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INTEGRABLE SYSTEMS IN TOPOLOGICAL FIELD THEORY

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ABSTRACT

Integrability of the system of PDE for dependence on coupling parameters of the (tree-level) primary partition function in massive topological field theories, being imposed by the associativity of the perturbed primary chiral algebra, is proved. In conformal case it is shown that all the topological field theories are classified as solutions of a universal high order Painlevé-type equation. Another integrable hierarchy (of systems of hydrodynamic type) is shown to describe coupling to gravity of the matter sector of any topological field theory. Different multicritical models with the given structure of primary correlators are identified with particular self-similar solutions of the hierarchy. The partition function of any of the models is calculated as the corresponding τ-function of the hierarchy.

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Introduction

The recent progress in low dimensional QFT essentially was related with the application [1] of the machinery of integrable systems of KdV-type in studying of matrix models. An alternative approach to two-dimensional gravity was suggested by Witten [2] basing on the ideas of topological field theory.

Topological field theories are solvable models without local, propagating degrees of freedom [3]. After the identification [2] of one-matrix models with the topological theory of 2D gravity a lot of remarkable links of two-dimensional gravity and topological field theories were found [4-6]. It was shown [5] that the multi-matrix models can be identified to some matter systems coupled to topological gravity. These matter systems proved to be [7] twisted versions of the $N = 2$ minimal superconformal models [8]. Deep relations of these models to catastrophe theory was studied in [9]. Correlation functions in minimal topological field theories were calculated by Dijkgraaf, E. Verlinde and H. Verlinde [10] using the Landau-Ginzburg (LG) potentials machinery [11]. (This calculation was analyzed from the point of view of catastrophe theory in [12].) The observation of [10] concerning relation of topological A_n minimal models to generalized KdV-hierarchy (or the Gelfand-Dikii (GD) hierarchy) was elucidated by Krichever [13]. He showed that the calculation of [10] has a natural interpretation as a part of the theory of the so-called dispersionless Lax hierarchy (the genus zero semi-classical limit of the Gelfand-Dikii hierarchy). He introduced also τ-functions of the dispersionless hierarchy and showed that the tree-level partition function of the minimal model (before coupling to gravity) coincides with the τ-function of a particular solution of a part of the dispersionless Lax hierarchy. So topological string theories can be considered as semi-classical approximation of ordinary string theories. The indications of [2,4,5,10] about equivalence (at tree-level) of topological string theory and ordinary string theory (in $d < 1$) can be interpreted therefore like statements about exactness (in some sense) of the semi-classical approximation of the KdV hierarchy.

The construction of [13] were generalized in [14] and, independently, in [15,16] using the genus g semiclassical approximation of the GD hierarchy (so-called Whitham-averaged GD hierarchy [17-20]). This gives [14] a multicut solution of the loop equations [21]. Also the corresponding "partition function" was proved [14] to satisfy a truncated version of the Virasoro constraints that were obtained in [22] for the partition function of two-dimensional gravity. The genus g variant of the LG model of [10] (here g is the "genus" of the LG superpotential) was constructed in [16] using the differential geometry of moduli space [15] and the Hamiltonian formulation [23,20] of the Whitham averaging procedure. The tree-level primary chiral algebra and the partition function of this model also was calculated in [15,16]. In Sect.4 (see below) I argue that these models can be obtained from minimal models as a result of "phase transition".

The main aim of this paper is to construct an "inverse spectral transform" for any topological QFT (at tree-level). It is shown that two types of integrable systems are hidden in a topological field theory. The first one coincides with the equations of associativity of the primary chiral algebra (before coupling to gravity). The second integrable system determines the dependence of correlators on the coupling parameters (including all the descendant couplings).

In this introduction I'll give an outline of the results of the paper. I will be considering (only at tree-level) a general topological field theory [2,3,10] with N primary bosonic fields $\phi_1, \ldots, \phi_N$. The symbol $\langle\phi_{\alpha_1}\phi_{\alpha_2}\ldots\rangle$ will denote the genus zero correlation functions. These do not depend on positions of operators $\phi_{\alpha_1}, \phi_{\alpha_2}, \ldots$ The double-point correlators

$$\langle\phi_\alpha\phi_\beta\rangle = \eta_{\alpha\beta} = \eta_{\beta\alpha} \tag{I.1}$$

determine a non-degenerate scalar product on the space of primaries. The triple correlators

$$c_{\alpha\beta\gamma} = \langle\phi_\alpha\,\phi_\beta\,\phi_\gamma\rangle \tag{I.2}$$

determines the structure of the operator algebra (or primary chiral algebra) of the model,

$$\phi_\alpha \cdot \phi_\beta = c_{\alpha\beta}^{\ \ \gamma} \phi_\gamma \ , \qquad c_{\alpha\beta}^{\ \ \gamma} = \eta^{\gamma\epsilon} c_{\alpha\beta\epsilon} \ , \qquad (\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1} \tag{I.3}$$

$$\langle \dots \phi_\alpha \phi_\beta \dots \rangle = c_{\alpha\beta}^{\ \ \gamma} \langle \dots \phi_\gamma \dots \rangle \ . \tag{I.4}$$

This is a commutative associative algebra A of dimension N with a unity ϕ_1,

$$c_{1\alpha\beta} = \eta_{\alpha\beta} \ , \qquad c_{1\ \beta}^{\ \alpha} = \delta^\alpha_\beta \ . \tag{I.5}$$

The symmetry of the tensor $c_{\alpha\beta\gamma}$ is equivalent to invariance of the scalar product $(I.1)$ on A:

$$\langle ab, \, c \rangle = \langle a, \, bc \rangle \ , \qquad \text{for} \ \ a, \, b, \, c \in A \ . \tag{I.6}$$

I recall that such algebras A are called Fröbenius algebras.

In fact one has to consider a N-parametric family of primary operator algebras $A = A(t), \quad t = (t^1, \dots, t^N)$, of the form

$$\begin{aligned} c_{\alpha\beta\gamma} &= c_{\alpha\beta\gamma}(t) \\ \eta_{\alpha\beta} &= \text{const.} \end{aligned} \tag{I.7}$$

satisfying all the previous conditions. As it was shown in [10], the perturbed correlators $(I.2)$ can be represented in the form

$$c_{\alpha\beta\gamma}(t) = \partial_\alpha \, \partial_\beta \, \partial_\gamma \, F(t) \ , \tag{I.8}$$

$\partial_\alpha = \partial/\partial t^\alpha$, where the function $F(t)$ is the primary free energy. The conditions of associativity of the perturbed primary chiral algebra give a system of nonlinear PDE for the free energy $F(t)$. I'll call this system as Witten - Dijkgraaf - E. Verlinde - H. Verlinde (WDVV) equations since the idea to classify topological theories solving differential equations of associativity of primary operator algebra seems to belong to Witten [2] (general topological models with two primaries were studied in [2] using this approach).

Formal integrability of the WDVV equations follows from the "commutation representations": the compatibility condition of the linear system for the vector function ξ

$$\partial_\alpha \xi_\beta = z\, c_\alpha{}^\gamma{}_\beta \xi_\gamma \tag{I.9a}$$

with the constraint of symmetry of the tensor

$$c_{\alpha\beta\gamma} = \eta_{\gamma\epsilon}\, c_{\alpha\beta}^{\epsilon} \tag{I.9b}$$

and the normalization

$$c_1{}^\gamma{}_\beta = \delta^\gamma_\beta \tag{I.9c}$$

are equivalent to the WDVV equations. Here z is a spectral parameter. In this paper inverse spectral transform for the WDVV system is constructed with a genericity assumption for the perturbed Frõbenius algebra $A(t)$. I assume that (locally) $A(t)$ has no nilpotents[1] (or, equivalently, it can be decomposed into a direct sum of 1-dimensional Frõbenius algebras). Identically indecomposable deformations of Frõbenius algebras will be considered in next publications (note that in topological sigma-models [2,5] where $A(t)$ is the "quantized" cohomology ring of the target space, the deformation $A(t)$ is indecomposable - see below, Sect.4, example 2). Under this decomposability assumption the WDVV equations proved to be gauge equivalent (in the sense of [24]) to the integrable system

[1]These topological theories can be called as massive ones. I am acknowledged to S. Ceccotti for explanation of this point.

$$\partial_k \gamma_{ij}(u) = \gamma_{ik}(u)\gamma_{kj}(u) \ , \qquad i,\ j,\ k \ \text{ are distinct,}$$

$$\sum_{k=1}^{N} \partial_k\, \gamma_{ij}(u) = 0 \ , \qquad (I.10)$$

$$\gamma_{ji}(u) = \gamma_{ij}(u) \qquad \text{are defined only for } i \neq j \ .$$

Here $u = (u^1, \ldots, u^N)$ are new co-ordinates on the coupling space, $u = u(t)$, $\partial_i = \partial/\partial u^i$. They are defined by the following representation of the perturbed structure constants

$$c_{\alpha\beta}^{\ \gamma}(t) = \sum_{i=1}^{N} \frac{\partial t^\gamma}{\partial u^i} \frac{\partial u^i}{\partial t^\alpha} \frac{\partial u^i}{\partial t^\beta} \ , \qquad (I.11)$$

(proof of the existence of such co-ordinates is the crucial step in construction of the gauge equivalence).

The functions $\gamma_{ij}(u)$ are expressed via components of the metric $\eta_{\alpha\beta}$ in the co-ordinates $u^1, \ldots, u^N$

$$g_{ij}(u) = \eta_{\alpha\beta} \frac{\partial t^\alpha}{\partial u^i} \frac{\partial t^\beta}{\partial u^j} \equiv g_{ii}(u)\delta_{ij} \ , \qquad (I.12a)$$

$$\gamma_{ij}(u) = \frac{\partial_j \sqrt{g_{ii}(u)}}{\sqrt{g_{jj}(u)}} = \gamma_{ji}(u) \ , \qquad i \neq j \ . \qquad (I.12b)$$

The system $(I.10)$ is nothing but the conditions of vanishing of the curvature of the metric $(I.12)$.

The system $(I.12)$ is familiar in the soliton theory [26-29]. It is equivalent to $(1+1)$-evolutionary system of $\frac{N(N-1)}{2}$ PDE (and dependence on $u^1, \ldots, u^N$ is determined by some particular isospectral symmetries of the system) being equivalent [26] to the so-called pure imaginary reduction of the N-wave interaction system (for $N \geq 3$; for $N = 2$ $(I.10)$ is a linear system). The inverse spectral transform for $(I.10)$ can be constructed in standard way [28, 29, 26]. The corresponding solution of the WDVV is determined by a solution of $(I.10)$ with a N-dimensional ambiguity (by solving the linear problem giving the commutation representation for $(I.10)$). These results are represented in Sect.1 below. Interpretation of the linear operators $(I.9)$ as an affine connection on the coupling space (depending on z) was very useful in the proofs and calculations of Sections 1, 2.

In fact self-similar solutions of the WDVV equations are of special importance since they describe perturbed chiral ring of topological conformal field theories. In the decomposable case they are classified by solutions of the similarity reduction of the system $(I.10)$

$$\gamma_{ij}(cu) = c^{-1}\gamma_{ij}(u) \ . \tag{$I.12$}$$

This reduction is a system of ODE of order $N(N-1)/2$ (nonlinear for $N > 2$). For $N = 3$ this system was proved to be equivalent [30] to a particular case of the Painlevé-VI equation. For any $N \geq 3$ the isomonodromic deformation method [31] is developed in Sect.3 below for $(I.10)$, $(I.12)$. The scaling dimensions of the model are shown to coincide with the monodromy indices in $z = 0$ of the corresponding linear ODE in z (with rational coefficients). All the primary correlators are expressed in quadratures via these "high-order Painlevé transcendents".

Let us consider now coupling of the model to gravity [2,3,5,10]. Here one has infinite number of fields $\sigma_q(\phi_\alpha)$, $q = 0, 1, \ldots$, where the fields $\sigma_0(\phi_\alpha)$ can be identified with the primaries ϕ_α and $\sigma_q(\phi_\alpha)$ for $q \geq 1$ are called as gravitational descendants of ϕ_α. The operator $\sigma_0(\phi_1)$ usually is denoted by P and is called as the puncture operator of the model. The tree-level correlators of these operators depend on infinite family of coupling parameters $T^{\alpha,q}$ ("descendant couplings") in such a way that

$$\langle \sigma_{q_1}(\phi_{\alpha_1})\sigma_{q_2}(\phi_{\alpha_2})\ldots\rangle = \frac{\partial}{\partial T^{\alpha_1,q_1}}\frac{\partial}{\partial T^{\alpha_2,q_2}}\ldots \log Z(T) \ . \tag{$I.13$}$$

Here $Z(T)$ is the partition function of the topological model. The N-dimensional subspace $T^{\alpha,q} = 0$, $q \geq 1$ (the couplings $T^{\alpha,0}$ are arbitrary) in the phase space of all couplings is called as small phase space. For the correlators on the small phase space one should have

$$\langle \mathcal{P}\phi_\alpha\rangle = t_\alpha = \eta_{\alpha\beta}t^\beta = \eta_{\alpha\beta}T^{\beta,0} , \tag{I.15}$$

$$\langle \phi_\alpha \, \phi_\beta \, \phi_\gamma\rangle = c_{\alpha\beta\gamma}(t) . \tag{I.16}$$

Correlators on all the phase space are determined by the recursion relations

$$\langle \sigma_p(\phi_\alpha)\phi_A\phi_B\rangle = \langle \sigma_{p-1}(\phi_\alpha)\phi_\lambda\rangle\eta^{\lambda\mu}\langle \phi_\mu \, \phi_A \, \phi_B\rangle \tag{I.17}$$

for any operators $\phi_A = \sigma_q(\phi_B)$, $\phi_B = \sigma_r(\phi_\gamma)$ of the model, and their dependence on T is determined by the string equation

$$\sum_{p\geq 1,\alpha} T^{\alpha,p}\partial_{T^{\alpha,p-1}}\partial_{T^{\mu,0}} F(T) + T_{\mu,0} = t_\mu , \qquad \mu = 1, \ldots , N . \tag{I.18}$$

