 , \qquad (13)$$

where $\psi(0)$ is a constant spinor, which translates into the Clifford language the solution[2] found by BZ for their analogous system of equations. In the case of solution (13), it holds:

$$v(\tau) \equiv \dot{x}(\tau) = \frac{pH}{m^2}\,[v(0) - \frac{pH}{m^2}]\cos(2m\tau) + \frac{\dot{v}(0)}{2m}\sin(2m\tau) , \qquad (14)$$

which clearly shows the presence of an internal helical motion (*i.e.*, at the classical limit, of the zbw phenomenon).

In eq.(14) we have $H = v \cdot p =$ constant. If the constant is chosen to be m:

$$H = p \cdot v = m , \qquad (15)$$

and —more important— if now $\psi(x)$ is a DH spinor *field* such that its *restriction* to the world–line σ yields $\psi(\tau)$, namely $\psi_{|\sigma}(x) = \psi(\tau)$, *then* eq.(13) writes

$$\psi(x) \;=\; \cos(p\cdot x)\,\psi(0) \;+\; \sin(p\cdot x)\,\gamma_0\,\psi(0)\,\gamma_0\gamma_1\gamma_2\;, \tag{13'}$$

which now is a quantum wave-function, solution, as we are going to see, of the Dirac equation! In fact, it is:

$$\dot{\psi} \;\equiv\; \frac{\mathrm{d}\psi}{\mathrm{d}\tau} \;=\; v^\mu \partial_\mu \psi \;=\; (v\cdot\partial)\,\psi\;, \tag{16}$$

and, for any eigen-spinors ψ of $\hat{p}\psi \equiv \partial\psi\,\gamma_1\gamma_2 = p\psi$, one gets immediately[7] that eq.(9) transforms into the ordinary *Dirac equation* in its Dirac–Hestenes form:[5–8]

$$\partial\psi\gamma_1\gamma_2 \;+\; m\psi\gamma_0 \;=\; 0\;. \tag{17}$$

Notice once more that, while eq.(9) refers to $\psi = \psi(\tau)$, on the contrary the quantum equation (17) refers to the spinor field $\psi = \psi(x)$ [such that $\psi_{|\sigma}(x) = \psi(\tau)$]. Then, eq.(13') is an actual solution of eq.(17); while eq.(13) —if you want— can be said to be a solution of eq.(17) *when* this equation is *restricted* along the stream–line σ (*i.e.*, the world-line of the sub-microscopic object $\mathcal{Q}$). When passing from the classical to the quantum interpretation, one has to pass from considering a single helical path to consider a *congruence* of helical paths.

Let us go back, for a moment, to the system (6)–(9). We may notice that —using eq.(16)— the [total derivative] equation (6) can be rewritten[7] in the [partial derivative] noticeable form:

$$v\cdot\partial\psi\gamma_1\gamma_2 + \pi\psi\gamma_0 \;=\; 0\;, \tag{6'}$$

which is a *non-linear*, Dirac–like, quantum equation.

3. *Time–like cylindrical helix: How to construct solutions of the Dirac equation describing time–like helical motions* – Let us prove, now, that solution $\psi(x)$ of eq.(17), which reduces to the $\psi(\tau)$ given by eq.(13) on the world–line σ of the sub–particle $\mathcal{Q}$, is indeed a superposition$^{(**)}$ of positive and negative energy states (*i.e.*, of particle and antiparticle states[15]). Indeed, $\psi(\tau)$ can be written as:

$$\psi(\tau) \;=\; \tfrac{1}{2}[\psi(0) + \gamma_0\psi(0)\gamma_0]\;\exp(\gamma_1\gamma_2 m\tau) \;+\; \tfrac{1}{2}[\psi(0) - \gamma_0\psi(0)\gamma_0]\;\exp(-\gamma_1\gamma_2 m\tau)\,. \tag{18}$$

But quantities

$$\tfrac{1}{2}[\psi \pm \gamma_0\psi\gamma_0] \;\equiv\; \Lambda_\pm(\psi) \tag{19}$$

are nothing but the positive and negative energy *projection operators*[14] $\Lambda_+,\ \Lambda_- \in \mathrm{End}(\mathbb{R}_{1,3})$, respectively; so that eq.(18) can read

$$\psi(\tau) \;=\; \psi_+(0)\;\exp(\gamma_1\gamma_2 m\tau) \;+\; \psi_-(0)\;\exp(-\gamma_1\gamma_2 m\tau) \tag{20}$$

where $\psi_\pm(0) \equiv \Lambda_\pm[\psi(0)]$, which proves our claim. It follows that any $\psi(x)$, such that $\psi_{|\sigma}(x) = \psi(\tau)$, is a solution of the DH equation (17).

