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Free Maxwell Equations, Dirac Equation, and Non-Dispersive de Broglie Wave–packets(*)

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ABSTRACT: In this paper we show that some elements of de Broglie's double solution theory seem to arise rather naturally from the *equivalence* between the free Maxwell equations and a non-linear Heisenberg–like spinor equation (NLSE). Such an equivalence is proved below, in the Clifford bundle formalism, by making recourse to the Rainich–Misner–Wheeler theorem. Our NLSE admits various types of interesting solutions. First, it admits for instance plane–wave solutions which solve also the free Dirac–Hestenes equation (representing the ordinary Dirac equation in the Clifford bundle). Second, our NLSE admits other solutions which are as well solutions of the (linear) spinor equation for magnetic monopoles by Lochak. Finally, it admits a third kind of solutions (non-dispersive de Broglie spinor wave–packets), such that each one of their spinor components satisfies also the equation by Gueret and Vigier, containing a non-linear term of the quantum potential type. A possible conclusion is that the reality of the de Broglie waves ought to be taken more seriously.

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1. INTRODUCTION

Louis de Broglie made several important contributions to modern physics. His legacy is a vast set of original ideas which are presently being revisited, as, for example, in the works by Barut[1], Gueret and Vigier[2], and Mackinnon[3]. It is the purpose of this paper to present some of our results in connection with de Broglie's theory of double solution.

As extensively discussed by Hestenes,[4] any interpretation of non-relativistic quantum mechanics must be consistent in particular with the interpretation of Dirac's equation, the relativistic quantum–mechanical equation for the electron. To this aim, we wish to have recourse to the natural, suitable, powerful language of Clifford algebras. Thus we shall adopt throughout this paper the Clifford bundle of the differential forms over Minkowski space-time.

In what follows, the starting point is the equivalence between the *free* Maxwell equations, for a non-null field, and a non-linear Heisenberg–like spinor equation (NLSE); such an equivalence is proved below, in the Clifford bundle formalism, by the Rainich–Misner–Wheeler theorem. Afterwards, we show that our NLSE admits various types of interesting solutions. First, it admits (plane–wave) solutions, which solve also the free Dirac–Hestenes equation, representing the ordinary Dirac equation in the Clifford bundle; and is expected to possess (more complicated) solutions satisfying also the Dirac–Hestenes equation with an external electromagnetic field. Second, we shall show that our NLSE has other solutions that solve as well the (linear) spinor equation for magnetic monopoles by Lochak. Finally, our NLSE will be shown to admit a third kind of solutions (non-dispersive de Broglie spinor wave–packets), such that each one of their spinor components satisfies also the equation by Gueret and Vigier, containing a non-linear term of the quantum potential type.

Let us summarize the content of this article. In Sect.**2** we briefly review the Clifford bundle approach to Maxwell and Dirac equations. Then, in Sects.**3** and **4** we prove the equivalence of Maxwell equations and the NLSE, and later on discuss the plane–wave solutions of the latter, as well as its solutions solving also Lochak's equation. In Sect.**5** we construct —which is our main aim— the NLSE solutions which are non-dispersive de Broglie wave–packets. Some conclusions are finally drawn in Sect.**6**.

2. MAXWELL AND DIRAC EQUATIONS IN THE CLIFFORD BUNDLE

Let $\mathcal{C}\ell(M,\widehat{g})$ denote the Clifford bundle of the differential forms over Minkowski space-time. The space-time algebra $I\!R_{1,3}$ is the typical fiber of this bundle[5,6,7]. Cross–sections $e \in \sec TM$ $[\gamma \in \sec T^*M]$ of the tangent TM [cotangent T^*M] bundle are called 1-vector (1-form) fields. Let be $\{e_\mu\} \in \sec TM$ a basis of TM and $\{\gamma^\mu\} \in \sec T^*M = \sec \Lambda^1 M \subset \sec \mathcal{C}\ell(M,\widehat{g})$ the dual basis, satisfying $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$; with $\eta^{\mu\nu} =$ diag(1,-1,-1,-1). The Dirac operator ∂, acting on sections of $\mathcal{C}\ell(M,\widehat{g})$, is: $\partial = d - \delta$, where d is the differential and δ the Hodge codifferential operator, so that: $\partial = \gamma^\mu \nabla_\mu$, where ∇ is the Levi–Civita connection of $g = \eta_{\mu\nu}\gamma^\mu \otimes \gamma^\nu$ ($\hat{g} = \eta^{\mu\nu} e_\mu \otimes e_\nu$). We can choose for simplicity $\{\gamma^\mu\}$ such that $\nabla_\mu = \partial_\mu$; thus $\partial = \gamma^\mu \partial_\mu$.