The main observation of Sect.2 (see below) is that this procedure of "switching on topological gravity" has a natural interpretation in the theory of an integrable hierarchy of PDE of the form

$$\partial_{T^{\alpha,p}} t^\gamma = c_{(\alpha,p)}{}^\gamma{}_{,\beta}(t)\,\partial_X t^\beta \tag{I.19}$$

(system of hydrodynamic type) being constructed by the primary operator algebra. Here

$$c_{(\alpha,0)}{}^\gamma{}_\beta(t) = c_{\alpha\beta}^{\gamma}(t) , \tag{I.20}$$

X is the "cosmological constant", the equations (I.19) for $p \geq 1$ are constructed using an appropriate recursion operator. The systems (I.19) are Hamiltonian systems on the loop space $\mathcal{L}M$ (where M is the coupling space) w.r.t. the Poisson bracket

$$\{t^\alpha(X), \, t^\beta(Y)\} = \eta^{\alpha\beta}\delta'(X - Y) . \tag{I.21}$$

Their Hamiltonians $H_{\alpha,p}$ have the form

$$H_{\alpha,p} = \int \langle \mathcal{P}\sigma_{p+1}(\phi_\alpha)\rangle dX . \tag{I.22}$$

The densities of the Hamiltonians (as functions on t) are determined from the linear system (I.9) in the form

$$x_\alpha(t,z) = \sum_{p=0}^{\infty} \langle \mathcal{P} \sigma_p(\phi_\alpha) \rangle z^p \ ,$$

$$\partial_\beta x_\alpha = \xi_\beta \quad \text{enjoys } (I.9) \ ,$$

$$x_\alpha(t,0) = t_\alpha \ . \qquad (I.23)$$

The double correlators (again as functions of t) have the form

$$(z+w) \sum z^p w^q \langle \sigma_p(\phi_\alpha) \sigma_q(\phi_\beta) \rangle = \langle \nabla x_\alpha(t,z), \nabla x_\beta(t,w) \rangle - \eta_{\alpha\beta} \qquad (I.24)$$

(here $\nabla = (\eta^{\alpha\beta}\partial_\beta)$ is the gradient; the scalar product of gradients via the scalar product $\eta_{\alpha\beta}$). The recursion relations (I.17) proved to be a consequence of the recursion procedure of constructing the hierarchy (I.19).

The last step is in determining o dependence of the special amplitudes $t_\alpha = \langle \mathcal{P} \phi_\alpha \rangle$ on the coupling parameters $T^{\alpha,p}$. This dependence is given as a particular solution of the hierarchy (I.19). This solution can be specified as the unique solution of (I.19) being defined for any $T^{1,1}$ for sufficiently small $T^{\alpha,p}$ (for $(\alpha,p) \neq (1,1)$). The τ-function of this particular solution proved to coincide with the partition function of the topological field theory. The "generalized hodograph transform" [25] for solving integrable systems of hydrodynamic type (being represented in the variational form [26]) immediately gives the string equations (I.18).

In the conformal case the dependence $t = t(T)$ is given also by a particular self-similar solution of the hierarchy (I.19). For the solution $\sigma_1(\phi_1)$ is the marginal operator (adding of it to the action does not change the scaling dimensions). Other self-similar solutions of (I.19) (with other $\sigma_p(\phi_a)$ as marginals), that can be constructed using the idea of [5,10] also are of importance in the theory. They give dependence of the correlators on the coupling parameters in different multi-critical models [5,10]. Note that the importance of self-similar solutions of the Whitham hierarchy in the dispersive hydrodynamics[2] was realized many years ago [32] (see also [28,20]). Generic solution of the hierarchy (I.19) can be considered, therefore, as an interpolation between different multicritical models.

I hope also that the hydrodynamic nature of the hierarchy (I.19) (together with some ideas of dispersive hydrodynamics [20,28,32]) might be instructive in studying of global properties of correlators in topological conformal field theories (see the end of Sect.4 for the discussion of a sort of "phase transitions" in multicritical models).

So the hierarchy (I.19) in topological field theory plays the role similar to the role of KdV-type hierarchies do in QFT basing on matrix models. It is interesting to find conditions (in terms of the primary operator algebra) providing that the hierarchy (I.19) can be obtained as a semiclassical limit (or can be obtained by the averaging procedure) of some integrable system of KdV-type. A variant of such specifications using so-called "strong Liouvillean property" [20] of the Hamiltonian formalism of the averaged systems is conjectured in Sect.2.

In this paper I didn't consider the recursion relations determining the high-genus correlators. Also it seems very interesting to find an appropriate variant of the "truncated Virasoro constraints" of [14] in general topological field theory. I hope to do it in forthcoming publications.

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[2]It is interesting that the numerical solution of the string equations in the matrix model with the criticality $k = 3$ has an amazing similarity to the dispersionless shock-wave [20,28,32]. I am acknowledged to S.P. Novikov for paying my attention to this point.

1. Geometry of primary operator algebra in topological field theory

Let A be a N-dimensional commutative associative algebra over C with a unity e. It is called *Fröbenius* algebra if a nondegenerate C-bilinear invariant scalar product $\langle\, ,\, \rangle$ on A exists:

$$\langle ab,\, c\rangle = \langle a,\, bc\rangle\ , \qquad a,\, b,\, c \in A\ . \tag{1.1}$$

Remark. Let us define a linear functional on A by the formula

$$\omega_e(a) = \langle e,\, a\rangle\ . \tag{1.2}$$

Then the invariant scalar product $\langle\, ,\, \rangle$ can be written in the form

$$\langle a,\, b\rangle = \omega_e(ab)\ . \tag{1.3}$$

And for any linear functional $\omega \in A^*$ the scalar product

$$\langle a,\, b\rangle_\omega = \omega(ab) \tag{1.4}$$

is invariant. It is nondegenerate for generic ω. Any invariant C-bilinear form on A can be represented in the form (1.4).

Let e_α, $\alpha = 1, \ldots, N$ be any basis in A such that $e_1 = e$. Let

$$\langle e_\alpha,\, e_\beta\rangle = \eta_{\alpha\beta} \tag{1.5a}$$

$$e_\alpha e_\beta = c_\alpha{}^\gamma{}_\beta e_\gamma \tag{1.5b}$$

(the summation over repeated indices is assumed).

The matrix $\eta_{\alpha\beta}$ and the structure constants $c_\alpha{}^\gamma{}_\beta$ satisfy the following conditions:

$$\eta_{\beta\alpha} = \eta_{\alpha\beta} \ , \qquad \det(\eta_{\alpha\beta}) \neq 0 \ , \tag{1.6a}$$

$$c_{\alpha}{}^{\epsilon}{}_{\beta}\, c_{\epsilon}{}^{\delta}{}_{\gamma} = c_{\alpha}{}^{\delta}{}_{\epsilon}\, c_{\beta}{}^{\epsilon}{}_{\gamma} \tag{1.6b}$$

(associativity),

$$c_1{}^{\alpha}{}_{\beta} = \delta^{\alpha}_{\beta} \tag{1.6c}$$

(normalization $e_1 = e$),

$$c_{\alpha\beta\gamma} = c_{\alpha}{}^{\epsilon}{}_{\beta}\eta_{\epsilon\gamma} = c_{\beta\alpha\gamma} = c_{\alpha\gamma\beta} \tag{1.6d}$$

(commutativity and invariance of the scalar product).

The operators

$$T_{\alpha} = (c_{\alpha}{}^{\beta}{}_{\gamma})$$

of (left) multiplication form an exact N-dimensional representation of A. The Fröbenius algebra is called *decomposable* if the operator

$$T = x^{\alpha} T_{\alpha}$$

has simple eigenvalues for generic $x^1, \ldots, x^N$. A decomposable Fröbenius algebra A is isomorphic to a direct sum of one-dimensional Fröbenius algebras

$$f_i f_j = \delta_{ij} f_i \ , \qquad \langle f_i, f_j \rangle = \delta_{ij} \ . \tag{1.7}$$

Decomposability of a Fröbenius algebra is an open property.

Deformation of a Fröbenius algebra is a k-parametric family $c_{\alpha}{}^{\gamma}{}_{\beta}(t)$, $\eta_{\alpha\beta}(t)$, $t = (t^1, \ldots, t^k)$, satisfying (1.6).

Definition 1. A N-parametric deformation of a N-dimensional Fröbenius algebra A is called *potential deformation* if

$$\partial_\gamma \eta_{\alpha\beta} \equiv 0 \ , \tag{1.8a}$$

$\partial_\gamma = \partial/\partial t^\gamma$,

$$c_{1\ \beta}^{\ \alpha} \equiv \delta^\alpha_\beta \ , \tag{1.8b}$$

and a potential function $F(t)$ exists such that

$$c_{\alpha\beta\gamma}(t) = \partial_\alpha \, \partial_\beta \, \partial_\gamma \, F(t) \ . \tag{1.8c}$$

Note that

$$\partial_1 \, \partial_\alpha \, \partial_\beta \, F(t) = \eta_{\alpha\beta} \tag{1.8d}$$

(this follows from (1.8c), (1.6c)).

The problem of classification of potential deformations of Fröbenius algebras is equivalent to a complicated system of nonlinear PDE for the potential function $F(t)$ (the primary free energy) being obtained by substitution (1.8c), (1.8d) to (1.6) (in fact, only equations (1.6b) of associativity are nontrivial). This system was discussed first for $N = 2$ by Witten [2], and the conditions (1.8) were obtained by Dijkgraaf, E. and H. Verlinde, so I'll call it as *Witten - Dijkgraaf - Verlinde - Verlinde (WDVV) equations*. The aim of this section is to construct "inverse spectral transform" for this system.

Let us construct a representation of the WDVV equations in a form of compatibility conditions of a over-determined linear system.

Proposition 1.1 Let $c_\alpha{}^\gamma{}_\beta(t)$ is a family of functions of $t = (t^1, \dots, t^N)$ satisfying (1.6c) and (1.6d) for some constant $(\eta_{\alpha\beta})$. Then $(c_\alpha{}^\gamma{}_\beta(t),\, \eta_{\alpha\beta})$ is a potential deformation of a Fröbenius algebra iff the following linear system depending on a spectral parameter z is compatible.

$$\partial_\alpha \xi_\beta = z c_\alpha{}^\gamma{}_\beta \, \xi_\gamma \,, \qquad \alpha,\, \beta = 1, \dots, N \,. \tag{1.9}$$

Proof. The compatibility $\partial_{\alpha'}\partial_{\alpha''} = \partial_{\alpha''}\partial_{\alpha'}$ of the system (1.9) is equivalent to the equations

$$c_{\alpha'}{}^\gamma{}_\beta \, c_{\alpha''}{}^\epsilon{}_\gamma = c_{\alpha''}{}^\gamma{}_\beta \, c_{\alpha'}{}^\epsilon{}_\gamma \,, \tag{1.10a}$$

$$\partial_{\alpha''} c_{\alpha'}{}^\gamma{}_\beta = \partial_{\alpha'} c_{\alpha''}{}^\gamma{}_\beta \,. \tag{1.10b}$$

The first one together with the symmetry (1.6d) implies the associativity. The same symmetry and the second equation provide existence of a potential $F(t)$. The proposition is proved.

Unfortunately I don't know how one can use the commutative representation (1.9) for integration of the WDVV equations. What I'm going to do is to construct a gauge equivalence of the linear problem (1.9) to a more familiar in the theory of integrable systems "commutative representation" (i.e., to construct a gauge equivalence [24] of the WDVV equations to a more familiar integrable system).

It turns out that a geometric interpretation of the linear system (1.9) will be very useful in constructing of such a gauge equivalence (it also will be very useful in the next section in calculation of all the correlators in the topological field theory with given primary chiral algebra).

Let M be the space of (complex) parameters $t = (t^1, \ldots, t^N)$ (the coupling space) of a deformation $(c_{\alpha}{}^{\gamma}{}_{\beta}(t), \eta_{\alpha\beta})$. Let us introduce a multiplication on the fibers of the tangent bundle TM by the formula

$$\partial_\alpha \cdot \partial_\beta \Big|_t = c_{\alpha}{}^{\gamma}{}_{\beta}(t)\partial_\gamma \,. \tag{1.11}$$

$\partial_\alpha = \partial/\partial t^\alpha$. This provides in the space of all vector fields on M a structure of a commutative associative algebra over the ring $\mathcal{F}(M)$ of functions on M. The unity vector field $\partial = \partial_1$ is specified on M. Also a metric

$$ds^2 = \eta_{\alpha\beta}\, dt^\alpha\, dt^\beta \tag{1.12}$$

is determined on M. In other words, a structure of a Fröbenius $\mathcal{F}(M)$-algebra is specified on the space of vector fields Vect (M).

This point of view is instructive to give a co-ordinate free reformulation of the main problem.

Let M be a N-dimensional Riemann[3] manifold M with a metric ds^2 and with a structure of algebra with a unity ∂ over $\mathcal{F}(M)$ in the space of vector fields:

$$(X \cdot Y|_t)^k = c_i{}^k{}_j(t)\, X^i(t)\, Y^j(t) \tag{1.13}$$

such that the scalar product ds^2 is invariant (see (1.1)) with respect to this multiplication. Let $\nabla_X Y$ be the Levi-Civita covariant derivative for the metric ds^2. Let $\tilde{\nabla}$ be a new covariant derivative depending on a parameter z of the form

$$\tilde{\nabla}_X Y = \nabla_X Y + z X \cdot Y \tag{1.14a}$$

or, equivalently, on 1-forms ω

[3] I mean that ds^2 is a non-degenerate quadratic form on TM (not necessary positive definite). For complex manifold M ds^2 is a complex quadratic form.

$$\tilde{\nabla}_X \omega(Y) = \nabla_X \omega(Y) - z\omega(X \cdot Y) \ . \qquad (1.14b)$$

Definition 2. M is a Fröbenius manifold if: 1) the connection $\tilde{\nabla}$ is symmetric and has zero curvature for any z; 2) the unity vector field ∂ is constant with respect to ∇ (i.e. $\nabla_X \partial = 0$ for any X).

