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SPIN AND ELECTRON STRUCTURE†

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ABSTRACT: In recent times, various independent approaches led to the surprising conclusion that it exists a classical limit of spin (at least in the case of the electron) and that it is related to a helical motion. Considering the possible interest of the previous results, we purpose in this paper to explore their physical and mathematical meaning, by the natural and powerful language of Clifford algebras (which, incidentally, will allow us to unify those different approaches). We meet that the ordinary electron is associated to the mean motion of a point-like "constituent" $\mathcal{Q}$, whose tajectory is —at the classical limit— a cylindrical helix. In particular, we find that the sub-microscopic object $\mathcal{Q}$ obeys a new, non-linear Dirac-like equation, such that —when averaging over an internal cycle, and thus passing to the center-of-mass motion— it transforms into the ordinary Dirac equation (valid, of course, for the electron as a whole).

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1. *Spin and helical motion* – In recent times various different, independent approaches led to the surprising conclusion that it exists a classical limit of spin (at least in the case of the electron) and that it is related to a helical motion.

In particular, Barut and Zanghi[1] considered a classical electron as characterized, besides by the usual pair of conjugate variables (x^μ, p_μ), by a second pair of conjugate classical *spinor* variables $(z, i\bar{z})$ representing internal degrees of freedom, and used as invariant time parameter τ the proper time of the center of mass. Quantity z was a Dirac spinor, while $\bar{z} \equiv z^\dagger \gamma^0$. Then, they introduced for a point–like particle the classical lagrangian $[c = 1]$

$$\mathcal{L} \; = \; \tfrac{1}{2}\lambda i(\dot{\bar{z}}z - \bar{z}\dot{z}) \; + \; p_\mu(\dot{x}^\mu - \bar{z}\gamma^\mu z) \; + \; eA_\mu(x)\bar{z}\gamma^\mu z \quad , \tag{1}$$

where λ has the dimension of an action, γ^μ are the Dirac matrices, and the particle velocity is

$$v^\mu \equiv \bar{z}\gamma^\mu z \; . \tag{1'}$$

We are not writing down explicitly the spinorial indices of $\bar{z}$ and z. Let us consider the simple case of a *free* electron $(A_\mu = 0)$. Then:

$$z(\tau) \;\; = \;\; [\cos m\tau \; + \; i\gamma^\mu \frac{p_\mu}{m} \sin m\tau] \; z(0) \;\; , \tag{2a}$$

$$\bar{z}(\tau) \;\; = \;\; \bar{z}(0) \; [\cos m\tau \; - \; i\gamma^\mu \frac{p_\mu}{m} \sin m\tau] \;\; , \tag{2b}$$

and $p_\mu =$ constant; $p^2 = m^2$; $H = p_\mu \bar{z}\gamma^\mu z \equiv p_\mu v^\mu$; and finally:

$$\dot{x}_\mu \; = \; v_\mu \; = \; \frac{p_\mu}{m^2}H \; + \; [\dot{x}_\mu(0) \; - \; \frac{p_\mu}{m^2}H] \cos 2m\tau \; + \; \frac{\ddot{x}_\mu}{2m} \sin 2m\tau \tag{2c}$$

(which in ref.[1] appeared with two misprints). In connection with this "free" general solution, let us remark that H is a constant of motion so that we can set $H = m$. Solution (2) exhibits the classical analog of the phenomenon known as "zitterbewegung" (zbw): in fact, the velocity $v_\mu \equiv \dot{x}_\mu$ contains the (expected) term p_μ/m plus a term describing an oscillatory motion with the characteristic frequency $\omega = 2m$. The velocity of the *center of mass* will be given by $W_\mu = p_\mu/m$. Notice incidentally that, instead of adopting the variables z and $\bar{z}$, one can work in terms of the spin variables, *i.e.* in terms of the dynamical variables $(x_\mu, v_\mu, \pi_\mu, S_{\mu\nu})$, where:

$$v_\mu = \dot{x}_\mu; \quad \pi_\mu = p_\mu - eA_\mu; \quad S_{\mu\nu} = \frac{1}{4}i\bar{z}[\gamma_\mu, \gamma_\nu]z \; ,$$

so that $\dot{S}_{\mu\nu} = \pi_\mu v_\nu - \pi_\nu v_\mu$ and $\dot{v}_\mu = 4S_{\mu\nu}\pi^\nu$. In the case of a free electron, by varying the action corresponding to $\mathcal{L}$ one finds as generator of space-time rotations the conserved quantity $J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$, where $S_{\mu\nu}$ is just the particle spin.[#1]