The representative of the *Maxwell equations* in $C\ell(M, \hat{g})$ is then:

$$\partial F = J \,, \tag{1}$$

where the electromagnetic field is $F \in \sec \Lambda^2 M \subset \sec C\ell(M, \hat{g})$, while the electric current $J \in \sec \Lambda^1 M \subset \sec C\ell(M, \hat{g})$. This form of Maxwell equations is probably due to Riesz[8]. When $J = 0$, the *free* Maxwell equations assume of course the simple form $\partial F = 0$.

The representative in $C\ell(M, \hat{g})$ of the *Dirac equation*, in the presence of electromagnetic field, is:

$$\partial \psi \gamma^1 \gamma^2 + \frac{mc}{\hbar} \psi \gamma^0 + \frac{e}{\hbar c} A \psi = 0 \,, \tag{2}$$

which is due to Hestenes[9,10]. The object $\psi \in \sec(\Lambda^0 M + \Lambda^2 M + \Lambda^4 M) \subset \sec C\ell(M, \hat{g})$ is an "operator spinor" in the terminology of ref.[11]; and quantity $A \in \sec \Lambda^1 M \subset \sec C\ell(M, \hat{g})$ is the electromagnetic potential.

We call the spinor ψ a *Dirac–Hestenes* (DH) spinor field. Let us mention, however, that due to its basic role we could call it the "*fundamental* spinor field". In fact, the ordinary (covariant) Dirac spinor field in $C\ell(M, \hat{g})$ is nothing but the algebraic spinor field $\Psi = \psi e$, quantity e being an idempotent, of the form $e = \frac{1}{2}(1 + \gamma^0)$, as shown in[11]. And any other spinors, used by physicists, have in $C\ell(M, \hat{g})$ an analogous representation.

In ref.[12] it can be found the proof that eq.(1) and eq.(2) are indeed the representatives of the usual Maxwell and Dirac equations in $C\ell(M, \hat{g})$. The DH spinor field can be written in the canonical form

$$\psi = \rho^{1/2} e^{\gamma^5 \beta / 2} R \,, \tag{3}$$

where $\rho, \beta \in \sec \Lambda^0 M \subset \sec C\ell(M, \hat{g})$; and $R \in Spin_+(1,3) \simeq SL(2, \mathbb{C})$, which implies $RR^* = R^* R = 1$, where $*$ (called reversion) is the principal anti-automorphism in $\mathbb{R}_{1,3}$: so that a Lorentz transformation of an arbitrary element $a \in \sec C\ell(M, \hat{g})$ is given by $a \longmapsto RaR^*$. Finally, $\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$ is the volume element.

3. ABOUT THE (FUNDAMENTAL) DIRAC–HESTENES SPINORS

In order to prove the equivalence of the free Maxwell equations (without sources) with a Heisenberg–like spinor equation, we need first the following result:[13,14]

THE RAINICH-MISNER-WHEELER THEOREM: Let us define as "extremal field" any electromagnetic field for which the magnetic [electric] field is zero and the electric [magnetic] field is parallel to one coordinate axis. Then, at any point of Minkowski space-time any non-null electromagnetic field can be reduced to an extremal field by a Lorentz transformation and a duality rotation.

An easy, new proof of this theorem, using Clifford algebras, can be found in ref.[15]. Now, let us observe that under a Lorentz transformation the electromagnetic field F transforms into $F' = LFL^*$ and that under a duality rotation by an angle α the field F' transforms into $F'' = e^{\alpha\gamma^5}F'$. Therefore $F'' = e^{\alpha\gamma^5}LFL^*$ is an extremal field. Since a duality rotation by $\pi/2$ transforms electric into magnetic field, and vice-versa, we can choose the extremal field to be a magnetic field with the spatial direction $\vec{\sigma}^3$; that is to say: $F'' = -\hat{\imath}H\vec{\sigma}^3$, where $\hat{\imath} = \vec{\sigma}^1\vec{\sigma}^2\vec{\sigma}^3$ and $\vec{\sigma}^i = \gamma^i\gamma^0$; $(i = 1,2,3)$. But $-\hat{\imath}H\vec{\sigma}^3 = H\gamma^1\gamma^2$, so that