Proposition 1.2. Any solution of the WDVV equations determines a Fröbenius manifold via the formulae (1.10), (1.11). Conversely, the metric ds^2 on a Fröbenius manifold is flat. In the corresponding flat co-ordinates t^α the metric and the multiplication have the form (1.12), (1.11) where $(c_\alpha{}^\gamma{}_\beta(t),\ \eta_{\alpha\beta})$ is a potential deformation of a Fröbenius algebra (i.e. it determines a solution of the WDVV equations).

Proof. Symmetry of $\tilde{\nabla}$ is equivalent to commutativity of the multiplication of vector fields. The metric ds^2 is flat since vanishing of the curvature of $\nabla = \tilde{\nabla}|_{z=0}$. In the flat co-ordinates t^α for ds^2 vanishing of the curvature of $\tilde{\nabla}$ needs as (1.10). The proposition is proved.

Let us explain the differential-geometric sense of the "free energy" $F(t)$. The family (1.14) of the flat symmetric connections $\tilde{\nabla}$ depending on the parameter z generates a deformation of the N-dimensional space with the metric ds^2. It turns out that the displacement vector of the deformation coincides with the gradient of the function $F(t)$. So the strain tensor of the deformation coincides with the Hessian of $F(t)$.

More precisely, let

$$x^\alpha = x^\alpha(t,z) = t^\alpha + zv^\alpha(t) + O(z^2) \qquad (1.15)$$

be flat co-ordinates for the connection $\tilde{\nabla}$. They are specified by condition of vanishing of the covariant Hessian $\tilde{\nabla}_\alpha \tilde{\nabla}_\beta x = 0$, or, equivalently, by the system

$$\partial_\alpha\, \partial_\beta x^\gamma = z\, c_\alpha{}^\epsilon{}_\beta\, \partial_\epsilon x^\gamma \ . \qquad (1.16)$$

The (infinitesimal) displacement vector $v^\alpha(t)$ is determined uniquely up to a transformation of the form

$$v^\alpha(t) \mapsto v^\alpha(t) + T^\alpha_\beta t^\beta$$

for any constant matrix (T^α_β).

Proposition 1.3. Gradient of the "free energy" $F(t)$ coincides with the displacement vector

$$\eta^{\alpha\mu}\partial_\mu F(t) = v^\alpha(t) \ . \tag{1.17}$$

Proof. The equation (1.16) for the vector v reads

$$\partial_\alpha \partial_\beta v^\gamma = c_\alpha{}^\gamma{}_\beta \ .$$

This proves (1.17).

The flat co-ordinates $x^\alpha(t,z)$ will be useful also in the next section. Note that gradients $\xi_\beta = \partial_\beta x^\alpha$ of these flat co-ordinates enjoy the system (1.9).

Let us return to investigation of the WDVV equation. The main idea of it is in choosing of a special co-ordinate system on Frõbenius manifold M in which the multiplication (1.13) of vector fields is determined by a constant structure constants (but the metric tensor is not constant). This I can do under additional assumption on the deformed Frõbenius algebra. From this moment in this section I will consider only deformations of *decomposable* Frõbenius algebras (it is sufficient to assume decomposability of $c_\alpha{}^\gamma{}_\beta(t)$ for some fixed t).

Main lemma 1. For any potential deformation of a decomposable Frõbenius algebra canonical local co-ordinates $u^i = u^i(t)$, $i = 1, \ldots, N$ exist such that

$$\partial_i \cdot \partial_j = \delta_{ij} \cdot \partial_i \ , \tag{1.18a}$$

$$\partial_i = \partial/\partial u^i \ . \tag{1.18b}$$

Proof. Due to openness of the decomposability property locally N linearly independent smooth vector fields $\partial_1, \dots, \partial_N$ exist satisfying (1.18a) (the idempotents of the algebra). Let the commutators of the fields have the form

$$[\partial_i, \partial_j] = f_{ij}^k \partial_k$$

for some functions f_{ij}^k on M. The Christoffel symbols for the connection ∇ are determined by the formula

$$\nabla_{\partial_j} \partial_j = \Gamma_{ij}^k \partial_k \ .$$

Vanishing of the Riemann curvature tensor for the connection $\tilde{\nabla}$ gives

$$\Gamma_{kj}^l \delta_i^l + \Gamma_{ki}^l \delta_{kj} - \Gamma_{ki}^l \delta_j^l - \Gamma_{kj}^l \delta_{ki} = f_{ij}^l \delta_k^l \ . \tag{1.19}$$

For $l = k$ this gives $f_{ij}^k = 0$. Lemma is proved.

Now let us pay attention at the invariant scalar product ds^2. What are the features of it in the canonical co-ordinates $u^1, \dots, u^N$?

To explain these properties of ds^2 I have to give some not well-known constructions of classical theory of curvilinear orthogonal co-ordinates. I recall that a diagonal metric

$$ds^2 = \sum_{i=1}^N g_{ij}(u)(du^i)^2 \tag{1.20}$$

determines curvilinear orthogonal co-ordinates in some Euclidean space iff the curvature of it vanishes. The metric (1.20) is called *Egoroff metric* [30] (see also [25], [26]) if the *rotation coefficients*

$$\gamma_{ij}(u) = \frac{\partial_j \sqrt{g_{ii}(u)}}{\sqrt{g_{jj}(u)}} \ , \qquad i \neq j \tag{1.21}$$

satisfy the symmetry condition

$$\gamma_{ji}(u) = \gamma_{ij}(u) \ . \tag{1.22}$$

Equivalently, a potential $V = V(u)$ exists such that

$$g_{ii}(u) = \partial_i V(u) , \qquad i = 1, \dots , N . \tag{1.23}$$

Vanishing of the curvature of the Egoroff metric (1.20) is equivalent to the following system

$$\partial_k \gamma_{ij} = \gamma_{ik}\gamma_{kj} \quad \text{for distinct } \; i,\, j,\, k \tag{1.24a}$$

$$\partial \gamma_{ij} = 0 , \qquad i \neq j \tag{1.24b}$$

where

$$\partial = \sum_{i=1}^{N} \partial_i . \tag{1.25}$$

It is easy to see that (1.24) can be reduced to a $(1+1)-$PDE, i.e. a solution $\gamma_{ij}(u) = \gamma_{ji}(u)$ is specified uniquely by fixation of $N(N-1)/2$ functions of one variable.

It was shown in [26] that (1.24) is an integrable system (without the symmetry $\gamma_{ji} = \gamma_{ij}$ it was studied in [27]). It is equivalent to the compatibility conditions of the system

$$\partial_j \psi_i = \gamma_{ij}\psi_j , \qquad i \neq j \tag{1.26a}$$

$$\partial \psi_i = z\psi_i , \tag{1.26b}$$

z is a spectral parameter. The relation of (1.24) to the N-waves interaction system is explained in [26].

The Egoroff zero-curvature metric (1.20) is called *∂-invariant* if

$$\partial g_{ii}(u) = 0 , \qquad i = 1, \dots , N . \tag{1.27}$$

It can be specified uniquely by its rotation coefficients and by N arbitrary constants via solving the linear system (1.26) for $z = 0$. The same is true for the corresponding flat co-ordinates. More precisely, let us consider the linear system

$$\partial_j \psi_i = \gamma_{ij} \psi_j \ , \qquad i \neq j \tag{1.28a}$$

$$\partial \psi_i = 0 \tag{1.28b}$$

for some solution $\gamma_{ij}(u) = \gamma_{ji}(u)$ of (1.24). It is easy to see that (1.28) is equivalent to a linear system of N linear ODE of the first order.

As it follows from (1.21), (1.27) $\psi_i = \sqrt{g_{ii}}$ is a solution of (1.28). Conversely, any solution ψ_{i1} of (1.28) determines a ∂-invariant Egoroff metric with the same rotation coefficients by the formula

$$g_{ii} = (\psi_{i1})^2 \ . \tag{1.29}$$

Let $\psi_{i1}(u), \dots, \psi_{iN}(u)$ be a basis in the space of solutions of (1.28). Note that the scalar product

$$\eta_{\alpha\beta} = \sum_{i=1}^{N} \psi_{i\alpha}(u)\, \psi_{i\beta}(u) \tag{1.30}$$

is non-degenerate and does not depend on u. Then the flat co-ordinates $t^1, \dots, t^N$ are determined by quadratures from the system

$$\partial_i t^\alpha = \psi_{i1} \psi_i{}^\alpha \equiv \sqrt{g_{ii}} \psi_i{}^\alpha \tag{1.31a}$$

where

$$\psi_i{}^\alpha = \eta^{\alpha\beta} \psi_{i\beta} \ , \qquad (\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1} \ . \tag{1.31b}$$

Note that

$$t_1 = \eta_{1\alpha} t^\alpha = V \qquad (1.32)$$

is the potential (1.23) of the metric.

Main lemma 2. The invariant metric ds^2 on a Frõbenius manifold M in the canonical co-ordinates $u^1, \ldots, u^N$ is a ∂-invariant Egoroff metric of zero curvature.

Proof. Let ω be a 1-form on M of the form

$$\omega(X) = \langle \partial, X \rangle \ . \qquad (1.33)$$

Here ∂ is the unity vector field. It has the form (1.25) in the canonical co-ordinates. From (1.3) and (1.18) one has

$$ds^2 = \sum_{i=1}^{N} \omega(\partial_i)\,(du^i)^2 \ .$$

This metric has zero curvature since it is constant in the co-ordinates t^α. The potential (1.23) for the metric $g_{ii}(u) = \omega(\partial_i)$ exists since the 1-form

$$\omega = \omega(\partial_i) du^i = \eta_{1\alpha} dt^\alpha$$

is closed. From the covariant constancy of the vector field ∂ it follows that

$$\sum_{k=1}^{N} \Gamma^j_{ik} = 0 \quad \text{for any } i,\, j \ .$$

From this and from identity $\nabla_k g_{ij} \equiv 0$ the ∂-invariance (1.27) follows. Lemma is proved.

Theorem 1. Any solution of the WDVV equations (in the decomposable case) is determined by a solution $\gamma_{ij}(u) = \gamma_{ji}(u)$ of the integrable system (1.24) and by N arbitrary constants by the formulae (1.29) - (1.31) and

$$c_{\alpha\beta\gamma}(t) = \sum_{i=1}^{N} \frac{\psi_{i\alpha}\,\psi_{i\beta}\,\psi_{i\gamma}}{\psi_{i1}} \ . \qquad (1.34)$$

Proof. Almost everything was proved in the two main lemmas. One needs to verify the equation (1.19) (for $f_{ij}{}^k = 0$) for any Egoroff metric. This can be done straightforward. To prove the formula (1.34) one can use that, by definition (1.18),

$$c_\alpha{}^\gamma{}_\beta(t) = \sum_{i=1}^N \frac{\partial u^i}{\partial t^\alpha} \frac{\partial u^i}{\partial t^\beta} \frac{\partial t^\gamma}{\partial u^i} . \tag{1.35}$$

The orthogonality conditions

$$\frac{\partial t^\gamma}{\partial u^i} = g_{ii} \frac{\partial u^i}{\partial t^\mu} \eta^{\mu\gamma} , \qquad i = 1, \ldots , N \tag{1.36}$$

together with (1.31) and (1.35) give (1.34). The theorem is proved.

Corollary. Any solution of the WDVV equation depends on $N(N-1)/2$ arbitrary functions of one variable and also on N arbitrary constants.

The following statement explains in what sense the system (1.24) is gauge equivalent to the WDVV equations (cf. [24]).

Proposition 1.4. The transformation

$$\xi_\alpha = \sum_{i=1}^N \sqrt{g_{ii}} \frac{\partial u^i}{\partial t^\alpha} \psi_i \tag{1.37a}$$

or, equivalently,

$$\psi_i = g_{ii}^{-1/2} \frac{\partial t^\alpha}{\partial u^i} \xi_\alpha \tag{1.37b}$$

transforms any solution ψ_i of (1.26) to a solution ξ_α of (1.9) and vice versa.

The proof is straightforward.

Remark. Existence of the canonical diagonal co-ordinates $u^1, \ldots , u^N$ can be proved even without the normalization (1.8b). The metric $g_{ii}(u)$ is specified in the form (1.29) by the rotation coefficients and by a solution $\psi_i(u)$ of the system (1.26a). General solution of (1.26a) depends on N arbitrary functions of one variable. If a global assumption on the behaviour of the functions $g_{ii}(u)$ is imposed then one can represent $\sqrt{g_{ii}(u)}$ as a linear combination of eigenfunctions of the spectral problem (1.26). For the deformations of Fröbenius algebras having been constructed in the appendix to [15] (for the case of decomposable Fröbenius algebra) all the rotation coefficients γ_{ij} vanish identically. The normalization condition (1.8b) in this example does not fulfill.