Let us explicitly observe that solution (2c) is the equation of a space-time cylindrical helix, *i.e.* it represents in 3–space a helical motion. Let us also stress that this motion describes the point–like particle spin at a classical level. In fact, such a classical system has been shown to describe, after quantization, the Dirac electron. Namely, Barut and Pavšič[2] started from the classical hamiltonian corresponding to eq.(1):

$$\mathcal{H} \;=\; \bar{z}\gamma^{\mu}z(p_{\mu} - eA_{\mu})\;; \tag{3}$$

passed to its quantum version, in which the above quantities are regarded as operators; and considered in the Schroedinger picture the equation

$$i\frac{\partial\varphi}{\partial\tau} \;=\; \mathcal{H}\varphi \tag{4}$$

where $\varphi = \varphi(\tau, x, z)$ is the wave function, and the operators in eq.(3) are $p_{\mu} \longrightarrow -i\partial/\partial x^{\mu}$ and $i\bar{z} \longrightarrow -i\partial/\partial z$. The wave function φ can be expanded in the z variable, around $z = 0$, as follows:

$$\varphi(\tau, x, z) \;=\; \bar{\Phi}(\tau, x) \;+\; \bar{\psi}_{\alpha}(\tau, x)z^{\alpha} \;+\; \bar{\psi}_{\alpha\beta}(\tau, x)z^{\alpha}z^{\beta} + ...$$

where α, β are spinorial indices. Quantity $\bar{\psi} \equiv \psi^{\dagger}\gamma^{0}$ is the Dirac adjoint spinor, which a charge $\bar{e}$ and a mass $\bar{m}$ are attributed to. By disregarding the spin–zero term $\bar{\Phi}(\tau, x)$, and retaining only the second (spin $\frac{1}{2}$) term, thus neglecting also the higher–spin terms, from eq.(4) they then derived the equation

$$i\frac{\partial\bar{\psi}}{\partial\tau} \;=\; (i\partial_{\mu} + \bar{e}A_{\mu})\bar{\psi}\gamma_{\mu}\;\;. \tag{5}$$

Taking the Dirac adjoint of such an equation, with[3] $\bar{e} = -e;\ \bar{m} = -m$, we end up just with the Dirac equation! This confirms, by the way, that the term $\bar{\psi}_{\alpha}z^{\alpha}$ refers to the spin $\frac{1}{2}$ case, i.e. to the case of the electron.[#2]

[#1] Alternatively, Barut and Zanghi,[1] in order to study the internal dynamics of the considered (classical) particle, did split x_{μ} and $v_{\mu} \equiv \dot{x}_{\mu}$ as follows: $x_{\mu} \equiv X_{\mu} + Q_{\mu}$; $v_{\mu} \equiv W_{\mu} + U_{\mu}$ (in such a way that by definition $W_{\mu} = \dot{X}_{\mu}$). In the particular case of a *free* particle, $\dot{W}_{\mu} = 0$; $W_{\mu} = p_{\mu}/m$. One can now interpret X_{μ} and p_{μ} as the c.m. coordinates, and Q_{μ} and $P_{\mu} \equiv mU_{\mu}$ as the *relative* position and momentum, respectively. For a free particle, then, one finds that the internal variables are coordinates oscillating with the zbw frequency $2m$; and that, again, the total angular momentum $J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$ is a constant of motion, quantities $S_{\mu\nu}$ being the spin variables.

[#2] For the case of higher spins, cf. ref.[4].

An alternative approach leading to a classical description of particles with spin is the one by Pavšič,[5,6] who made recourse to the (extrinsic) curvature of the particle world–line in Minkowski space; so that his starting, classical lagrangian [α,β = constants]

$$\mathcal{L} \; = \; \sqrt{\dot{x}^2}\,(\alpha - \beta K^2); \;\; K^\mu \; \equiv \; \frac{1}{\sqrt{\dot{x}^2}}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}}\right) \tag{6}$$

contained the extra (kinematical) term $\beta\sqrt{\dot{x}^2}\,K^2$. The conserved generator of rotations, belonging to lagrangian (6), is once more $J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$, where now, however, $S_{\mu\nu} = \dot{x}_\mu p^{(2)}_\nu - \dot{x}_\nu p^{(2)}_\mu$, while $p^{(2)}_\nu \equiv \partial\mathcal{L}/\partial\ddot{x}^\mu = 2\beta\ddot{x}_\mu/\sqrt{\dot{x}^2}$ is the second–order canonical momentum, conjugated to $\dot{x}^\mu$. The equations of motion, which correspond to eq.(6) for constant K^2, do again admit as solution[5,6] the *helical* motion with the zbw frequency $\omega = 2m$. For a suitable choice of the constant K^2, and when the affine parameter τ is the c.m. proper–time, the equations of motion result to be [$v_\mu \equiv \dot{x}_\mu$]

$$\dot{v}_\mu \; = \; 4S_{\mu\nu}p^\nu; \;\; \dot{S}_{\mu\nu} \; = \; \pi_\mu v_\nu - \pi_\nu v_\mu \; ,$$

i.e., they are the same (for the free case: $A_\mu = 0$) as in the Barut–Zanghi (BZ) model. Moreover, the constraint due to reparametrization invariance can be written as $p_\mu v^\mu - m \; = \; 0$, which reminds us of the Dirac equation; and Poisson brackets are obtained which obey the same algebra as the Dirac γ matrices.[#3] After quantization, the Dirac equation was actually derived in refs.[5,6].