$$e^{\alpha\gamma^5}LFL^* = H\gamma^1\gamma^2\,. \tag{4}$$

If we take $\beta = -\alpha$ and $R = L^*$ it follows from eq.(4) that

$$F = H\,e^{\beta\gamma^5}R\gamma^1\gamma^2R^* \tag{5}$$

and, by taking $H \equiv b\rho$ with $\rho \geq 0$, we have

$$F = b\psi\gamma^1\gamma^2\psi^* \tag{6}$$

where ψ has the form given in eq.(3). Eq.(6) is a rather important result: it permits us to interpret the DH spinor field on the basis of the above discussion, where we identified $H = b\rho$, $\alpha = -\beta$, $L = R^*$ (we wrote, a priori, $H = b\rho$ for a matter of convenience).

Namely, the DH field ψ is the (spinorial) operator that transforms an extremal field into a non-null electromagnetic field. And in fact eq.(3) reveals that ψ is just, and nothing but, the product of an operation R, a duality transformation $e^{\gamma^5\beta/2}$ and a dilation $\rho^{1/2}$. Conversey, eq.(6) shows the strict relation existing between a non-null electromagnetic field F and a DH spinor ψ.

For future convenience, let us observe that the definition $H \equiv b\rho$ implies b to be a *scalar* function (since H is the magnetic field *magnitude*).

4. THE EQUIVALENCE OF MAXWELL AND DIRAC EQUATIONS

Let us examine the free Maxwell equations:

$$\partial F = 0\,. \tag{7}$$

We want to look for solutions of eq.(7) of the form (6); since eq.(6) is valid when F is non-null ($F^2 \neq 0$) then plane–wave solutions of eq.(7) are excluded (since in this case $F^2 = 0$). But one trivial non-null solution of eq.(7) is $F =$ constant. This will be certainly the case when b and ρ in eq.(6) are constants (even if this is not a necessary assumption, as we shall see in the next Section). Let us suppose as a first instance that b, ρ and β be constants. If we use eq.(6) in eq.(7), we immediately obtain the non-linear *Heisenberg–like* spinor equation:

$$\partial\psi\gamma^1\gamma^2 + \mathcal{F}(\psi) = 0 \tag{8}$$

with

$$\mathcal{F}(\psi) = \gamma^\mu\psi\gamma^1\gamma^2(\partial_\mu\psi^*)\psi(\psi\psi^*)^{-1}\,. \tag{9}$$

From $RR^* = 1$ it follows that

$$\partial_\mu R = \frac{1}{2}\Omega_\mu R \tag{10}$$

where

$$\Omega_\mu = 2(\partial_\mu R)R^*\,. \tag{11}$$

Since we have supposed ρ and β constant, eq.(10) can be written as

$$\partial_\mu\psi = \frac{1}{2}\Omega_\mu\psi\,. \tag{12}$$

If we introduce eq.(12) into eq.(9) and define the 2-form S as

$$S \equiv \frac{\hbar}{2}R\gamma^1\gamma^2 R^*\,, \tag{13}$$

where the constant $\hbar$ will be identified with the (reduced) Planck constant, then eq.(8) adquires the relevant (non-linear) form:

$$\partial\psi\gamma^1\gamma^2 - \frac{1}{\hbar}\gamma^\mu S\Omega_\mu\psi = 0\,. \tag{14}$$

Eq.(14) is an interesting result: it is equivalent to the free Maxwell equations (7), under the above assumptions. The 2-form S will be called a spin 2-form.

Now, both S and Ω_μ are 2-forms. Thus, the product $S\Omega_\mu$ in eq.(14) results in the sum of a scalar, a 2-form and a pseudo-scalar; that is:

$$S\Omega_\mu = -p_\mu + E_{\mu,\alpha\beta}(\gamma^\alpha{\wedge}\gamma^\beta) + \gamma^5 r_\mu\,, \tag{15}$$

where $p_\mu, E_{\mu,\alpha\beta}$ and r_μ are scalars.