I end this section with discussion of potential deformations of indecomposable Fröbenius algebras. One can consider a particular case of deformations for which a co-ordinate system $u^1, \ldots, u^N$ in the coupling space M exists such that the multiplication (1.11) in these co-ordinates has constant structure coefficients

$$\partial_i \cdot \partial_j = c_{ij}{}^k \partial_k \, , \tag{1.38a}$$

$$\partial_i = \partial / \partial u^i \, , \tag{1.38b}$$

$$c_{ij}{}^k = \text{const.} \tag{1.38c}$$

(Probably, this is the general case. But this still should be proved.) Here $c_{ij}{}^k$ are the structure constants of a fixed Fröbenius algebra A_0. The invariant scalar product (1.12) of the deformation in these co-ordinates has the form

$$ds^2 = c_{ij}{}^k \omega_k(u) \, du^i \, du^j \, . \tag{1.39}$$

This follows from (1.3). The 1-form

$$\omega = \omega_k(u)\, du^k \tag{1.40a}$$

is defined as

$$\omega_k(u) = \langle \partial \, , \, \partial_k \rangle \tag{1.40b}$$

where ∂ is the unity of the deformation (cf. the proof of Main lemma 2). The 1-form ω is closed

$$d\omega = 0 \, . \tag{1.40c}$$

Vanishing of the curvature of the metric (1.39), (1.40) reads as a system of PDE for the functions $\omega_k(u)$. It depends on the Frõbenius algebra A_0 as on parameters. For the decomposable algebra A_0 this system is equivalent to (1.24).

It would be interesting to construct IST for this zero curvature system for any Frõbenius algebra A_0. Another interesting problem is to prove that vanishing of the curvature of (1.39), (1.40) provides existence of a potential F for the deformation (such that $c_{ijk} = c_{ij}^{\ \ p}\, c_{pk}^{\ \ q}\, \omega_q(u) = \nabla_i \nabla_j \nabla_k F(u)$). This will give an extension of the theorem 1 to deformations of arbitrary Frõbenius algebras (i.e. identically indecomposable). As it was noted in the Introduction, this might be of use for classification of topological sigma models. I hope to do it in forthcoming publications.

2. Integrable Hamiltonian hierarchies of hydrodynamic type, their solutions and τ-functions, and topological amplitudes.

Let us fix a solution of the WDVV equations. In other words, let us assume that the dependence of all the (tree-level) primary correlators on the special amplitudes

$$t_\alpha = \langle P \phi_\alpha \rangle , \quad \alpha = 1, \dots , N$$

is given on the "small phase space" $T^{\alpha,p} = 0$, $p > 0$ (i.e. on the coupling space M) of a model of topological field theory (I recall that $t_\alpha = \eta_{\alpha\beta} T^{\beta,0}$ for $T^{\alpha,p} = 0$, $p > 0$). How one can calculate all the tree-level correlators of the model on the whole phase space with arbitrary couplings $T^{\alpha,p}$? Here I'll show that the dependence of the special correlators $\langle P \phi_\alpha \rangle$ on the coupling constants $T^{\beta,p}$ is determined by a hierarchy of integrable Hamiltonian PDE systems with M as the targets space. Following the idea of [13] I define the τ-function of the hierarchy. The particular solution of the hierarchy is specified for which τ coincides with the tree-level partition function of the model of the topological field theory. The genus zero recursion relations of [2] for the correlators of the model are identified with the recursion operator of the hierarchy. And the "generalized hodograph transform" of [25] (being represented in the variational form of [26]) for solving the hierarchy proves to coincide with the string equations [2,10] (or "pre-string" equations in the terminology of [2]).

As it will be shown in Sect.4 for the topological A_n minimal model [10] my hierarchy coincides with the dispersionless Lax - Gelfand - Dikii hierarchy (essentially it follows from [13]). And for the model of [16] it coincides with the Whitham-type hierarchy being obtained by averaging over g-gap solutions of the Lax - Gelfand - Dikii hierarchy.

I start with recalling some ideas from the Hamiltonian theory [20,23] of systems of hydrodynamic type.

Let M be any manifold and $v^1, \dots , v^N$ any local co-ordinates on M. I recall that the formula

$$\{v^a(X),\, v^b(Y)\} = g^{ab}(v(X))\delta'(X-Y) + b^{ab}_c(v)v^c_X\delta(X-Y) \tag{2.1}$$

determines a Poisson bracket on the loop space[4] $\mathcal{L}M$ of smooth functions $v^a(X)$, $X \in S^1$ (Poisson brackets of hydrodynamic type) iff the tensor

$$g^{ab}(v) = g^{ba}(v) \tag{2.2}$$

determines a flat metric on M (the matrix $(g^{ab}(v))$ is assumed to be nondegenerate), and the coefficients $b^{ab}_c(v)$ can be represented in the form

$$b^{ab}_c(v) = -g^{ad}(v)\,\Gamma^b_{dc}(v) \tag{2.3}$$

where $\Gamma^b_{dc}(v)$ are the Christoffel symbols of the Levi-Civita connection ∇_i for the metric

$$ds^2 = g_{ab}(v)\,dv^a\,dv^b\ , \qquad (g_{ab}(v)) = (g^{ab}(v))^{-1} \tag{2.4}$$

(see [20]). As in [20] I shall consider as Hamiltonians only

$$H = \int h(v(X))dX\ , \tag{2.5}$$

$\int = \int_0^{2\pi}$, ("functionals of hydrodynamic type"); the density $h = h(v)$ does not depend on derivatives. So any function $h(v)$ on M determines a Hamiltonian system on $\mathcal{L}M$

$$\partial_T v^a(X) = \{v^a(X)\,, \textstyle\int h(v(Y)dY\} = w^a_b(v)\partial_X v^b\ , \tag{2.6a}$$

$$w^a_b(v) = \nabla^a\,\nabla_b\,h(v) \tag{2.6b}$$

(a Hamiltonian system of hydrodynamic type).

[4]Components of the loop space $\mathcal{L}M$ are numerated by conjugate classes of the fundamental group $\pi_1(M)$. Here only formal theory of Poisson brackets is considered. All the statements are proved for the component of $\mathcal{L}M$ consisting of loops of trivial homotopy class.

The class of Hamiltonian systems of hydrodynamic type is invariant under changes of co-ordinates on M. The following three types of co-ordinate systems are of special use in the theory of Hamiltonian systems of hydrodynamic type (see [20] for details).

1) Flat co-ordinates t^α, $\alpha = 1, \ldots, N$,

$$ds^2 = \eta_{\alpha\beta}\, dt^\alpha\, dt^\beta \ , \qquad (\eta_{\alpha\beta}) = \text{const.} \ , \tag{2.7a}$$

$$\{t^\alpha(X), t^\beta(Y)\} = \eta^{\alpha\beta}\delta'(X-Y) \ , \quad (\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1} \ . \tag{2.7b}$$

The functionals

$$\int t^\alpha(X) dX \ , \qquad \alpha = 1, \ldots, N \tag{2.8}$$

span the annihilator of $\{\ ,\ \}$. The Hamiltonian system (2.6) in the flat co-ordinates has the form

$$\partial_T t^\alpha = \eta^{\alpha\beta}\, \partial_\beta\, \partial_\gamma\, h(t) \partial_X t^\gamma \ , \tag{2.9}$$

$\partial_\beta = \partial / \partial t^\beta$.

2) Curvilinear orthogonal co-ordinates u^i,

$$ds^2 = \sum_{i=1}^{N} g_{ii}(u)\, (du^i)^2 \ . \tag{2.10}$$

These are of special importance in the theory of integrability of systems of hydrodynamic type ([25], see also [20]). A system of orthogonal co-ordinates $u^1, \ldots, u^N$ specifies a maximal Lagrangian subspace $\mathcal{H} \subset$ Funct $(\mathcal{L}M)$ of functionals of hydrodynamic type (2.5). The densities $h(u) \in \mathcal{H}$ are specified by the condition of diagonality of the covariant Hessian

$$\nabla_i\, \nabla_j h(u) = 0 \quad \text{for} \quad i \neq j \ . \tag{2.11}$$

Equivalently, the vector-function

$$\psi_i{}^h = g_{ii}^{-1/2}\, \partial_i h \ , \tag{2.12}$$

$\partial_i = \partial/\partial u^i$, satisfies the system (cf. (1.26a))

$$\partial_j \psi_i{}^h = \gamma_{ji}\psi_j{}^h \qquad \text{for } i \neq j \; . \tag{2.13}$$

Here γ_{ij} are the rotation coefficients of the metric (2.10). The corresponding commuting Hamiltonian systems (2.6) are diagonal,

$$\partial_\tau u^i(X) = \{u^i(X), \int h(u)dY\} = w^i(u)\partial_X u^i(X) , \tag{2.14}$$

$i = 1, \ldots, N, \quad h \in \mathcal{H}$, so the variables $u^1, \ldots, u^N$ are the common Riemann invariants for them (and $w^1(u), \ldots, w^N(u)$ are the corresponding characteristic speeds). All the systems (2.14) are integrable [25] (the algorithm of integration in the form of [26] will be given below).

3). "Physical" (or Liouvillean) co-ordinates v^a in which the Poisson bracket (2.6) have the form

$$\{v^a(X), v^b(Y)\} = [q^{ab}(v(Y)) + q^{ba}(v(X))]\delta'(X-Y) , \tag{2.15}$$

for some matrix $q^{ab}(u)$. So the metric g^{ab} and the connection b^{ab}_c can be represented in the form

$$g^{ab}(v) = q^{ab}(v) + q^{ba}(v) , \tag{2.16a}$$

$$b^{ab}_c(v) = \partial q^{ab}(v)/\partial v^c . \tag{2.16b}$$

These mean that the functionals

$$\int v^a(X)\, dX , \qquad a = 1, \ldots, N . \tag{2.17}$$

commute pairwise. So they are the standard dependent variables in the equations of hydrodynamics of an ideal fluid (possibly, with inner degrees of freedom). Also the Hamiltonian averaging procedure of [23,20] always provides a Liouvillean co-ordinate system for the averaged system. The particular case of linear functions $q^{ab}(v)$ is of special interest due to its relation to vector analogues of the Virasoro algebra [34]. It can be shown (see below) that the string amplitudes

$$v_\alpha = \langle \mathcal{P} \sigma_{k_\alpha}(\phi_\alpha) \rangle \tag{2.18}$$

for any $k_1, \ldots, k_N \geq 0$ (here $\mathcal{P}$ is the puncture operator) are physical co-ordinates on the coupling space. (I recall that the flat co-ordinates on M are the amplitudes $t_\alpha = \langle \mathcal{P} \phi_\alpha \rangle$.)

Let us fix a solution of the WDVV equations (i.e. a potential deformation $c_\alpha{}^\gamma{}_\beta(t)$, $\eta_{\alpha\beta}$ of a Fröbenius algebra). The metric

$$ds^2 = \eta_{\alpha\beta}\, dt^\alpha\, dt^\beta \tag{2.19}$$

determines a Poisson bracket (2.7). Let us consider a family of systems of hydrodynamic type

$$\partial_{T^\alpha} t^\beta = c_\alpha{}^\beta{}_\gamma(t) \partial_X t^\gamma\ , \qquad \alpha = 1, \ldots, N\ . \tag{2.20}$$

(Note that $\partial_{T^1} t^\beta = \partial_X t^\beta$. So T^1 can be identified with X.)

This system can be rewritten in an elegant co-ordinate-free way using the multiplication (1.11) of vector-fields on the coupling space M: a map $f : \mathbf{R}^2_{X,T^\alpha} \to M$ enjoys the equation (2.20) if

$$\partial_{T^\alpha} f = \partial_\alpha \cdot \partial_X f \tag{2.21}$$

(we consider here $\partial_{T^\alpha} f(X, T^\alpha)$ and $\partial_X f(X, T^\alpha)$ as vector-fields on M).

Proposition 2.1. The systems (2.20) commute pairwise, and they are Hamiltonian with respect to the P.B. (2.7) with the (density of) Hamiltonians being equal to $F_\alpha(t)$ respectively. Conversely, if the systems (2.20) commute pairwise and are Hamiltonian system w.r.t. the P.B. (2.7) and their Hamiltonians are derivatives of a function $F(t)$, then $F(t)$ enjoys the WDVV equations and $c_{\alpha\ \gamma}^{\ \beta}(t)$, $\eta_{\alpha\beta}$ is the correspondent potential deformation.

Proof. The "direct" part of the proposition can be verified straightforward from (2.9). Conversely, if the system (2.20) is a Hamiltonian one with the Hamiltonian

$$H_\alpha = \int F_\alpha \, dX \, , \tag{2.22}$$

then

$$c_{\alpha\ \gamma}^{\ \beta} = \eta^{\beta\lambda} F_{\alpha\gamma\lambda} \, . \tag{2.23}$$

So

$$c_{\alpha\beta\gamma} = \eta_{\gamma\epsilon} \, c_{\alpha\ \beta}^{\ \epsilon} \tag{2.24}$$

is a symmetric tensor. The condition of commutativity of the flows (2.20) is equivalent to the associativity condition for the structure constants (2.23). The proposition is proved.

Note that the functional $\int F_1 dX = \frac{1}{2} \int \eta_{\alpha\beta} t^\alpha t^\beta \, dX$ generates the spatial translations $\partial_{T^1} = \partial_X$. The proposition provides another reformulation of the WDVV equations: a function $F(t)$ on a space with a metric $ds^2 = \eta_{\alpha\beta} \, dt^\alpha \, dt^\beta$ satisfies the WDVV equations iff the Legendre transform

$$t^\alpha \mapsto v_\alpha(t) = F_\alpha(t) \tag{2.25}$$

provides Liouvillean co-ordinates for the metric ds^2, and, particularly, $F_1(t)$ is the momentum density.

Remark. It will be shown below that the form (2.25) for the transform from the flat co-ordinates to Liouville co-ordinates provides existence of infinite number of conservation laws of the equations (2.20). Therefore, this provides an existence of a rich family of nonlinear changes of co-ordinates conserving the Liouville form (2.16) of the P.B. (a *priori*, (2.16) admits only affine transformations of the co-ordinates v). Existence of such nonlinear transformations is a feature of hierarchies of hydrodynamic type being obtained from Hamiltonian hierarchies of KdV-type by the averaging procedure [20]. Another feature ("strong Liouvillean property" [20]) of the Hamiltonian formalism of the averaged systems is in compatibility of the Liouville form (2.16) of the averaged Poisson bracket with the restriction onto any affine subspace (the affine structure here is determined by the physical co-ordinates!). This does not hold for any solution of the WDVV equations. Explicit form of the constraint on F will be given in the forthcoming publication.