Let us finally mention that the *same* classical equations of motion (and the same Poisson–bracket algebra) have been found also in a third approach, which consists in adding to the ordinary lagrangian an extra term containing Grassmann variables.[8] Even in this case, the Dirac equation was gotten after quantization.

2. *About the electron structure* – Considering the interest of the previous results (which suggest in particular that helical motion as the classical limit of spin), we purpose to explore their physical meaning more deeply, by the very natural —and powerful— lan-

[#3] This conclusion, referring to a half–integer spin, has been criticized in refs.[7] by noticing a seeming analogy with the case of the ordinary orbital angular momentum (which admits integer spin values only). We believe such arguments to not apply to the present situation, where spin is the orbital momentum in the $q^\mu \equiv \dot{x}^\mu$ variables space; since the corresponding, canonically conjugate momentum $p^{(2)}_\mu$ is never a constant of motion. [Namely, when we consider a congruence of world–lines which are solutions of the equations of motion, one finds a non–zero curl of the field $p^{(2)}_\mu(q)$, so that the phase $\int p^{(2)}_\mu \mathrm{d}q^\mu$ is not a single–valued function. Thus, a basis of single–valued wave functions does not exist, nor the operator representation $p^{(2)}_\mu \longrightarrow -i\partial/\partial q^\mu$; and the ordinary arguments about orbital angular momentum do no longer apply].

guage of the Clifford algebras:[9,10] in particular of the "space-time algebra (STA)" $\mathbb{R}_{1,3}$. First of all, let us preliminarily clarify *why* Barut and Zanghi had to introduce the Dirac spinors z in their lagrangian, by recalling that the classical world–line σ of a spinning point–like particle must be individuated —besides by the ordinary coordinates x^μ (and the conjugate momenta p_μ)— by the Frenet tetrad[11]

$$e_\mu \;=\; R\gamma_\mu\tilde{R} \;=\; \Lambda^\nu_\mu\gamma_\nu\,; \quad \Lambda^\nu_\mu \in L^\uparrow_+ \tag{7}$$

where e_0 is parallel to the particle velocity v (even more, $e_0 = v$ when we use as parameter τ the particle proper–time); the tilde represents the *reversion*#4; and $R = R(\tau)$ is a "Lorentz rotation" [more precisely, $R \in \mathrm{Spin}_+(1,3)$, and a Lorentz transform of quantity a is given by $a' = Ra\tilde{R}$]. Moreover $R\tilde{R} = \tilde{R}R = 1$. The Clifford STA fundamental unit–vectors γ_μ should not be confused with the Dirac *matrices* γ_μ. Let us also recall that, while the orthonormal vectors $\gamma_\mu \equiv \partial/\partial x^\mu$ constitute a *global* tetrad in Minkowski space-time (associated with a given inertial observer), on the contrary the Frenet tetrad e_μ is defined only along σ, in such a way that e_0 is tangent to σ. At last, it is: $\gamma^\mu = \eta^{\mu\nu}\gamma_\nu$, and $\gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3$.

Notice that $R(\tau)$ does contain all the essential information carried by a Dirac spinor. In fact, out of R, a "Dirac–Hestenes" (DH) spinor[12] ψ_{DH} can be constructed as follows:

$$\psi_{\mathrm{DH}} \;=\; \rho^{\frac{1}{2}}\mathrm{e}^{\beta\gamma_5/2}R \tag{8}$$

where ρ is a normalization factor; and $\mathrm{e}^{\beta\gamma_5/2} = +1$ for the electron (and -1 for the positron); while, if ε is a primitive idempotent of the STA, any Dirac spinor ψ_{D} can be represented in our STA as:[13]

$$\psi_{\mathrm{D}} \;=\; \psi_{\mathrm{DH}}\,\varepsilon\,. \tag{9}$$

For instance, the Dirac spinor z introduced by BZ is obtained from the DH spinor#5 by the choice $\varepsilon = \frac{1}{2}(1+\gamma_0)$. Incidentally, the Frenet frame can also write $\rho e_\mu = \psi_{\mathrm{DH}}\gamma_\mu\tilde{\psi}_{\mathrm{DH}}$.

Let us stress that, to specify how does the Frenet tetrad rotate as τ varies, one has to single out a particular $R(\tau)$, and therefore a DH spinor ψ_{DH}, and eventually a Dirac spinor ψ_{D}. This makes intuitively clear why the BZ Dirac–spinor z provides a good description of the "spin motion" of a classical (point–like) particle.

Let us now repeat what precedes on a more formal ground. In the following, unless differently stated, we shall indicate the DH spinors ψ_{DH} simply by ψ.

#4 The main anti-automorphism in $\mathbb{R}_{1,3}$ (called reversion), denoted by the tilde, is such that $\widetilde{AB} = \tilde{B}\tilde{A}$, and $\tilde{A} = A$ when A is a scalar or a vector, while $\tilde{F} = -F$ when F is a 2-vector.