Let us *first* suppose that $S\Omega_\mu$ possesses only a scalar part; then

$$\gamma^\mu S\Omega_\mu = -p_\mu\gamma^\mu \equiv -p\,. \tag{16}$$

Now, given the velocity field v, defined as

$$v \equiv R\gamma^0 R^* \tag{17}$$

so that $\rho v = \psi\gamma^0\tilde{\psi}^*$, let us define the mass m in such a way that

$$p\psi \equiv e^{\beta\gamma^5} mcv\psi \, , \tag{18}$$

wherefrom it follows:

$$p\psi = mc\psi\gamma^0 \, . \tag{19}$$

When we insert eq.(16) into eq.(14) and then use eq.(19), we eventually end up with a *linear* equation:

$$\partial\psi\gamma^1\gamma^2 + \frac{mc}{\hbar}\psi\gamma^0 = 0 \tag{20}$$

which is just the Dirac–Hestenes equation for a free particle (electron). It is trivial to verify that for the plane–wave solutions of equation (20) our assumption that $S\Omega_\mu$ possesses only a scalar part is indeed satisfied. In other words, all the plane–wave solutions of the fundamental equation (14) correspond to a scalar $S\Omega_\mu$, *i.e.*, they obey also eq.(20).

Let us recall at this point that in ref.[18] we showed such plane–wave solutions to be associated with rotations in the $\gamma^1\gamma^2$ plane; and that *mass* itself —according to our definition of it— coincides in the present approach with the kinetic energy of that rotation: a result already met by Hestenes[16], starting from a different point of view. We shall discuss elsewhere[20] about the origin of the electron *spin*.

One further remark, this time concerning the definition of p given by eq.(18), has to be put forth. Equation (18) may suggest to introduce a new mass, M, such that $p\psi = Mcv\psi$. Such a mass M can be introduced as $M \equiv e^{\beta\gamma^5}m = m\cos\beta + \gamma^5 m\sin\beta$. If we want M to be real, we must have $\sin\beta = 0$, that is: $\beta = 0$ or $\beta = \pi$, which just distinguish electrons from positons in Dirac theory (cf. ref.[16]). This leads us to define M in the following way:

$$M = \langle e^{\beta\gamma^5} m \rangle_0 = m\cos\beta \, , \tag{21}$$

where the angle brackets mean the scalar part. Moreover, we may evaluate that $\psi\tilde{\psi}^* = \rho e^{\beta\gamma^5} = \Omega_1 + \gamma^5\Omega_2$, where Ω_1 and Ω_2 (scalar functions of x^μ) are the Lorentz invariants of Dirac theory (so that $\Omega_1 = \rho\cos\beta$, $\Omega_2 = \rho\sin\beta$ and $\rho = \sqrt{\Omega_1^2 + \Omega_2^2}$). As a consequence, equation (21) gets the form

$$M = \frac{m}{\sqrt{1 + \frac{\Omega_2^2}{\Omega_1^2}}} \tag{22}$$

Notice that, when β —contrarily to our previous assumptions— is variable, then M becomes a *variable mass.* Quantity β appears to be variable whenever the DH field interacts with an external electromagnetic field.(**) In such a case, it is interesting that eq.(22) coincides with a well-known formula[17] by de Broglie (even if in de Broglie's work it is m

0(**) Let us mention, however, that in the Hydrogen atom case —besides the known solutions with *variable* β— new solutions with $\beta = 0$ have been recently found. The interpretation of β seems still to deserve further attention.

and not M the variable mass).

Let us *go on* assuming, for the moment, b, ρ and β to be still constant [cf. eq.(8)]. However, instead of supposing that $S\Omega_\mu$ possesses only a scalar part, let us use its general expression, eq.(15). One can write:

$$E_{\mu,\alpha\beta}\gamma^\mu(\gamma^\lambda{\wedge}\gamma^\beta) = -\frac{e}{c}A_\mu\gamma^\mu - \gamma^5\frac{g}{c}B_\mu\gamma^\mu = -\frac{e}{c}A - \gamma^5\frac{g}{c}B \ , \tag{23}$$

where A_μ and B_μ are

$$\frac{e}{c}A_\mu \equiv \eta_{\nu\sigma}E_{\nu,\mu\sigma} \tag{24}$$

$$\frac{g}{c}B_\mu \equiv \eta_{\mu\nu}\varepsilon_{\nu\sigma\rho\tau}E_{\sigma,\rho\tau} \tag{25}$$