The topological counterparts of the matrix-type model seem to have this feature. This might give a nontrivial procedure of restriction of a solution of the WDVV equation to some subspaces in the coupling space (affine subspaces in the physical co-ordinates (2.25)).

Let us construct conservation laws for the systems (2.20).

I am going to show that the flat co-ordinates for the connection (1.14) are the generating functions for the conservation laws we need. This gives also a recursion operator for the conservation laws.

Proposition 2.2. For any $\alpha = 1, \ldots, N$ formal series

$$x^\alpha = x^\alpha(t, z) = \sum_{p=0}^{\infty} h^{\alpha,p}(t) z^p \ , \qquad h^{\alpha,0} = t^\alpha \ , \tag{2.26}$$

exist such that

$$\partial_\beta \, \partial_\gamma \, x^\alpha = z \, c_{\beta}{}^{\epsilon}{}_{\gamma} \, \partial_\epsilon \, x^\alpha \ . \tag{2.27}$$

Proof. For the coefficients $h^{\alpha,p}(t)$ the system (2.27) gives the recursion relations

$$\partial_\beta \, \partial_\gamma \, h^{\alpha,p+1}(t) = c_{\beta}{}^{\epsilon}{}_{\gamma} \, \partial_\epsilon \, h^{\alpha,p}(t) \ , \qquad p = 0, 1, \ldots \tag{2.28}$$

Solvability of them follows from the conditions

$$c_{\gamma}{}^{\epsilon}{}_{\beta} = c_{\beta}{}^{\epsilon}{}_{\gamma} \ , \qquad \partial_\lambda c_{\beta}{}^{\epsilon}{}_{\gamma} = \partial_\beta c_{\lambda}{}^{\epsilon}{}_{\gamma} \ .$$

The proposition is proved.

For calculations it is convenient to rewrite equation (2.27) in the form

$$\partial_\beta(\partial_\gamma x^\alpha) = z(\partial_\beta \cdot \partial_\gamma)x^\alpha \tag{2.27'}$$

(the multiplication 1.11 is used in the r.h.s.).

It is more convenient to use linear combinations of the functions (2.26)

$$x_\alpha(t,z) = \eta_{\alpha\beta}x^\beta(t,z) = \sum_{p=0}^{\infty} h_{\alpha,p}(t)z^p \ , \tag{2.29a}$$

$$h_{\alpha,p}(t) = \eta_{\alpha\beta}\, h^{\beta,p}(t) \ . \tag{2.29b}$$

I recall (see the previous sect.) that

$$h_{\alpha,1}(t) = F_\alpha(t) \ . \tag{2.30}$$

Particularly,

$$h_{1,1}(t) = \frac{1}{2}\eta_{\alpha\beta}t^\alpha t^\beta \ . \tag{2.31}$$

The solution $x_\alpha(t,z)$ can be normalized in such a way that

$$\langle \nabla x_\alpha(t,z)\, ,\, \nabla x_\beta(t,-z)\rangle = \eta_{\alpha\beta}\, , \tag{2.32}$$

$$\partial_1 x_\alpha(t,z) = z x_\alpha(t,z) + \eta_{1\alpha} \ . \tag{2.33}$$

Here ∇ means the gradient and $\langle\, ,\, \rangle$ the scalar product w.r.t. $(\eta_{\alpha\beta})$:

$$\nabla^\lambda x_\alpha(t,z) = \eta^{\lambda\mu}\partial_\mu x_\alpha(t,z) \ , \tag{2.34}$$

$$\langle \nabla x_\alpha(t,z)\, ,\, \nabla x_\beta(t,w)\rangle = \eta_{\lambda\mu}\nabla^\lambda x_\alpha(t,z)\nabla^\mu x_\beta(t,w) = \eta^{\lambda\mu}\partial_\lambda x_\alpha(t,z)\partial_\mu x_\beta(t,w) \tag{2.35}$$

The ambiguity of the above definition of the functions $x_\alpha(t,z)$ has the form

$$x_\alpha(t,z) \to T_\alpha^\lambda(z)x_\lambda(t,z) \tag{2.36}$$

where $T_\alpha^\lambda(z)$ are any power series in z with constant coefficients satisfying the conditions

$$T_\alpha^\lambda(0) = \delta_\alpha^\lambda \tag{2.37}$$

$$\eta_{\lambda\mu} T_\alpha^\lambda(z) T_\beta^\mu(-z) = \eta_{\alpha\beta} \ . \tag{2.38}$$

Expressions of the functions $h_{\alpha,p}(t)$ via eigenfunctions of the gauge-equivalent linear problem (1.26) will be given in the appendix to this section.

The gradients of the functions $h_{\alpha,p}(t)$ obey different bilinear identities that wil be used in the forthcoming pages. These can be summarized in the following "generating identity"

$$\nabla\langle\nabla x_\alpha(t,z),\ \nabla x_\beta(t,w)\rangle = (z+w)\nabla x_\alpha(t,z)\cdot\nabla x_\beta(t,w) \ . \tag{2.39}$$

Here the product (1.11) is used in the r.h.s.

It is interesting that the commutators of the gradient vector fields also can be expressed via the same multiplication:

$$[\nabla x_\alpha(t,z),\ \nabla x_\beta(t,w)] = (w-z)\nabla x_\alpha(t,z)\cdot\nabla x_\beta(t,w) \ . \tag{2.40}$$

Proposition 2.3 For any solution $t = t(T^1,\dots,T^N,X)$ of the system (2.20) the following identities hold:

$$\partial_{T^\alpha} x_\beta(t,z) = \partial_X[z^{-1}(\partial_\alpha x_\beta(t,z) - \eta_{\alpha\beta})] \ . \tag{2.41}$$

Conversely, the equation (2.41) implies (2.20).

Proof. For the l.h.s. of (2.41) one has

$$\partial_{T^\alpha} x_\beta(t;z) = \partial_\epsilon x_\beta(t;z) c_\alpha^{\ \epsilon}{}_\gamma \, \partial_X t^\gamma \ . \tag{2.42}$$

Since the functions $x_\beta(t;z)$ satisfy the system (2.27), (2.42) can be rewritten as

$$\partial_{T^\alpha} x_\beta(t;z) = z^{-1}\,\partial_\gamma\,\partial_\alpha\,x_\beta(t;z)\,\partial_X t^\gamma \ .$$

The normalization condition

$$\partial_\alpha x_\beta(t;0) = \eta_{\alpha\beta}$$

completes the proof.

The representation (2.42) of the equations (2.20) is of the Flaschka - Forest - McLaughlin (FFM) type [18] (but it does not coincide with the FFM representation even for the original case of the Whitham equations!).

Corollary. The functions

$$h_{\beta,p}(t) = \operatorname*{res}_{z=0} z^{-p-1} x_\beta(t;z)\,, \qquad p = 0,1,\ldots \tag{2.43}$$

for any $\beta = 1,\ldots,N$, $p = 0,1,\ldots$, are densities of conservation laws for the equations (2.20):

$$\partial_{T^\alpha} h_{\beta,p}(t) = \partial_X f_{\alpha\beta,p}(t)\,, \tag{2.44a}$$

$$f_{\alpha\beta,p}(t) = \operatorname*{res}_{z=0} z^{-p-2}\, \partial_\alpha\, x_\beta\,(t,z)\,. \tag{2.44b}$$

Commutativity of these conservation laws w.r.t. the P.B. (2.7) follows from the following statement.

Proposition 2.4 The P.B. (2.7) of the functionals $x_\alpha(t(X),z)$ have the following Liouvillean form

$$\{x_\alpha(t(X),z_1),\, x_\beta(t(Y),\, z_2)\} = [q_{\alpha\beta}(t(Y);z_1,z_2) + q_{\beta\alpha}(t(X);z_2,z_1)]\delta'(X-Y) \tag{2.45a}$$

where

$$q_{\alpha\beta}(t;z_1;z_2) = \frac{z_2}{z_1+z_2}\langle \nabla x_\alpha(t,z_1),\, \nabla x_\beta(t,z_2)\rangle \ . \tag{2.45b}$$

Proof. For the derivatives of $q_{\alpha\beta}(t;z_1,z_2)$ one has from (2.39)

$$\nabla q_{\alpha\beta}(t;z_1,z_2) = z_2 \nabla x_\alpha(t,z_1)\cdot \nabla x_\beta(t,z_2) \ .$$

And the l.h.s. of (2.45a) has the form

$$\begin{aligned}\{x_\alpha(t(X),z_1),\, x_\beta(t(Y),z_2)\} &= \langle \nabla x_\alpha(t(X),z_1),\, \nabla x_\beta(t(Y),z_2)\rangle \delta'(X-Y) = \\ &= \langle \nabla x_\alpha(t(X)),\, z_1),\, \nabla x_\beta(t(X),\, z_2)\rangle \delta'(X-Y) + \\ &\quad + z_2\langle \nabla x_\alpha(t,z_1)\cdot \nabla x_\beta(t,z_2),\, \partial_X t\rangle \delta(X-Y) \ .\end{aligned}$$

This completes the proof.

Corollary. Functionals with the densities $h_{\alpha,p}(t)$ commute pairwise.

The commuting Hamiltonians

$$H_{\alpha,p} = \int h_{\alpha,p+1}(t(X))dX, \qquad \alpha = 1,\ldots,N; \qquad p = -1,\,0,\,1,\,\ldots, \tag{2.46}$$

generate a hierarchy of commuting Hamiltonian systems of hydrodynamic type

$$\partial_{T^{\alpha,p}} t^\beta(X) = \{t^\beta(X),\, H_{\alpha,p}\} = c_{(\alpha,p)}{}^{\beta}{}_{\gamma}\partial_X t^\gamma \ , \qquad p = 0,\,1,\,\ldots \tag{2.47a}$$

(the functionals $H_{\alpha,-1}$ span the annihilator of the P.B. (2.7)) where

$$c_{(\alpha,p)}{}^{\beta}{}_{\gamma} = \eta^{\alpha\lambda}\,\partial_\lambda\,\partial_\gamma h_{\alpha,p+1} = \eta^{\mu\lambda}\,\partial_\lambda h_{\alpha,p}\, c_{\mu}{}^{\beta}{}_{\gamma} \ . \tag{2.47b}$$

So the system (2.47) can be rewritten like (2.21) using the multiplication by the gradient vector field $\nabla h_{\alpha,p}$

$$\partial_{T^{\alpha,p}} f = \nabla h_{\alpha,p}\cdot \partial_X f \tag{2.47c}$$

(this is an equation for a map $f : \mathbf{R}^2_{X,T^{\alpha,p}} \to M$ solving the system (2.47)). For $p = 0$ one has

$$c_{(\alpha,0)}{}^{\beta}{}_{\gamma} = c_{\alpha}{}^{\beta}{}_{\gamma} \,. \tag{2.48}$$

So the system (2.47) coincides with (2.20) for $p = 0$, $T^{(\alpha,0)} \equiv T^{\alpha}$. The formula (2.47b) gives a recursion procedure for constructing the system (2.47) on the basis of the system (2.20):

$$\partial_{T^{\alpha,p}} = \eta^{\mu\lambda}\partial_{\lambda}\, h_{\alpha,p}\, \partial_{T^{\mu}} \,. \tag{2.49}$$

For $p = 1$ the equations (2.47) read

$$\partial_{T^{\alpha,1}} t^{\beta} = c_{\gamma}^{\mu\beta}\, F_{\mu\alpha}\, \partial_X t^{\gamma} \,. \tag{2.50}$$

The particular value $\alpha = 1$ is of special importance

$$\partial_{T^{1,1}} t^{\beta} = c_{\lambda}{}^{\beta}{}_{\gamma}\, t^{\lambda}\, \partial_X t^{\gamma} \,, \tag{2.51}$$

the density of the corresponding Hamiltonian $H_{1,1}$ equals

$$h_{1,2} = F_{\alpha} t^{\alpha} - 2F \,. \tag{2.52}$$

For the generating function $x_{\beta}(t,z)$ for the densities of the conservation laws (2.43) one obtains

$$\partial_{T^{\alpha,p}}\, x_{\beta}(t,z) = \partial_X \operatorname*{res}_{w=0}\, w^{-p-1} z^{-1} \langle \nabla x_{\alpha}(t,w),\, \nabla x_{\beta}(t,z)\rangle \,. \tag{2.53}$$

This provides a FFM-type representation of all the hierarchy (2.47).

Commutativity of the systems (2.47) provides the commutativity of the operators $c_{(\alpha,p)}{}^{\beta}{}_{\gamma}$,

$$c_{(\alpha,p)}{}^{\lambda}{}_{\mu}\, c_{(\beta,q)}{}^{\mu}{}_{\nu} = c_{(\beta,q)}{}^{\lambda}{}_{\mu}\, c_{(\alpha,p)}{}^{\mu}{}_{\nu} \,. \tag{2.54}$$

All these operators are symmetric with respect to the scalar product $(\eta_{\alpha\beta})$.

In the decomposable case the commutativity (2.54) implies diagonality of all the systems (2.47) in the canonical diagonal co-ordinates $u^1, \ldots, u^N$.

It can be proved (see Appendix below) that in this case linear combinations of the functionals $H_{\alpha,p}$ form a complete set of conservation laws of the system (2.20) (or, of the hierarchy (2.47)). In other words, they span the canonical Lagrangian subspace $\mathcal{H}$ (see above). It would be interesting to investigate completeness of the conservation laws $H_{\alpha,p}$ for identically indecomposable deformation.