#5 The DH spinors can be regarded as the *parent* spinors, since all the other spinors of common use among physicists are got from them by operating as in eq.(9). We might call them "the *fundamental* spinors".

Let us translate the BZ lagrangian into the Clifford language.

In eq.(1) quantity $z^{\mathrm{T}} = (z_1 \;\; z_2 \;\; z_3 \;\; z_4)$ is a Dirac spinor $\tau \longrightarrow z(\tau)$, and $\bar{z} = z^{\dagger}\gamma^0$. To perform our translation, we need a matrix representation of the Clifford space-time algebra; this can be implemented by *representing* the fundamental Clifford vectors $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ by the *ordinary* Dirac matrices $\boldsymbol{\gamma}_\mu$. Choosing:

$$\gamma_0 \longrightarrow \boldsymbol{\gamma}_0 \equiv \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \; ; \quad \gamma_i \longrightarrow \boldsymbol{\gamma}_i \equiv \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} ,$$

the representative in $I\!R_{1,3}$ of Barut–Zanghi's quantity z is

$$z \longrightarrow \Psi \equiv \psi\,\varepsilon \; ; \quad \varepsilon \equiv \tfrac{1}{2}(1+\gamma_0) \tag{10}$$

where ψ, and $\tilde{\psi}$, are represented by

$$\psi = \begin{pmatrix} z_1 & -\bar{z}_2 & z_3 & \bar{z}_4 \\ z_2 & \bar{z}_1 & z_4 & -\bar{z}_3 \\ z_3 & \bar{z}_4 & z_1 & -\bar{z}_2 \\ z_4 & -\bar{z}_3 & z_2 & \bar{z}_1 \end{pmatrix} \; ; \quad \tilde{\psi} = \begin{pmatrix} -\bar{z}_1 & \bar{z}_2 & -z_3 & -z_4 \\ -z_2 & z_1 & -z_4 & z_3 \\ -\bar{z}_3 & -\bar{z}_4 & \bar{z}_1 & -\bar{z}_2 \\ -z_4 & z_3 & z_2 & z_1 \end{pmatrix} . \tag{11}$$

The translation of the various terms in eq.(1) is then:

$$\begin{array}{lcl} \frac{1}{2}i(\dot{\bar{z}}z - z\dot{z}) & \longrightarrow & \langle\tilde{\psi}\dot{\psi}\gamma_1\gamma_2\rangle_0 \\ p_\mu(\dot{x}^\mu - \bar{z}\gamma^\mu z) & \longrightarrow & \langle p(\dot{x} - \psi\gamma_0\tilde{\psi})\rangle_0 \\ eA_\mu\bar{z}\gamma^\mu z & \longrightarrow & e\,\langle A\psi\gamma_0\tilde{\psi}\rangle_0 \, , \end{array}$$

where $\langle\quad\rangle_0$ means "the scalar part" of the Clifford product. Thus, the lagrangian $\mathcal{L}$ in the Clifford formalism is

$$\mathcal{L} \;=\; \langle\tilde{\psi}\dot{\psi}\gamma_1\gamma_2 \;+\; p(\dot{x} - \psi\gamma_0\tilde{\psi}) \;+\; eA\psi\gamma_0\tilde{\psi}\rangle_0 \, , \tag{12}$$

which is analogous, incidentally, to Krüger's lagrangian[14] (apart from a missprint).

As we are going to see, by "quantizing" it, also in the present formalism it is possible (and, actually, quite easy) to derive from $\mathcal{L}$ the Dirac–Hestenes equation:[#6]

$$\partial\,\psi(x)\,\gamma_1\gamma_2 \;+\; m\,\psi(x)\,\gamma_0 \;+\; e\,A(x)\,\psi(x) \;=\; 0 \, , \tag{13}$$

which is nothing but the ordinary Dirac equation written down in the Clifford formalism.[9,12] Quantity $\partial = \gamma^\mu\partial_\mu$ is the Dirac operator. Let us notice that p in eq.(12)

[#6] Observe that in eq.(12) it is $\psi = \psi(\tau)$, while in eq.(13) we have $\psi = \psi(x)$ with $\psi(x)$ such that its *restriction* to the world–line σ coincides with $\psi(\tau)$. Below, we shall meet the same situation, for instance, when passing from eq.(14a) to eqs.(16)–(16').

can be regarded as a Lagrange multiplier, when the velocity $v = \dot{x}$ is represented by $\psi\gamma_0\tilde{\psi}$. The dynamical variables are then $(\psi, \tilde{\psi}, x, p)$, and the Euler–Lagrange equations yield a system of three independent equations:

$$\dot{\psi}\gamma_1\gamma_2 + \pi\psi\gamma_0 = 0 \tag{14a}$$
$$\dot{x} = \psi\gamma_0\tilde{\psi} \tag{14b}$$
$$\dot{\pi} = eF \cdot \dot{x} \tag{14c}$$

where $F \equiv \partial \wedge A$ is the electromagnetic field (a bivector, in Hestenes' language) and $\pi \equiv p - eA$ is the kinetic momentum. [Notice incidentally, from eq.(14b), that $\dot{x}^2 = \rho(\tau)$ so that the parameter τ cannot be the proper time of the considered particle, unless $\rho(\tau) = 1$. On the contrary, τ *can be* the c.m. proper–time].