with $\eta_{\mu\nu}$ =diag$(+1,-1,-1,-1)$, and $\varepsilon_{\nu\sigma\rho\tau} = +1$ (-1) for even (odd) permutations of [0,1,2,3] while $\varepsilon_{\nu\sigma\rho\tau} = 0$ when two indexes are equal. Notice that A and $\gamma^5 B$ appear to play the same role played by the two electromagnetic potentials when *external* (electromagnetic) fields are present: *i.e.*, played by the potentials associated to an electric charge e and a magnetic monopole $\gamma^5 g$, respectively.[19] However, our quantities A and $\gamma^5 B$, given by eqs.(24)–(25), seem to refer to an *"internal" field* (cf. also refs.[20]). If we now define, in analogy with eq.(18),

$$r\psi \equiv e^{\beta\gamma^5}\mu c v\psi = \mu c\psi\gamma^0 \tag{26}$$

where $r = r_\mu\gamma^\mu$ [cf. eq.(15)], then eq.(14) assumes the noticeable ("complete") form:

$$\partial\psi\gamma^1\gamma^2 + (m+\gamma^5\mu)\frac{c}{\hbar}\psi\gamma^0 + (eA+\gamma^5 gB)\frac{1}{\hbar c}\psi = 0\,. \tag{27}$$

We derived eq.(27) under some assumptions: in particular, that ρ and β were constants; however, we shall see in the next section that it is possible to eliminate those restrictions. So we can discuss, and interpret, eq.(27) without worrying about them.

Let us first notice that, when $\mu = 0$ and $g = 0$ or $B = 0$, equation (27) reduces to

$$\partial\psi\gamma^1\gamma^2 + \frac{mc}{\hbar}\psi\gamma^0 + \frac{e}{\hbar c}A\psi = 0 \tag{28}$$

which is *formally* identical to the Dirac–Hestenes equation (2). We might therefore claim that the presence of a *minimal coupling* to some "internal" electromagnetic field can be regarded as associated with the existence of the 2-form term in the product of S and Ω_μ [eq.(15)].

Second, if $m = 0$ and $e = 0$ or $A = 0$, eq.(27) reduces to

$$\partial\psi\gamma^1\gamma^2 + \gamma^5\mu\frac{c}{\hbar}\psi\gamma^0 + \gamma^5\frac{g}{\hbar c}B\psi = 0 \tag{29}$$

which is an equation of the type studied by Lochak[21] as a wave equation for a magnetic monopole: this can easily be seen by writing $\mu c\psi\gamma^0 = \frac{1}{2}c\mathcal{M}(\Omega_1 - \gamma^5\Omega_2)\psi\gamma^0$, with $\mathcal{M} \equiv 2\mu e^{\beta\gamma^5}/\rho$. Here we can note again that the (minimal) coupling with the pseudo-potential $\gamma^5 B$ comes from the 2-form part of $S\Omega_\mu$, while the magnetic monopole mass term comes from its pseudo-scalar part.

A similar analysis was performed by Daviau[22,23]; however, in writing down $F = \psi\gamma^1\gamma^2\psi^*$, that author attempted at associating an electromagnetic field with Dirac's waves[22], which certainly is not the case in our approach (which, on the contrary, made recourse to the Rainich–Misner–Wheeler theorem and its consequences). Very interesting and pioneering work in the same direction has been done by Campolattaro,[24] even if by means of the traditional tensor and spinor calculus (which is often intricated, and sometimes does not help the physical interpretation). Let us mention that in ref.[15] we did prove the equivalence of our eq.(8) with the non-linear spinor equation forwarded by Campolattaro[24] as equivalent to the Maxwell equations.

5. NON-DISPERSIVE DE BROGLIE WAVE–PACKETS FROM THE MAXWELL EQUATIONS

In this Section let us pass finally to the most general case, by *eliminating* the restriction that b, ρ and β be constants. In this (third) case, instead of eq.(8), we obtain:

$$\partial\psi\gamma^1\gamma^2 + \mathcal{F}(\psi) = -(\partial \log b)\psi\gamma^1\gamma^2\,, \tag{30}$$

which *generalizes* the non-linear Heisenberg–like equation (8). Now, by using eq.(10), one gets

$$\partial_\mu\psi = [\partial_\mu \log(\rho e^{\beta\gamma^5})^{1/2}]\psi + \frac{1}{2}\Omega_\mu\psi \tag{31}$$

which, when introduced in eq.(3) and after using eq.(13), results in:

$$\partial\psi\gamma^1\gamma^2 - \frac{1}{\hbar}\gamma^\mu S\Omega_\mu\psi = -(\partial\log b)\psi\gamma^1\gamma^2 - [\partial\log(\rho e^{\beta\gamma^5})^{1/2}]\psi\gamma^1\gamma^2\,. \tag{32}$$