Let us proceed to construction of solutions of the hierarchy (2.47). It turns out that, in some sense, it has only one solution. The others formally can be obtained by shifts along the $T^{\alpha,p}$-axis. At least it can be proved using [25] for the decomposable case. To construct it, let us use the obvious scaling group of symmetries of the systems (2.47):

$$X \mapsto cX\,, \quad T^{\alpha,p} \mapsto cT^{\alpha,p}\,, \quad t^\beta \mapsto t^\beta\,. \tag{2.55}$$

Let us denote by $T = (T^{\alpha,p})$ the infinite vector with the co-ordinates $T^{\alpha,p}$.

Proposition 2.5. The hierarchy (2.47) in the domain

$$T^{1,1} = \varepsilon\,, \quad X, T^{\alpha,p} = o(\varepsilon) \quad \text{for} \quad (\alpha,p) \neq (1,1)\,, \quad \varepsilon \to 0 \tag{2.56}$$

has a nonconstant solution $t^\beta = t^\beta(X,T)$ being invariant with respect to the scaling transformations (2.55). It can be found in implicit form from the "variational principle"

$$\nabla[\Phi_T(t) + Xt_1] = 0 \tag{2.57}$$

where

$$\Phi_T(t) = \sum_{\alpha,p} T^{\alpha,p}\, h_{\alpha,p}(t)\,. \tag{2.58}$$

Proof. For scaling invariant solutions of (2.47) one has

$$\left(X\delta^\beta_\gamma + \sum c_{(\alpha,p)}{}^\beta{}_\gamma(t)\right)\partial_X t^\gamma = 0\,, \quad \beta = 1, \ldots, N\,.$$

Using (2.47b), this system can be represented in the form (2.57). In the domain (2.56) one has

$$\partial_\mu \partial_\nu [\Phi_T(t) + X t_1] = \varepsilon \eta_{\mu\nu} + o(\varepsilon) \ . \tag{2.59}$$

Hence the solution $t = t(X,T)$ locally is unique. Therefore it satisfies (2.47). The proposition is proved.

The variable X can be omitted in the solution (2.57) (since it can be restored by a shift $T^{1,0} \mapsto T^{1,0} + X$).

Let us construct the "τ-function" of the hierarchy (2.47) (cf. [13,14], [16]) for the particular solution (2.57). It is defined by the formula

$$\log \tau_0(T) = \frac{1}{2} \operatorname*{res}_{z=0} \operatorname*{res}_{w=0} (z+w)^{-1} \sum z^{-r-1} w^{-s-1} T^{\lambda,r} T^{\mu,s} \left[\langle \nabla x_\lambda(t,z), \nabla x_\mu(t,w) \rangle - \eta_{\lambda\mu} \rangle \right]|_{t=t(T)} \ . \tag{2.60}$$

Here the functions $x_\alpha(t,z)$ are assumed to enjoy the normalization (2.32).

Proposition 2.6. The τ-function (2.60) satisfies the following equations:

$$\partial_{T^{\alpha,p}} \partial_{T^{\beta,q}} \log \tau_0(T) = \operatorname*{res}_{z=0} \operatorname*{res}_{w=0} (z+w)^{-1} z^{-p-1} w^{-q-1} \langle \nabla x_\alpha(t,z), \nabla x_\beta(t,w) \rangle \ . \tag{2.61}$$

Proof. The equations (2.57) can be represented in the form

$$\left[\operatorname*{res}_{z=0} \sum z^{-r-1} T^{\lambda,r} \nabla x_\lambda(t,z) \right]_{t=t(T)} = 0 \ .$$

Using (2.39), one has

$$\begin{aligned} \partial_{T^{\alpha,p}} \log \tau_0 = & \operatorname*{res}_{z=0} \operatorname*{res}_{w=0} z^{-p-1} (z+w)^{-1} \sum w^{-s-1} T^{\mu,s} \\ & \left[\langle \nabla x_\alpha(t,z), \nabla x_\mu(t,w) \rangle - \eta_{\alpha\mu} \right] + \\ & + \frac{1}{2} \operatorname*{res}_{z=0} \operatorname*{res}_{w=0} \sum z^{-r-1} w^{-s-1} T^{\lambda,r} T^{\mu,s} \langle \nabla x_\lambda(t,z) \cdot \nabla x_\mu(t,w), \partial_{T^{\alpha,p}} t \rangle = \\ & = \operatorname*{res}_{z=0} \operatorname*{res}_{w=0} z^{-p-1} (z+w)^{-1} \sum w^{-s-1} T^{\mu,s} \left[\langle \nabla x_\alpha(t,z), \nabla x_\mu(t,w) \rangle - \eta_{\alpha\mu} \right] \ . \end{aligned}$$

The derivative of this expression w.r.t. $T^{\beta,q}$ can be calculated in similar way. The proposition is proved.

Let us introduce a notation for the generating function of the second derivatives of $\log \tau_0$:

$$\begin{aligned} V_{\alpha\beta}(t;z,w) &= (z+w)^{-1}[\langle \nabla x_\alpha(t,z),\, \nabla x_\beta(t,w)\rangle - \eta_{\alpha\beta}] = \\ &= \sum_{p,q} V_{(\alpha,p),(\beta,q)}(t)\, z^p\, w^q\ . \end{aligned} \tag{2.62}$$

The proposition implies

$$\partial_{T^{\alpha,p}}\, \partial_{T^{\beta,q}} \log \tau_0 = V_{(\alpha,p),(\beta,q)}\ . \tag{2.63}$$

Note that the P.B. (2.7) of the functionals $h_{\alpha,p}(t(X))$ have the form

$$\{h_{\alpha,p}(t(X)),\, h_{\beta,q}(t(Y))\} = [V_{(\alpha,p),(\beta,q-1)}(t(Y)) + V_{(\beta,p),(\alpha,q-1)}(t(X))]\delta'(X-Y)\ . \tag{2.64}$$

For the coefficients $V_{(\alpha,p),(\beta,q)}$ one has formulae

$$V_{(\alpha,0),(\beta,0)} = F_{\alpha\beta}\ , \tag{2.65a}$$

$$V_{(\alpha,p),(1,0)} = h_{\alpha,p}\ , \tag{2.65b}$$

$$V_{(\alpha,p),(\beta,0)} = \partial_\beta\, h_{\alpha,p+1}\ , \tag{2.65c}$$

$$V_{(\alpha,p),(1,1)} = (t^\lambda \partial_\lambda - 1)\, h_{\alpha,p+1}\ , \tag{2.65d}$$

$$V_{(\alpha,0),(1,1)} = F_{\alpha\lambda}t^\lambda - F_\alpha\ , \tag{2.65e}$$

$$V_{(1,1),(1,1)} = F_{\lambda\mu}t^\lambda t^\mu - 2F_\lambda t^\lambda + 2F\ . \tag{2.65f}$$

Also one has an identity

$$\operatorname*{res}_{w=0}\ w^{-1} V_{\alpha 1}(t;z,w) = x_\alpha(t,z)\ . \tag{2.66}$$

The recurrence relation (2.49) reads

$$\partial_{T^{\alpha,p}} = \eta^{\mu\lambda}\, \partial_{T^{\alpha,p-1}}\, \partial_{T^{\lambda,0}} \log \tau_0\, \partial_{T^{\mu,0}} \ . \tag{2.67}$$

Particularly, one obtains recurrence relation for derivatives of the τ-functions $\tau = \tau_0$

$$\partial_{T^{\alpha,p}}\, \partial_{T^{\beta,q}}\, \partial_{T^{\gamma,r}} \log\tau = \left(\partial_{T^{\alpha,p-1}}\, \partial_{T^{\lambda,0}} \log\tau\right)\, \eta^{\lambda\mu}\, \partial_{T^{\mu,0}}\, \partial_{T^{\beta,q}}\, \partial_{T^{\gamma,r}} \log\tau \ . \tag{2.68}$$

The identities

$$\partial_\mu \left(\sum T^{\alpha,p}\, h_{\alpha,p}\right) = 0 \ , \qquad \mu = 1, \ldots, N$$

for finding the solution (2.57) read

$$\sum_{p\geq 1} T^{\alpha,p}\, \partial_{T^{\alpha,p-1}}\, \partial_{T^{\mu,0}} \log\tau_0 + T_{\mu,0} = 0 \ . \tag{2.69}$$

Also the τ-function satisfies the Euler identity

$$\sum T^{\alpha,p}\, \partial_{T^{\alpha,p}} \log\tau_0 = 2\log\tau_0 \ . \tag{2.70}$$

Let us introduce explicitly the shift

$$T^{1,1} \mapsto T^{1,1} + 1 \tag{2.71}$$

into the solution (2.57). Thus the functions $t^\beta(T)$ are specified now by the system

$$\nabla\hat{\Phi}_T(t) = 0 \ , \tag{2.72a}$$

$$\hat{\Phi}_T(t) = \Phi_T(t) - \tfrac{1}{2}\eta_{\alpha\beta}\, t^\alpha t^\beta \ . \tag{2.72b}$$

They are well-defined for sufficiently small T. The solution $t = t(T)$ can be considered as the fixed point of the gradient map $t \mapsto \nabla\Phi_T(t)$,

$$\nabla\Phi_T(t) = t \ . \tag{2.73}$$

For the vector fields $\partial_{T^{\alpha,p}} t$ from (2.73) and (2.28) one obtains

$$\partial_{T^{\alpha,p}} t = \frac{\nabla h_{\alpha,p}}{\partial - \sum_{q\geq 1} T^{\gamma,q} \nabla h_{\gamma,q-1}} . \tag{2.73'}$$

Since $\partial = \partial_{\alpha=1}$ is the unity for the multiplication (1.11), the denominator is an invertible vector field in some neighbourhood of the "small phase space" $T^{\gamma,q} = 0$, $q > 0$.

The formula (2.60) for the τ-function should be modified by the same shift:

$$\begin{aligned} \log \tau_{1,1}(T) &= \frac{1}{2} \operatorname*{res}_{z=0} \operatorname*{res}_{w=0} \sum z^{-r-1} w^{-s-1} T^{\lambda,r} T^{\mu,s} V_{\lambda\mu}(t(T); z, w) - \\ &- \operatorname*{res}_{z=0} \operatorname*{res}_{w=0} w^{-2} \sum z^{-r-1} T^{\lambda,r} V_{\lambda 1}(t(T); z, w) + \frac{1}{2} V_{(1,1),(1,1)}(t(T)) . \end{aligned} \tag{2.74}$$

The identity (2.63) (but not (2.70)) still is valid for this τ-function.

Let us consider the restriction of the solution (2.72) onto the N-dimensional vectors

$$T_0 = (T^{1,0}, \ldots, T^{N,0}, 0, \ldots) . \tag{2.75}$$

For these vectors one has

$$\hat{\Phi}_{T_0}(t) = \eta_{\alpha\beta} T^{\alpha,0} t^{\beta} - \frac{1}{2} \eta_{\alpha\beta} t^{\alpha} t^{\beta} . \tag{2.76}$$

This gives an obvious solution

$$T^{\alpha,0} = t^{\alpha}, \qquad \alpha = 1, \ldots, N \tag{2.77}$$

of the system (2.20). So a solution $t = t(T)$ of (2.72) can be specified in such a way that

$$t^{\alpha}(T) \approx T^{\alpha,0} \quad \text{for small} \quad T - T_0 . \tag{2.78}$$

Further in this section I shall use only this solution of (2.72).

Proposition 2.7. The τ-function (2.74) being restricted onto the subspace

(2.75) coincides with the primary partition function:

$$\log \tau_{1,1}(T_0) = F(T_0) \ . \tag{2.79}$$

Proof. This follows from the formulae (2.65). This proposition can be considered as a hint to consider the τ-function (2.74) for any T as the (tree-level) partition function of a model of topological field theory coupled to topological gravity with given primary operator algebra. The additional arguments are provided by the equations (2.68) and (2.69). The equation (2.68) still is valid for the τ-function (2.74). It can be considered as the Witten's recursion relation [2] for the tree-level correlators in topological field theory

$$\langle \sigma_p(\phi_\alpha)\phi_A\,\phi_B\rangle = \langle \sigma_{p-1}(\phi_\alpha)\phi_\lambda\rangle\eta^{\lambda\mu}\langle \phi_\mu\,\phi_A\,\phi_B\rangle \tag{2.80}$$

where

$$\phi_A = \sigma_q(\phi_\beta)\ , \qquad \phi_B = \sigma_r(\phi_\gamma)\ ,$$

and the correlators of the descendants $\sigma_p(\phi_\alpha)$ are defined by the derivatives of $\log \tau$:

$$\langle \sigma_p(\phi_\alpha)\phi_\lambda\rangle = \partial_{T^{\alpha,p}}\,\partial_{T^{\lambda,0}}\,\log \tau_{1,1}\ , \tag{2.81a}$$

$$\langle \sigma_p(\phi_\alpha)\sigma_q(\phi_\beta)\sigma_r(\phi_\gamma)\rangle = \partial_{T^{\alpha,p}}\,\partial_{T^{\beta,q}}\,\partial_{T^{\gamma,r}}\,\log \tau_{1,1}\ . \tag{2.81b}$$

So the variables $T^{\alpha,p}$ can be considered as the descendant couplings. Note that for the puncture operator $P = \sigma_0(\phi_1)$ the correlators $\langle P\,\phi_\alpha\rangle$ have the form

$$\langle P\,\phi_\alpha\rangle = \partial_{T^{(1,0)}}\,\partial_{T^{\alpha,0)}}\,\log\,\tau \equiv t_\alpha \tag{2.82}$$

(the identity for any T, cf. [10]). So $\langle P\,\phi_\alpha\rangle$ are the flat co-ordinates on the coupling space.