At this point, let us consider a velocity vector *field* $V(x)$ together with its integral lines (or stream–lines). Be σ the stream–line along which a particle moves (*i.e.*, the particle world–line). Then, V is required to be such that its restriction to the world–line σ is the ordinary velocity $v = v(\tau)$ of the considered particle.

If we moreover recall[15,16] that any Lorentz "rotation" R can be written $R = e^{\mathcal{F}}$, where $\mathcal{F}$ is a *bivector*, then along any stream–line σ we shall have:[8]

$$\dot{R} \equiv \frac{\mathrm{d}R}{\mathrm{d}\tau} = \frac{1}{2}v^\mu \Omega_\mu R = \frac{1}{2}\Omega R\,, \tag{15}$$

with $\partial_\mu R = \Omega_\mu R/2$, where $\Omega_\mu \equiv 2\partial_\mu \mathcal{F}$, and where $\Omega \equiv v^\mu \Omega_\mu$ is the angular–velocity bivector (also known, in differential geometry, as the "Darboux bivector"). Therefore, for the tangent vector *along any line* σ we obtain the relevant relation:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = v^\mu \partial_\mu = v \cdot \partial$$

The [total derivative] equation (14a) thus becomes:[#6]

$$v \cdot \partial\psi\gamma_1\gamma_2 + \pi\psi\gamma_0 = 0\,, \tag{16}$$

which is a non-linear [partial derivative] equation, as it is easily seen by using eq.(14b) and rewriting it in the noticeable form

$$(\psi\gamma_0\tilde{\psi}) \cdot \partial\psi\gamma_1\gamma_2 + \pi\psi\gamma_0 = 0\,. \tag{16'}$$

Equation (16') constitutes a new *non-linear* Dirac–like equation.

Let us pass now to the *free* case ($A_\mu = 0$), when eq.(14a) may be written

$$\dot{\psi}\gamma_1\gamma_2 + p\psi\gamma_0 = 0\,, \tag{14'a}$$

and admits some simple solutions. Actually, in this case p is constant [cf. eq.(14c)] and one can choose the γ_μ frame so that $p = m\gamma_0$ is a constant vector in the direction γ_0.

Since $\dot{x} = \psi\gamma_0\tilde{\psi}$, it follows that

$$v = \frac{1}{m}\psi p\tilde{\psi}\,; \qquad \frac{p}{m} = \psi^{-1}v\tilde{\psi}^{-1}\,. \tag{17}$$

The mean value of v over a zitterbewegung period is then given by the remarcable relation

$$\langle v\rangle_{\rm zbw} = \psi^{-1}v\tilde{\psi}^{-1} \tag{18}$$

which resembles the ordinary quantum-mechanical mean value for the wave-function ψ^{-1} (recall that in Clifford algebra any ψ has its inverse).

Let us explicitly stress that, due to the first one $[z \longrightarrow \psi\varepsilon]$ of eqs.(10), the results found by BZ for z are valid as well for ψ in our formalism. For instance, for BZ [cf. eq.(1')] it was $v^\mu = \bar{z}\gamma^\mu z \equiv v^\mu_{\rm BZ}$, while in the Clifford formalism [cf. eq.(11)] it is $v = \psi\gamma^0\psi = \langle\gamma^0\tilde{\psi}\gamma^\mu\psi\rangle_0\gamma_\mu = v^\mu_{\rm BZ}\gamma_\mu$. As a consequence, σ refers in general to a cylindrical helix (for the free case) also in our formalism.

Going back to eq.(14'a), by the second one of eqs.(17) we finally obtain our non-linear (free) Dirac-like equation in the following form:

$$v\cdot\partial\psi\gamma_1\gamma_2 + m\psi^{-1}v\tilde{\psi}^{-1}\psi\gamma_0 = 0\,. \tag{19}$$

Equation (19) is the fundamental equation of motion, holding for the free particle world-line σ. Let us explicitly notice that, at a classical level, the spinorial equation (19) is valid for *the* particle world-line. When wishing to pass from the classical to the quantum level, a change in interpretation is necessary; and eq.(19) is to be regarded as valid for *a congruence* of world lines, *i.e.*, for a congruence of stream-lines of the velocity field $V = V(x)$. In the quantum case, the "particle" can follow any of those integral lines, with probability amplitude ρ.

In particular, eq.(19) will hold —as we have seen— for the (BZ) helical motion. Notice that, since [cf. eq.(8)] it is $\psi = \rho^{\frac{1}{2}}e^{\beta\gamma_5/2}R$, in the case of the helical path solution the Lorentz "rotation" R will be the product of a pure space rotation and a boost.