We can rewrite the l.h.s. of eq.(32) in order to have

$$\Big[\partial R\gamma^1\gamma^2 - \frac{1}{\hbar}\gamma^\mu S\Omega_\mu R\Big](\rho e^{\beta\gamma^5})^{1/2} = -(\partial\log b)\psi\gamma^1\gamma^2 - 2[\partial\log(\rho e^{\beta\gamma^5})^{1/2}]\psi\gamma^1\gamma^2\,. \tag{33}$$

The l.h.s. of eq.(33) vanishes, once R is required to satisfy eq.(14) (which was written in terms of ψ because ρ, β were there supposed to be constants); then we must have [K being a constant]:

$$\partial\log b = -2\partial\log(\rho e^{\beta\gamma^5})^{1/2} \Longleftrightarrow b = \frac{K}{\rho e^{\beta\gamma^5}} \tag{34}$$

which implies in eqs.(5),(6) that F is proportional to $R\gamma^1\gamma^2R^*$, just as in the case discussed in Sect.**3**. Equation (34), therefore, implies a (non-null) constant field F. Notice, incidentally, that in eq.(34) it must be either $\beta = 0$ or $\beta = \pi$, since b is a *scalar*; and this is a consequence of supposing R to obey the Dirac–Hestenes equation (14).

Putting eq.(34) into eq.(32), we get finally the generalized Heisenberg–like spinor equation:

$$\partial\psi\gamma^1\gamma^2 - \frac{1}{\hbar}\gamma^\mu S\Omega_\mu\psi = (\partial\log\psi_0)\psi\gamma^1\gamma^2 \tag{35}$$

where we set $\psi_0 \equiv (\rho e^{\beta\gamma^5})^{1/2}$. And again, *if* $S\Omega_\mu$ has only a scalar part, eq.(35) can be written —according to our previous discussion— as:

$$\partial\psi\gamma^1\gamma^2 + \frac{mc}{\hbar}\psi\gamma^0 = (\partial\log\psi_0)\psi\gamma^1\gamma^2 \tag{36}$$

which is a (non-linear) generalized Dirac-Hestenes equation. We have shown elsewhere[25] that, if one applies the Dirac operator ∂ to the above equation, one obtains $[\Box \equiv \partial^2]$:

$$\Box\psi + \left(\frac{mc}{\hbar}\right)^2\psi = \frac{\Box\psi_0}{\psi_0}\psi + W\psi \tag{37}$$

where

$$W\psi = \eta^{\mu\nu}(\partial_\nu\log\psi_0)\Omega_\mu\psi\,. \tag{38}$$

The term $W\psi$ can be easily[25] calculated in the rest frame; the result is that it vanishes, *i.e.*:

$$W\psi = 0\,. \tag{39}$$

Then eq.(37) assumes the interesting, simple form:

$$\Box\psi + \left(\frac{mc}{\hbar}\right)^2\psi = \frac{\Box\psi_0}{\psi_0}\psi\,. \tag{40}$$

Finally, if we multiply eq.(40) by the global idempotent $e = \frac{1}{2}(1+\gamma^0) \in \sec\mathcal{C}\ell(M,\widehat{g})$, then $\Psi = \psi e$ is the representation in $\mathcal{C}\ell(M,\widehat{g})$ of the standard Dirac covariant spinor field[12] and eq.(40) splits into various equations for its components; namely, for each one of the components ϕ of Ψ one has:

$$\Box\phi + \left(\frac{mc}{\hbar}\right)^2\phi = \frac{\Box\phi_0}{\phi_0}\phi\,, \tag{41}$$

where $\phi \equiv \phi_0 e^{-i\chi}$ and $\phi_0 \equiv \psi_0$. This is a *non-linear* Klein–Gordon equation, which exactly coincides with the equation proposed by Gueret and Vigier[2], and possesses *localized, non-dispersive* solutions. The term $\Box\phi_0/\phi_0$ is usually called the "quantum potential". Let us stress that Gueret and Vigier considered only one eq.(41), for a scalar (complex)

ϕ; whilst we met a different eq.(41) for each component ϕ of Ψ. In order to look for possible solutions of eq.(41), let us observe for example that, if

$$\Box\phi_0 = \left(\frac{mc}{\hbar}\right)^2 \phi_0 \,, \tag{42}$$

then:

$$\Box\phi = 0 \tag{43}$$

which is just the case discussed by Mackinnon[26]. In ref.[1], Barut did already show how to construct general, *localized*, non-dispersive solutions of the equation $\Box\phi = 0$. In particular, eq.(43) admits a non-trivial solution, representing a non-dispersive *soliton* (localized wave–packet) wich moves *undeformed* with subluminal speed.