The identity (2.69) after the shift (2.71) reads

$$\sum_{p\geq 1} T^{\alpha,p}\,\partial_{T^{\alpha,p-1}}\,\partial_{T^{\mu,0}}\log\tau_{1,1} + T_{\mu,0} = \partial_{T^{1,0}}\,\partial_{T^{\mu,0}}\log\tau_{1,1} \equiv t_\mu\ , \qquad (2.83a)$$

or, after integration

$$\sum_{p\geq 1} T^{\alpha,p}\,\partial_{T^{\alpha,p-1}}\log\tau_{1,1} + \frac{1}{2}\eta_{\alpha\beta}T^{\alpha,0}\,T^{\beta,0} = \partial_{T^{1,0}}\log\tau_{1,1}\ . \qquad (2.83b)$$

(vanishing of the integration constant can be proved by reducing to T_0 and using (2.49)).

This is nothing but the string equation [2,10].

Trivial example. Let us consider trivial deformation $c_{\alpha\beta\gamma}$ = const. of a N-dimensional Fröbenius algebra A. Let $e_1 = e, \ldots, e_N$ be a basis of the algebra,

$$e_\alpha e_\beta = c_{\alpha\ \beta}^{\ \gamma}\, e_\gamma\ .$$

The solutions $x_\alpha(t,z)$ of the equation (2.27) have the form

$$x_\alpha(t,z) = z^{-1}\langle e_\alpha,\, \exp zt - e\rangle\ , \qquad (2.84)$$

where

$$t = t^\lambda e_\lambda \in A\ . \qquad (2.85)$$

The function (2.62) has the form

$$V_{\alpha\beta}(t;z,w) = (z+w)^{-1}[\langle e_\alpha e_\beta,\, \exp(z+w)t\rangle - \langle e_\alpha,\, e_\beta\rangle]\ . \qquad (2.86)$$

The physical co-ordinates are

$$v_\alpha \equiv \partial_\alpha F = \frac{1}{2}\langle e_\alpha, (t)^2\rangle\ . \qquad (2.87)$$

Their P.B. (2.7) are linear (the so-called Lie-Poisson brackets)

$$\{v_\alpha(X), v_\beta(Y)\} = [c_{\alpha\ \beta}^{\ \epsilon} v_\epsilon(Y) + c_{\beta\ \alpha}^{\ \epsilon} v_\epsilon(X)]\delta'(X-Y) \qquad (2.88)$$

(see (2.64)). Therefore linear functionals of $v_\alpha(X)$ form an infinite dimensional Lie algebra. For $N = 1$ it coincides with the Lie algebra of one dimensional vector fields (i.e. the zero charge Virasoro algebra). For $N > 1$ the Lie algebras being dual to (2.88) were studied in [34] (also the quadratic transforms of the type (2.87) were used for reduction the P.B. (2.88) to a constant form). Also some non-commutative and non-associative generalization of Fröbenius algebras was proposed in [34]. This also gives rise to linear P.B. of the form (2.88) (but $c_{\beta\ \alpha}^{\ \epsilon} \neq c_{\alpha\ \beta}^{\ \epsilon}$). It would be interesting to investigate possible relations of these non-associative analogues of Fröbenius algebras to topological field theory.

The hierarchy (2.47) has the form

$$\partial_{T^{\alpha,p}} t = \frac{1}{p!}\, e_\alpha (t)^p\, \partial_X t \ . \tag{2.89}$$

Let us introduce vectors

$$T^p = T^{\alpha,p}\, e_\alpha \in A\ , \qquad p = 0, 1, \ldots \tag{2.90}$$

The τ-function (2.74) has the form

$$\begin{aligned} \log\tau &= \frac{1}{6}\langle e, (t)^3\rangle - \sum_p \frac{\langle T^p, (t)^{p+2}\rangle}{(p+2)p!} + \\ &+ \frac{1}{2}\sum_{p,q} \frac{\langle T^p T^q, (t)^{p+q+1}\rangle}{(p+q+1)p!q!}\ . \end{aligned} \tag{2.91}$$

Here the dependence $t = t(T)$ is determined by the "fixed-point equation"

$$G(t) = t\ , \tag{2.92a}$$

where

$$G(t) = \sum_{p=0}^{\infty} T^p \frac{(t)^p}{p!}\ . \tag{2.92b}$$

The solution has the well-known form

$$t = G(G(G(\ldots)))\ . \tag{2.93}$$

(infinite number of iterations).

Note that for the $T^{\alpha,p}$-derivatives can be found from (2.92) in the form

$$\partial_{T^{\alpha,p}} t = \frac{e_\alpha (t)^p / p!}{e - \sum_{s\geq 1} \frac{T^s (t)^{s-1}}{(s-1)!}} \tag{2.94}$$

(the denominator for small T is an invertible element of the Fröbenius algebra). The formulae (2.86) and (2.91) - (2.94) complete the solution of the topological model with constant primary correlators. So

$$\langle \sigma_p(\phi_\alpha)\, \sigma_q(\phi_\beta) \rangle = \frac{\langle e_\alpha\, e_\beta, (t)^{p+q+1} \rangle}{(p+q+1)\, p!\, q!}\ , \tag{2.95}$$

$$\langle \sigma_p(\phi_\alpha)\, \sigma_q(\phi_\beta)\, \sigma_r(\phi_\gamma) \rangle = \frac{1}{p!\, q! r!} \frac{\langle e_\alpha\, e_\beta\, e_\gamma, (t)^{p+q+r} \rangle}{e - \sum_{s\geq 1} \frac{T^s (t)^{s-1}}{(s-1)!}} \tag{2.96}$$

etc. For $N = 1$ the formulae (2.95), (2.96) coincide with the "pure gravity model" [2] (up to normalization of the coupling constants). For $N > 1$, particularly, one obtains the correlators for the K_3 model [5]. In this case $N = 24$, the Fröbenius algebra is the cohomology algebra of a generic K_3 surface. So it has generators P, $Q_1, \ldots, Q_{22}$, R in dimensions 0, 2 and 4 resp. The multiplication has the form

$$Q_i Q_j = \eta_{ij} R\ , \qquad P \text{ is the unity,} \qquad Q_i R = R^2 = 0\ . \tag{2.97}$$

Here (η_{ij}) is a nondegenerate symmetric matrix. The scalar product (the intersection number) has the form

$$\eta_{PR} = 1\ , \qquad \eta_{Q_i Q_j} = \eta_{ij}\ . \tag{2.98}$$

This is not a decomposable case, so the formulae (2.91) - (2.94) cannot be decoupled into a sum of N one-dimensional entries.

Examples of non-constant $c_{\alpha\beta\gamma}(t)$ will be considered in Sect.4.

Appendix 1. Inverse problem formulae for string correlators

Here I am going to express the τ-function in the decomposable case via solutions of the linear problem (1.26). It is sufficient to obtain an expression for the generating function (2.62).

Let us fix a basis $\psi_i{}^\alpha(u)$, $\alpha = 1, \ldots, N$ in the space of solutions of the linear system (1.28). Here $u = (u^1, \ldots, u^N)$ are the canonical co-ordinates (1.18). I recall that the basis $\psi_i{}^\alpha$ relates to the flat co-ordinates t^α and the scalar product $\eta_{\alpha\beta}$ by the formulae

$$\eta^{\alpha\beta} = \sum_{i=1}^{N} \psi_i{}^\alpha(u)\, \psi_i{}^\alpha(u)\,, \tag{A.1.1}$$

$$g_{ii}^{1/2} = \eta_{1\alpha}\, \psi_i{}^\alpha\,, \tag{A.1.2}$$

$$\partial_i t^\alpha = g_{ii}^{1/2}\, \psi_i{}^\alpha\,. \tag{A.1.3}$$

Let

$$\psi_{i\alpha}(u) = \eta_{\alpha\beta}\, \psi_i{}^\beta(u)\,, \qquad (\eta_{\alpha\beta}) = (\eta^{\alpha\beta})^{-1}\,, \tag{A.1.4}$$

and

$$\psi_{i\alpha}(u,z) = \sum_{p=0}^{\infty} \psi_{i(\alpha,p)}(u) z^p \tag{A.1.5}$$

be a solution of the system (1.26) being specified by the normalization

$$\psi_{i\alpha}(u,0) = \psi_{i\alpha}(u)\,. \tag{A.1.6}$$

It can be normalized also by the equations

$$\sum_{i=1}^{N} \psi_{i\alpha}(u,z)\,\psi_{i\beta}(u,-z) = \eta_{\alpha\beta}\ . \tag{A.1.7}$$

It relates to the functions $x_\alpha(t,z)$ by the formula

$$\psi_{i\alpha}(u,z) = g_{ii}^{-1/2}(u)\,\partial_i\, x_\alpha(t,z) \tag{A.1.8}$$

(here $t = t(u)$). For the coefficients $\psi_{i(\alpha,p)}$ this implies

$$\psi_{i,(\alpha,p)} = g_{ii}^{-1/2}(u)\,\partial_i\, h_{\alpha,p}\ . \tag{A.1.9}$$

Thus the gradient of $x_\alpha(t,z)$ in the diagonal co-ordinates $u^i, \ldots, u^N$ has the form

$$\nabla x_\alpha(t,z) = \sum_{i=1}^{N} \frac{\psi_{i\alpha}(u,z)}{\psi_{i(1,0)}(u)}\,\partial_i\ . \tag{A.1.10}$$

So the hierarchy (2.47) in the co-ordinates $u^1, \ldots, u^N$ have the form

$$\partial_{T^{\alpha,p}} u^i = \frac{\psi_{i(\alpha,p)}(u)}{\psi_{i(1,0)}(u)}\,\partial_X u^i\,, \qquad i = 1, \ldots, N\ . \tag{A.1.11}$$

For the generating formula (A.1.8) gives

$$V_{\alpha\beta}(u;z,w) = (z+w)^{-1}\left[\sum_{i=1}^{N} \psi_{i\alpha}(u,z)\,\psi_{i\beta}(u,w) - \eta_{\alpha\beta}\right]\ . \tag{A.1.12}$$

Particularly,

$$x_\alpha(u,z) = z^{-1}\left[\sum g_{ii}^{1/2}(u)\,\psi_{i\alpha}(u,z) - \eta_{\alpha 1}\right] . \qquad \text{(A.1.13)}$$

Completeness of the conservation laws $h_{\alpha,p}$ follows from the following statement.

Proposition A.1 Let $h_{\alpha,p}(u)$, $\alpha = 1, \ldots, N$, $p = 0, 1 \ldots$ be a family of densities of conservation laws of a Hamiltonian system of hydrodynamic type. Let us assume that the system have a diagonal form with different characteristic speeds in the co-ordinates $u^1, \ldots, u^N$ and the P.B. for the system have the form (2.10) where the metric ds^2 in the co-ordinates $u^1, \ldots, u^N$ is a diagonal Egoroff ∂-invariant metric. If the densities $h_{\alpha,p}(u)$ satisfies the recursion relations

$$\partial\, h_{\alpha,p+1} = h_{\alpha,p} , \qquad p \geq 0 , \qquad \text{(A.14a)}$$

where

$$\partial = \sum_{i=1}^{N} \partial_i , \qquad \text{(A.14b)}$$

and the densities $h_{\alpha,0} = t_\alpha$ span the annihilator of the P.B., then linear combinations of the functionals

$$\int h_{\alpha,p} dX \qquad \text{(A.1.15)}$$

form a dense subset in the space of all conservation laws of the system.

This can be proved using [25].

3. Self-similar solutions of the WDVV equations.

Let us look for solutions of the WDVV equations being self-similar with respect to some scaling transformations

$$t^\alpha \mapsto k^{1-q_\alpha} t^\alpha \ , \qquad \alpha = 1, \dots, N \tag{3.1a}$$

$$ds^2 = \eta_{\alpha\beta}\, dt^\alpha\, dt^\beta \mapsto k^{2-d}\, ds^2 \ , \tag{3.1b}$$

$$c_{\alpha\beta\gamma} \mapsto k^{q_\alpha+q_\beta+q_\gamma-l}\, c_{\alpha\beta\gamma} \tag{3.1c}$$

for some $q_1 = 0, q_2, \dots, q_N, d, l$ (it will be shown that $l = d$). Then the deformed Fröbenius algebra $\eta_{\alpha\beta}$, $c_\alpha{}^\gamma{}_\beta(t)$ can be considered as the perturbed chiral algebra of a model of topological conformal field theory. Here q_α are the charges of the primary fields and d is the dimension of the model. In the conformal point $t = 0$ (if it belongs to the coupling space) the equations (3.1) imply the tree-level superselection rules [10] for the primary correlators: $\langle \phi_{\alpha_1}\, \phi_{\alpha_2} \dots \rangle \neq 0$ only for $\alpha_1 + \alpha_2 + \dots = d$. In this case the WDVV equations can be reduced to a system of ODE. This ODE system will be investigated in this section.

Only decomposable case will be considered here. For a self-similar solution of the WDVV equation the rotation coefficients satisfy the similarity condition

$$\gamma_{ij}(ku) = k^{-1}\, \gamma_{ij}(u) \ . \tag{3.2}$$

So the similarity reduction of the gauge equivalent system (1.24) has the following "standardized" form:

$$\partial_k\, \gamma_{ij} = \gamma_{ik}\, \gamma_{kj} \qquad \text{for distinct } \ i, j, k = 1, \dots, N$$

$$\partial \gamma_{ij} = 0 \ , \qquad \partial = \sum_{i=1}^{N} \partial_i \ , \qquad \gamma_{ji} = \gamma_{ij} \ ,$$

$$\sum_{k=1}^{N} u^k \partial_k \gamma_{ij} = -\gamma_{ij} \ . \tag{3.3}$$

For $N = 2$ the system (3.3) is linear and can be solved easily (see Example 1 in the sect.4).