Now, it is clear that one can construct "helical" solutions of the DH equation with time–like velocity. As a concrete example, let us take

$$\psi(0) \equiv \sqrt{\rho_+} + \sqrt{\rho_-}\,\gamma_1\gamma_0 \,; \qquad \sqrt{\rho_\pm} \equiv \sqrt{\rho_\pm}(0)\,. \tag{21}$$

Since

$$\psi(0)\,\tilde{\psi}(0) \;=\; \rho_+ - \rho_- \tag{22}$$

we can put, for simplicity, $\rho_+ - \rho_- = 1$. In this case, $\psi_+(0) = \sqrt{\rho_+}$ and $\psi_-(0) = \sqrt{\rho_-}\,\gamma_1\gamma_0$. Then, by using eq.(14), from eq.(21) it follows that

$$v(0) \;=\; (\rho_+ + \rho_-)\gamma_0 + 2\sqrt{\rho_+\rho_-}\,\gamma_1 \,; \quad \dot{v}(0) \;=\; 4m\sqrt{\rho_+\rho_-}\,\gamma_2 \tag{23}$$

$$H \;=\; m(\rho_+ + \rho_-)\,, \tag{24}$$

and we end up with:

$$v(\tau) \;=\; (\rho_+ + \rho_-)\gamma_0 + 2\sqrt{\rho_+\rho_-}\,[\gamma_1\cos(2m\tau) + \gamma_2\sin(2m\tau)]\,, \tag{25}$$

in which it is actually $v^2(\tau) = 1$ (*time–like* case). For the spin bivector $S = \frac{\hbar}{2}\psi\gamma_2\gamma_1\tilde{\psi}$ and the spin vector $s = \frac{\hbar}{2}\psi\gamma_3\tilde{\psi}$, we have in this case

$$S \;=\; \frac{1}{2}(\rho_+ + \rho_-)\gamma_2\gamma_1 \;+\; \sqrt{\rho_+\rho_-}\,[\gamma_0\gamma_1\sin(2m\tau) - \gamma_0\gamma_2\cos(2m\tau)] \tag{26}$$

$$s \;=\; \frac{1}{2}\gamma_3\,; \qquad [\hbar = 1]\,. \tag{27}$$

The velocity $v(\tau) \equiv \mathrm{d}x/\mathrm{d}\tau$, with $x(\tau) = x^\mu(\tau)\gamma_\mu$, is easily integrated, from eq.(25), to give:

$$x(\tau) \;=\; (\rho_+ + \rho_-)\tau\gamma_0 \;+\; \frac{\sqrt{\rho_+\rho_-}}{m}\,[\gamma_1\sin(2m\tau) - \gamma_2\cos(2m\tau)] \;+\; x_0\,, \tag{28}$$

which is the parametric equation of a helix, whose diameter is $D = 2m\sqrt{\rho_+\rho_-}$. Equation (24) suggests to introduce a renormalized mass $M \equiv m(\rho_+ + \rho_-)$. If one assumed the maximum diameter of that helix to be the electron Compton wave–length, one would get for the new mass M the upper limit $M = m\sqrt{2}$.

It is worth observing that, from eq.(28), for $L \equiv x \wedge p$ we have:

$$L \;=\; \sqrt{\rho_+\rho_-}\,[\gamma_1\gamma_0\sin(2m\tau) \;-\; \gamma_2\gamma_0\cos(2m\tau)]\,, \tag{29}$$

where we neglected the constant contribution $mx_0 \wedge \gamma_0$. Notice that $\dot{L} \neq 0$, so that L alone is not conserved; however, in view of eq.(26), we obtain that the *total* angular momentum J *is* conserved:

$$J \equiv L + S = \frac{1}{2}(\rho_+ + \rho_-)\gamma_2\gamma_1 \; ; \qquad \dot{J} = 0 \; ; \tag{30}$$

which implies a *nutation* of the spin plane. Under the above assumption (that the maximum helix diameter be the electron Compton wave–length), one would get $J \leq \frac{\sqrt{2}}{2}\gamma_2\gamma_1$, that is to say $\mid J \mid \equiv [J \cdot \tilde{J}]^{1/2} \leq \frac{\sqrt{2}}{2}$.