It is important to observe that, if we replace v in eq.(19) by its mean value over a zbw period, eq.(18), then we end up with the Dirac–Hestenes equation [*i.e.*, the ordinary Dirac equation!], valid now for the center-of-mass world-line. In fact, since $\langle v\rangle_{\rm zbw} = p/m$, eq.(19) yields

$$p\cdot\partial\psi\gamma_1\gamma_2 + mp\psi\gamma_0 = 0\,,$$

that is to say, taking its scalar part,

$$\langle p(\partial\psi\gamma_1\gamma_2 + m\psi\gamma_0)\rangle_0 = 0\,,$$

which is satisfied once it holds the *ordinary* Dirac equation (in its Dirac–Hestenes form):

$$\partial\psi\gamma_1\gamma_2 + m\psi\gamma_0 = 0\,. \tag{20}$$

In conclusion, our non-linear Dirac–like equation (19) is a "sub-microscopic" quantum–relativistic equation, holding along the helical paths σ. On the contrary, the ordinary (linear) Dirac equation can be regarded as the equation describing the *mean motion* of the electron (since it has been obtained from eq.(19) by averaging over the zbw period).

The last point is confirmed by the fact that, in the free case, the "BZ equation", eq.(14'a), admits also a trivial solution σ_0, corresponding to rectilinear motion. And for this world–line the ordinary Dirac equation directly holds.

3. *A very simple solution* – In the free case, the equation

$$\dot{\psi}\gamma_1\gamma_2 + p\psi\gamma_0 = 0 \tag{14'a}$$

admits also a very simple solution (the limit of the ordinary helical paths when their radius r tends to zero), which —incidentally— escaped BZ's attention. In fact, let us recall that in this case eq.(14c) implies p to be constant, and we were thus able to choose $p = m\gamma_0$. Then, any pure *space* rotation is solution of eq.(14'a). Actually, if ψ is a pure space rotation, it holds $\psi\gamma_0\tilde{\psi} = \gamma_0\rho$, and eq.(14'a) becomes $\psi^{-1}\dot{\psi} = m\gamma_1\gamma_2$, so that one verifies

$$\psi = \rho^{\frac{1}{2}}\exp[-\gamma_2\gamma_1 m\tau] = \psi(0)\exp[-\gamma_2\gamma_1 m\tau] \tag{21}$$

to be a (very simple) solution of its. Moreover, from eq.(14b) it follows

$$v \equiv \dot{x} = \rho\gamma_0$$

and we can set $\rho = 1$, which confirms that our trivial solution (21) corresponds to rectilinear uniform motion, and that τ in this case is just the proper–time along the particle world–line σ_0.

But, recalling eq.(16), equation (14'a) in the free case may read

$$v \cdot \partial\psi\gamma_1\gamma_2 + mv\psi\gamma_0 = 0$$

and finally (taking its scalar part)

$$\langle v(\partial\psi\gamma_1\gamma_2 + m\psi\gamma_0)\rangle_0 = 0 \,,$$

which is satisfied once it holds the equation

$$\partial\psi\gamma_1\gamma_2 + m\psi\gamma_0 = 0 \tag{20'}$$

which, as expected, is just the Dirac equation in the Clifford formalism.

Before going on, we want to explicitly put forth the following observation, which can be relevant from the physical point of view. Let us first recall that in our formalism the

(Lorenz force) equation of motion for a charged particle moving with velocity w in the electromagnetic field F is

$$\dot{w} \; = \; \frac{e}{m} F \cdot w \; . \tag{22}$$

Now, for *all* the free–particle solution of the BZ model in the Clifford language, it holds the "Darboux relation":

$$\dot{e}_\mu \; = \; \Omega \cdot e_\mu \; , \tag{23}$$

so that the "sub–microscopic" point–like object $\mathcal{Q}$, moving along the helical path σ, is endowed [cf. eq.(15)] with the angular–velocity bivector

$$\Omega \; = \; \tfrac{1}{2} \, \dot{e}_\mu \wedge e^\mu \; = \; \tfrac{1}{2} \, \dot{e}_\mu e^\mu \; , \tag{24}$$

as it follows by recalling that e_μ can always be written, like in eq.(7), as $e_\mu \; = \; R\gamma_\mu\tilde{R}$. Finally, let us observe that eq.(23) yields in particular $\dot{e}_0 \; = \; \Omega \cdot e_0$, which is formally identical to eq.(22). It follows that the bivector field Ω can be regarded as a kind of *internal* electromagnetic–like field, which keeps the "sub–microscopic" object $\mathcal{Q}$ moving along the helix.[17] In other words, $\mathcal{Q}$ may be considered as confining itself along σ [*i.e.*, along a circular orbit, in the electron c.m.], via the generation of the internal, electromagnetic–like field

$$F_{\rm int} \; \equiv \; \frac{m}{2e} \, \dot{e}_\mu \wedge e^\mu \; .$$

4. *About Hestenes' interpretation* – In connection with our new equation (16'), or rather with its (free) form (19), we met solutions corresponding —in the free case— to helical motions with constant radius r; as well as a limiting solution, eq.(21), for $r \to 0$. We have seen above that the latter is a solution also of the ordinary (free) Dirac equation.