Indeed, it is a remarkable fact that such non-dispersive, localized solutions exist, and that ultimately they satisfy an equation —eq.(41)— which derives from the free Maxwell equations. Let us recall, here, that already in 1915 Bateman[27] had looked for "solitonic" solutions of Maxwell equations.

At last, we want to notice that —when we replace $S\Omega_\mu$ in eq.(35) by its full expression, eq.(15), containing a scalar, a 2-form and a pseudo-scalar part— then eq.(35) gets its most general form:

$$\partial\psi\gamma^1\gamma^2 + (m + \gamma^5\mu)\frac{c}{\hbar}\psi\gamma^0 + (eA + \gamma^5 gB)\frac{1}{\hbar c}\psi = (\partial\log\psi_0)\psi\gamma^1\gamma^2 \,. \tag{44}$$

Actually, eq.(27) is a particular case of eq.(44), valid when the non-linear term $(\partial\log\psi_0)\psi\gamma^1\gamma^2$ can be neglected.

6. CONCLUSIONS

Given a non-null electromagnetic field F, from the Rainich–Misner–Wheeler theorem we deduced eq.(6):

$$F = b\psi\gamma^1\gamma^2\psi^* \tag{45}$$

where ψ is a DH spinor field (whose canonical form, eq.(3), is $\psi = \rho^{1/2}e^{\gamma^5\beta/2}R$). Moreover we saw that, even supposing b, ρ and β to be *not* constant, the field F that solves the *free* Maxwell equations (without sources) $\partial F = 0$ is constant throughout space-time. Consequently, the equations $\partial F = 0$ require that ψ obeys the (generalized) non-linear Heisenberg–like spinor equation (35) or —in the *most* generalcase— the analogous equation (44), under the condition $F =$ constant.

On the other side, a (constant) F can be written:

$$F = K\Phi\gamma^1\gamma^2\Phi^* \tag{46}$$

where $b \equiv K/\rho e^{\beta\gamma^5}$; and where $\Phi \equiv R$ [cf. eq.(34)] is another DH spinor field which satisfies a (linear) Dirac–Hestenes equation like eq.(20) [as it is possible to verify, recalling[15] that any $R \in Spin_+(1,3)$ can be written as the exponential of a bivector].

Those spinor fields are then related by:

$$\psi = \rho^{1/2} e^{\gamma^5 \beta/2} \Phi \tag{47}$$

and in particular, for the "electron solutions" (*i.e.*, for $\beta = 0$), by:

$$\psi = \rho^{1/2} \Phi \tag{48}$$

which coincides with a well-known expression in de Broglie's theory of double solution. [For the positon, one would get $\psi = -\gamma^5 \rho^{1/2} \Phi$]. Let us recall that in such de Broglie's theory ψ was the electron (total) quantum–probabilistic wave function, but Φ (which obeys the Dirac–Hestenes equation) was a physical wave!

In our present approach, the DH spinor field Φ is just the *rotor* part of ψ and has therefore only 6 degrees of freedom, so as the electromagnetic field F (while ψ possesses 8 degrees of freedom: cf. eq.(6)). Actually, in the case here examined, the spinor field Φ could be ultimately of electromagnetic nature, as suggested by equations (45) and (48). For instance, the basic equation (6), or (45), shows the strict relation existing between the non-null electromagnetic field F (present in the absence of sources!) and the electron wave–function ψ.

Thus the reality of the de Broglie waves seems supported by our analysis.

We may conclude that many of de Broglie's ideas concerning the interpretation of quantum mechanics should be seriously revisited, while the language of Clifford algebras appears to be particularly convenient for that purpose. Our results, which have taken the free Maxwell equations $\partial F = 0$ as a starting–point, remind us of a conjecture made by Mackinnon[26], *i.e.*, that "equation $\Box \phi = 0$ may prove to be of more importance to quantum mechanics than has hitherto been supposed".

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