4. *How to construct solutions with light–like helical motions* – Finally, let us show how one can obtain solutions of the DH equation with speed of the helical motion equal to the light speed c. To this aim, it is enough to choose $\rho_+ = \rho_- = 1/2$, so that $\rho_+ - \rho_- = 0$. [In this case, $\psi(0) = (1 + \gamma_1\gamma_0)/\sqrt{2}$ is a singular spinor, actually a Majorana spinor since the charge conjugation operator C is such[14] that $C\psi = \psi\gamma_1\gamma_0$].

In fact, from eq.(14) one then gets

$$v(0) = \gamma_0 + \gamma_1 \; ; \qquad \dot{v}(0) = 2m\gamma_1 \tag{31}$$

$$H = m \; , \tag{32}$$

which yield for the velocity v:

$$v(\tau) = \gamma_0 + \gamma_1 \cos(2m\tau) + \gamma_2 \sin(2m\tau) \; , \tag{33}$$

in which it is now $v^2(\tau) = 0$ (*light–like* case). One can see that, after a convenient rotation, it is possible to write v as $v = e_0 - e_2$, like in Hestenes' papers.[5,6] Moreover, we now have:

$$S = \frac{1}{2}\gamma_2\gamma_1 + \frac{1}{2}[\gamma_0\gamma_1 \sin(2m\tau) - \gamma_0\gamma_2 \cos(2m\tau)] \tag{34}$$

$$s = \frac{1}{2}\gamma_3 \; ; \qquad [\hbar = 1] \; . \tag{35}$$

Integration of eq.(33), now, yields:

$$x(\tau) = \tau\gamma_0 + \frac{1}{2m}[\gamma_1 \sin(2m\tau) - \gamma_2 \cos(2m\tau)] + x_0 \; , \tag{36}$$

and one can verify that in this case the helix diameter is actually the Compton wave–length of the electron! For $L \equiv x \wedge p$ we obtain again

$$L = \frac{1}{2}[\gamma_1\gamma_0 \sin(2m\tau) - \gamma_2\gamma_0 \cos(2m\tau)] \; ; \qquad \dot{L} \neq 0 \; , \tag{37}$$

whilst the conserved quantity is

$$J \equiv L + S = \frac{1}{2}\gamma_2\gamma_1 \; ; \qquad \dot{J} = 0 \; ; \tag{38}$$

which again implies a nutation of the spin plane.

In conclusion, we showed how to construct solutions of Dirac equation associated (at the classical limit) to a helical motion with the light speed c. And, in particular, we got Hestenes' results[5] without his ad–hoc assumptions, by making recourse —however— to a superposition[(**)] of particle and antiparticle[15] solutions. Indeed, the part of the velocity $v(\tau)$ responsible for the zbw, namely $\sqrt{\rho_+\rho_-}\,[\gamma_1\cos(2m\tau)+\gamma_2\sin(2m\tau)]$, is given by an interference term $\psi_+\gamma_0\tilde{\psi}_- + \psi_-\gamma_0\tilde{\psi}_+$, in which quantities $\psi_+ = \sqrt{\rho_+}\,\exp(\gamma_1\gamma_2 m\tau)$ and $\psi_- = \sqrt{\rho_-}\,\gamma_1\gamma_0\,\exp(-\gamma_1\gamma_2 m\tau)$ are the positive–energy [particle] and negative–energy [antiparticle[15]] states.

5. *Acknowledgements* – The authors are grateful for discussions to R. Garattini, L. Lo Monaco, G.D. Maccarrone, J.E. Maiorino, E.C. de Oliveira, M. Pavšič, R. Pucci, F. Raciti, M. Sambataro, Q.A.G. de Souza, and particularly to G. Salesi and S. Sambataro.

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