Actually, the solution of the Dirac equation for a free electron in its rest frame can be written in the present formalism as:[9]

$$\psi(x) \; = \; \psi(0) \exp[-\gamma_2\gamma_1 m\tau] \tag{25}$$

which coincides with our eq.(21) along the world–line σ_0.

It is interesting to examine how in refs.[9], even if confronting themselves only with the usual (linear) Dirac equation and with eq.(25), those authors were led to propose for the electron the existence of internal helical motions. It was first noticed that, in correspondence with solution (25), it is $\dot{e}_0 \; = \; 0$; $\dot{e}_3 \; = \; 0$, so that e_0, e_3 are constants; while e_2, e_3 are *rotating*[9] in the e_2e_1 plane (*i.e.*, in the spin plane[9]) with the zbw frequency $\omega = 2m$ $[\hbar = c = 1]$:

$$\begin{aligned} e_1(\tau) \; &= \; e_1(0) \, \cos 2m\tau \; + \; e_2(0) \, \sin 2m\tau \\ e_2(\tau) \; &= \; e_2(0) \, \cos 2m\tau \; - \; e_1(0) \, \sin 2m\tau \end{aligned} \tag{26}$$

as follows from eq.(7) with $R = \exp[-\gamma_2\gamma_1 m\tau]$. Incidentally, by recalling that in the Clifford formalism the spin bivector S is given by

$$S = R\gamma_2\gamma_1\tilde{R}\frac{\hbar}{2} = e_2e_1\frac{\hbar}{2}\,, \tag{27}$$

whilst the angular–velocity bivector Ω is given by eq.(24), one then gets

$$p\cdot v = \Omega\cdot S = m\,, \tag{28}$$

which seems to suggest[9] the electron rest–mass to have a (sub-microscopic) kinetic origin! Hestenes could do nothing but asking himself (following Lorentz[18,19]): what is rotating?

If something was rotating inside the electron, since $v = e_0$ refers *in this case* to the electron mean motion, *i.e.*, is the velocity of the whole electron, in refs.[9] it was *assumed* for the velocity of the internal "constituent" $\mathcal{Q}$ the value

$$u = e_0 - e_2\,; \qquad e_0 \equiv v \tag{29}$$

which is a light–like quantity. Eq.(29) represents a null vector since ${e_0}^2 = +1$; ${e_2}^2 = -1$ (quite analogously, they could have chosen $u = e_0 - e_1$). Notice that the ordinary Dirac current will correspond to the average $\bar{u}$ of velocity u over a zbw period; *i.e.*, due to eqs.(26), to: $\bar{u} = e_0 = v$, as expected. However, if we set $u \equiv \dot{\zeta}$, then one gets[9]

$$\zeta(\tau) = (\mathrm{e}^{\Omega\tau} - 1)R_0 + \zeta_0\,, \tag{30}$$

which is just the parametric equation of a light–like helix $\zeta(\tau) = x(\tau) + R(\tau)$ centered on the stream–line σ_0 with radius R_{H} given by $[\omega = 2m]$:

$$R_{\mathrm{H}}(\tau) = \mathrm{e}^{\Omega\tau}R_0 = \frac{-e_1}{\omega} = \frac{-\dot{u}}{\omega^2}\,. \tag{31}$$

The parameters in eqs.(30)-(31) were chosen by Hestenes[9] in *such a way* that the helix diameter equals the Compton wavelength of the electron, and the angular momentum of the zbw motion yields the correct electron spin. It is possible, incidentally, that such a motion be also at the origin of the electric charge; in any case, we saw that, if the electron is associated with a clock-wise rotation, then the positron will be associated with an anti–clock-wise rotation, with respect to the motion direction.

Choice (29), of course, suits perfectly well with the standard discussions about the velocity operator[20] for the Dirac equation, and as a consequence does naturally allow considering that helical motion as the classical analog of zbw.

However, such approach by Hestenes, even if inspiring and rich of physical intuition, seems rather *ad hoc* and in need of a few assumptions [particularly with regard to eqs.(29) and (31)]. In our opinion, to get a sounder theoretical ground, it is to be linked with our eqs.(16'), (19) and the related discussion above.

In the original BZ model, there exist particular initial conditions yielding *as a limiting case* a light–like velocity $v^\mu = \bar{z}\gamma^\mu z$; that is, such that $v^\mu v_\mu = 0$, as it can be checked from eq.(2b). However, in our Clifford formalism we assumed $\psi\tilde{\psi} \neq 0 \longleftrightarrow \bar{z}z \neq 0$, and our v was bound to be time–like. We may, of course, allow for a light–like v; but, in this case, we have to change the explicit representation of the velocity vector: for instance, to comply with Hestenes' assumption (29), we have just to set: $v = \psi\gamma_0\tilde{\psi} - \psi\gamma_2\tilde{\psi}$, so that eq.(12) has to be rewritten accordingly as follows:

$$\mathcal{L} = \langle \tilde{\psi}\dot{\psi}\gamma_1\gamma_2 + p(\dot{x} - \psi\gamma_0\tilde{\psi} + \psi\gamma_2\tilde{\psi}) + eA(\psi\gamma_0\tilde{\psi} - \psi\gamma_2\tilde{\psi})\rangle_0 \, . \tag{12'}$$

But choice (29) of ref.[9] is just one possibility; for example, one might choose

$$u = e_0 - e_1 - e_2 \tag{32}$$

and in this case we would get for the rotating point–object $\mathcal{Q}$ a *space–like* velocity u, whose mean value $\bar{u}$ would still be

$$\bar{u} = e_0 \, .$$

This would correspond in our Clifford formalism to representing the velocity vector as $v = \psi\gamma_0\tilde{\psi} - \psi\gamma_1\tilde{\psi} - \psi\gamma_2\tilde{\psi}$, and modifying the lagrangian (12') accordingly.

In any case, one may observe that also with the choice (29) the light–like u results from the composition of a time–like velocity e_0 with a *space–like* velocity e_2. Some interesting work in this direction did already appear in refs.[21], by Campolattaro.

5. *Further remarks* – We mentioned, at the beginning, about further methods for introducing an helical motion as the classical limit of the "spin motion". We want here to show, at last, how to represent in the Clifford formalism the *extrinsic curvature* approach,[5,6] corresponding to lagrangian (6), due to its interest for the development of the present work.

Let us first recall that in classical differential geometry one defines[11] the Frenet frame $\{e_\mu\}$ of a non-null curve σ by the so–called Frenet equations, which with respect to proper time τ write [besides $\dot{x} = e_0 = v$]:

$$\ddot{x} = \dot{e}_0 = K_1 e^1 \, ; \quad \dot{e}_1 = -K_1 e^0 + K_2 e^2 \, ; \quad \dot{e}_2 = -K_2 e^1 + K_3 e^3 \, ; \quad \dot{e}_3 = -K_3 e^2 \tag{33}$$

where the i-th curvatures K_i (i=1,2,3) are scalar functions chosen in such a way that ${e_j}^2 = -1$, with j=1,2,3. Quantity K_1 is often called curvature, and K_2, K_3 torsions (recall that in the 3–dimensional space one meets only K_1 and K_2, called curvature and torsion, respectively). Inserting eqs.(33) into eq.(24), we get for the Darboux (angular-velocity) bivector:

$$\Omega = K_1 e^1 e^0 + K_2 e^2 e^1 + K_3 e^3 e^2 \, , \tag{34}$$

so that one can build the following scalar function

$$\Omega \cdot \Omega = K_1^2 - K_2^2 - K_3^2 = (\dot{e}_\mu \wedge e^\mu) \cdot (\dot{e}_\nu \wedge e^\nu) \, . \tag{34'}$$

At this point, one may notice that the square, K^2, of the "extrinsic curvature" entering eq.(6) is equal to $-K_1^2$, so that the lagrangian adopted in refs.[5] results —after the present analysis— to take advantage only of the first part,

$$\ddot{x}^2 \;=\; \dot{e}_0^2 \;=\; -K_1^2 \,,$$

of the Lorentz invariant (34'). On the contrary, in our formalism the whole invariant $\Omega \cdot \Omega$ suggests itself as the suitable, complete lagrangian for the problem at issue; and in future work we shall exploit it, in particular comparing the expected results with Plyushchay's.[22]

For the moment, let us stress here only the possibly important result that the lagrangian $\mathcal{L} \;=\; \Omega \cdot \Omega$ does coincide (factors apart) along the particle world–line σ with the auto–interaction term[23]

$$\theta^5 \; (\mathrm{d}\theta^\mu \wedge \theta_\mu) \cdot (\mathrm{d}\theta^\nu \wedge \theta_\nu)$$

of the *Einstein–Hilbert lagrangian density* written (in the Clifford bundle formalism) in terms of tetrads of 1-form fields θ^μ. Quantity $\theta^5 \equiv \theta^0\theta^1\theta^2\theta^3$ is the volume element.

Finally, we can examine within our formalism the third approach: that one utilizing Grassmann variables.[8] For instance, if we recall that the Grassmann product is nothing but the external part $A_\mathrm{r} \wedge B_\mathrm{s} \;=\; \langle A_\mathrm{r} B_\mathrm{s} \rangle_{|\mathrm{r}-\mathrm{s}|}$ of the Clifford product (where A_r, B_s are a r-vector and a s-vector, respectively), then the Ikemori lagrangian[8] can be immediately translated into the Clifford language and shown to be equivalent to the BZ lagrangian, apart from the constraint $p^2 = m^2$.

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