For the first nontrivial case $N = 3$ the ODE (3.3) can be written in the form

$$\Gamma'_{23} = \Gamma_{21}\Gamma_{13} \ , \qquad (z\Gamma_{13})' = -\Gamma_{12}\Gamma_{23} \ , \qquad [(z-1)\Gamma_{13}]' = \Gamma_{12}\Gamma_{23} \ . \tag{3.4a}$$

Here

$$z = \frac{u^1 - u^3}{u^2 - u^3} \ , \qquad \gamma_{ij}(u) = \frac{1}{u^2 - u^3}\Gamma_{ij}(z) = \gamma_{ji}(u) \ . \tag{3.4b}$$

The system (3.4a) can be reduced [30] to a system of the second order being equivalent to a particular case of the Painlevé-VI equation using the first integral

$$\Gamma_{23}^2 + (z\Gamma_{13})^2 + [(z-1)\Gamma_{13}]^2 = \text{const.} \tag{3.4c}$$

For any $N \geq 3$ one obtains a nonlinear system of ODE of order $N(N-1)/2$ (multicomponent generalization of the Painlevé-VI). The isomonodromic deformations method can be used for solving the system. I will show here how one can calculate the scaling dimensions q_α, d, l in the framework of the isomonodromic deformations theory. It will be shown that in the self-similar case one can express the deformation $(c_\alpha{}^\beta{}_\gamma(t), \eta_{\alpha\beta})$ via $\gamma_{ij}(u)$(by algebraic operations and quadratures.

Let us start with a commutation representation of (3.3).

Proposition 3.1 The system (3.3) is equivalent to the equations of compatibility of the linear problem (1.26) with the system

$$z\partial_z\psi = (zU - [U, \Gamma])\psi \ . \tag{3.5}$$

Here

$$\psi = (\psi_1, \ldots, \psi_N)^T , \tag{3.6a}$$

$$U = \mathrm{diag}\,(u^1, \ldots, u^N) , \tag{3.6b}$$

$$\Gamma = (\gamma_{ij}(u)) . \tag{3.6c}$$

Proof. The system (1.26) under the condition (3.2) is invariant with respect to the transformations

$$u \mapsto cu , \quad z \mapsto c^{-1}z , \quad \psi \mapsto \psi . \tag{3.7}$$

Hence it commutes with the operator

$$\sum_{i=1}^{N} u^i \partial_i - z\partial_z . \tag{3.8}$$

Action of this operator on the solutions of (1.26) can be written in the form (3.5). The proposition is proved.

The system (3.5) is a linear system of ODE with rational coefficients in z depending on the parameters $u^1, \ldots, u^N$. The standard corollary of the proposition (see, e.g. [31]) that the dependence of the coefficients on the parameters $u^1, \ldots, u^N$ is an isomonodromy deformation of the equation (3.5). So the monodromy matrices of (3.5) parametrize the general solution of (3.3). Any solution of (3.3) locally is an analytic function of the variables $u^1, \ldots, u^N$. Solutions of (3.3) with different reality conditions of the form $\gamma_{ij}^* = \pm\gamma_{ij}$ were constructed in [26]. See below Appendix for general solutions of (3.3).

If a solution $\gamma_{ij}(u)$ of (3.3) is defined in a neighbourhood of the diagonal hyperplanes $u^i = u^j$ then for $u^i - u^j \to 0$ one has

$$\gamma_{ij}(u) = \frac{\mu_{ij}}{u^i - u^j} + O(1) \tag{3.9a}$$

for some constants $\mu_{ij} = -\mu_{ji}$,

$$\gamma_{kl}(u) = O(1) \qquad \text{for} \qquad (k,l) \neq (i,j) \ . \tag{3.9b}$$

It would be interesting to investigate dependence of the parameters μ_{ij} on the monodromy matrix of (3.5). I hope to do it in the next paper.

Another corollary of the Prop.3.1 is very important here:

Proposition 3.2. Eigenvalues of the matrix $[U, \Gamma]$ are integrals of the system (3.3). If a vector-function $\psi = (\psi_1, \ldots, \psi_N)^T$ enjoys the system (1.28) and the similarity condition

$$\psi(cu) = c^p \psi(u) \tag{3.10}$$

then ψ is an eigenvector of the matrix $[U, \Gamma]$ with the eigenvalue $-p$.

Proof. From (1.26), (3.5) for $z = 0$ one obtains

$$\partial_k [U, \Gamma] = [[E_k, \Gamma], [U, \Gamma]], \qquad k = 1, \ldots, N \tag{3.11a}$$

where the matrix E_k has the form

$$(E_k)_{ij} = \delta_{ik}\, \delta_{kj} \ . \tag{3.11b}$$

Hence the eigenvalues of $[U, \Gamma]$ do not depend on u^k. From the Euler identity

$$\sum u^k \partial_k \psi = p\psi$$

for the homogeneous function ψ satisfying (1.26), for $z = 0$ one obtains

$$[U, \Gamma]\psi = -\psi \, .$$

The proposition is proved.

Note that the eigenvalues of $[U, \Gamma]$ are the local monodromy indices of (3.5) near the singular point $z = 0$.

The following statement is a rigidity theorem for *real* decomposable Fröbenius algebras with positive invariant scalar product.

Corollary. Any self-similar potential deformation of real decomposable Fröbenius algebra with positive invariant scalar product is a trivial one: $c_{\alpha}{}^{\gamma}{}_{\beta}(t) =$ const..

Proof. For positive metric ds^2 the canonical co-ordinates $u^1, \ldots, u^N$ are real (for real t). The diagonal entries $g_{ii}(u)$ of the metric in these co-ordinates also are real and positive. Hence the rotation coefficients $\gamma_{ij}(u) = \gamma_{ji}(u)$ are real, and $[U, \Gamma]$ is a real skew-symmetric matrix. All non-zero eigenvalues of the matrix are imaginary. So the matrix $[U, \Gamma]$ should equal zero identically. This means that the deformation is a trivial one.

Theorem 2. For any self-similar potential deformation of a decomposable Fröbenius algebra with the indices $q_1 = 0, q_2, \ldots, q_N, d, l$ (see (3.1)) the corresponding matrix $[U, \Gamma]$ is a diagonalizable one with the eigenvalues

$$\mu_\alpha = -\frac{d}{2} + q_\alpha \, , \qquad \alpha = 1, \ldots, N-1 \, , \qquad \mu_N = \frac{d}{2} \, . \tag{3.12a}$$

The corresponding eigenvectors ψ^α have the form

$$\psi_i{}^N = g_{ii}^{1/2}\,, \qquad \psi_i{}^\alpha = g_{ii}^{-1/2}\,\partial_i t^\alpha \qquad \alpha = 1, \ldots, N-1 \tag{3.12b}$$

and obey the normalization conditions

$$\eta^{\alpha\beta} = \sum_{i=1}^{N} \psi_i{}^\alpha\, \psi_i{}^\beta\,, \qquad (\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1}\,. \tag{3.12c}$$

Conversely, let $\Gamma = (\gamma_{ij})$ be any solution of the system (3.3) such that the matrix $[U,\Gamma]$ is a diagonalizable one. Let the eigenvalues $\mu_1, \ldots, \mu_N$ of the matrix are ordered in such a way that

$$\mu_{N-\alpha+1} = -\mu_\alpha\,. \tag{3.13}$$

Then a self-similar potential deformation $\eta_{\alpha\beta}$, $c_\alpha{}^\gamma{}_\beta(t))$ of a decomposable Fröbenius algebra with the canonical diagonal co-ordinates $u^1, \ldots, u^N$ exists with the indices d, l, $q_1, \ldots, q_N$ of the form

$$d = 2\mu_N\,; \qquad l = d\,; \qquad q_\alpha = \frac{d}{2} + \mu_\alpha\,. \tag{3.14}$$

If $\bar\psi^\alpha$ is any basis of eigenvectors of the matrix $[U,\Gamma]$,

$$[U,\Gamma]\bar\psi^\alpha = \mu_\alpha\,\bar\psi^\alpha\,, \tag{3.15}$$

then the metric g_{ii} and the flat co-ordinates t^α can be determined by the formulae (3.12b) where the vectors ψ^α have the form

$$\psi^\alpha = T^\alpha_\beta\,\bar\psi^\beta \tag{3.16}$$

for some matrix $T^\alpha_\beta = T^\alpha_\beta(u)$. If the eigenvalues μ_α of the matrix $[U,\Gamma]$ are simple then the matrix $T^\alpha_\beta(u)$ is a diagonal one, and its diagonal entries can be found by quadratures. The invariant scalar product $\eta_{\alpha\beta}$ has the form (3.12c), and the structure constants $c_\alpha{}^\gamma{}_\beta(t)$ have the form

$$c^{\alpha\beta\gamma} = \eta^{\alpha\lambda}\,\eta^{\beta\mu}\,c_\lambda{}^\gamma{}_\mu = \sum_{i=1}^{N} \frac{\psi_i{}^\alpha\,\psi_i{}^\beta\,\psi_i{}^\gamma}{\psi_i{}^N}\,. \tag{3.17}$$

Proof. For a self-similar deformation with the indices d, l, $q_1, \ldots, q_N$ the corresponding solutions ψ^α of the system (1.28) have the form (3.12b). They are homogeneous vector-functions of the weights $p_\alpha = -q_\alpha + \frac{d}{2}$. So the first part of the theorem follows from the Prop.3.2.

To prove the second part we use the equation (3.11a). The equation means that the operators

$$\partial_k - [E_k, \Gamma] , \qquad k = 1, \ldots, N , \tag{3.18}$$

commute with the matrix $[U, \Gamma]$. Also they commute pairwise because of (1.26). So they can be diagonalized simultaneously due to diagonalizability of the matrix $[U, \Gamma]$. The diagonalization procedure has the form (3.16). For the case of nondegeneracy of the spectrum $[U, \Gamma]$ one has for any α

$$(\partial_k - [E_k, \Gamma])\bar{\psi}^\alpha = f_k^\alpha \, \bar{\psi}^\alpha , \qquad k = 1, \ldots, N \tag{3.19}$$

for some functions $f_k^\alpha(u)$. The diagonal matrix $T_\beta^\alpha = T_\alpha^\alpha \cdot \delta_\beta^\alpha$ can be found by quadratures from the equations

$$\partial_k \log T_\alpha^\alpha = f_k^\alpha , \qquad k = 1, \ldots, N . \tag{3.20}$$

The theorem is proved.

Remark 1. If μ is a degenerate eigenvalue of the diagonalizable matrix $[U, \Gamma]$ of the multiplicity l, and $\bar{\psi}^{(1)}, \ldots, \bar{\psi}^{(l)}$ are corresponding linearly-independent eigenvectors then instead of (3.19) one will have

$$(\partial_k - [E_k, \Gamma])\bar{\psi}^{(a)} = \sum_{b=1}^{l} f_{k(b)}^{(a)} \, \bar{\psi}^b \tag{3.21}$$

for some matrices $f_{kb}^{(a)}$. These matrices commute pairwise for different k. If they can be diagonalized simultaneously then the corresponding basis $\psi^{(1)}, \ldots, \psi^{(l)}$ of solutions of (1.15b,c) also can be found by quadratures.

Remark 2. The functions f_k^α can be found explicitly. E.g., if an eigenvector $\bar\psi^\alpha$ of $[U,\Gamma]$ is normalized in such a way that

$$\sum_{i=1}^{N} \bar\psi_i^\alpha = 1 \tag{3.22a}$$

then

$$f_k^\alpha = \sum_{i=1}^{N} \gamma_{ik} \left(\bar\psi_i{}^\alpha - \bar\psi_k{}^\alpha\right) , \qquad k = 1, \ldots, N . \tag{3.22b}$$

The solutions $\psi_\alpha = (\psi_{i\alpha}(u,z))$ of the linear system (3.5), can be chosen in such a way that

$$\begin{aligned} \psi_{i\alpha}(cu, c^{-1}z) &= c\bar\mu_\alpha \, \psi_{i\alpha}(u,z) , \\ \bar\mu_\alpha &= q_\alpha - \frac{d}{2} . \end{aligned} \tag{3.23}$$

They satisfy the equation

$$z\partial_z \, \psi_\alpha = (zU - [U,\Gamma])\psi_\alpha - \bar\mu_\alpha \psi_\alpha . \tag{3.24}$$

For the coefficients

$$\psi_\alpha(u,z) = \sum_{p=0}^{\infty} \psi_{\alpha,p}(u) z^p , \qquad \psi_{\alpha,p}(u) = (\psi_{i(\alpha,p)}(u)) \tag{3.25}$$

one has

$$[U,\Gamma]\psi_{\alpha,0} = -\bar\mu_\alpha \, \psi_{\alpha,0} , \tag{3.26}$$

so

$$\psi_{i(\alpha,0)}(u) = \eta_{\alpha\beta} \, \psi_i{}^\beta(u) , \tag{3.27a}$$

$$([U,\Gamma] + p + \bar\mu_\alpha)\psi_{\alpha,p}(u) = U\psi_{\alpha,p-1} , \qquad p \geq 1 . \tag{3.27b}$$

In the nonresonant case where

$$\bar{\mu}_\alpha - \bar{\mu}_\beta + p \neq 0 , \qquad p = 1, 2 \ldots \tag{3.28a}$$

or, equivalently

$$q_\alpha - q_\beta + p \neq 0 , \quad p = 1, 2, \ldots , \quad \beta = 1, \ldots , N, \tag{3.28b}$$

this gives a recurrence procedure for finding the functions $\psi_{i\alpha}(u, z)$. The formula (A.1.13) then provides an expression for the functions $x_\alpha(z)$ (and, particularly, for the flat co-ordinates t_α), and the formula (A.1.12) gives the expression for the function $V_{\alpha\beta}(z, w)$.

The densities $h_{\alpha,p}(t)$ and the coefficients $V_{(\alpha,p),(\beta,q)}(t)$ with respect to the scaling transformations $t^\alpha \mapsto c^{1-q_\alpha} t^\alpha$ transform in the following way:

$$h_{\alpha,p} \mapsto c^{1+q_\alpha+p-d} \, h_{\alpha,p} , \tag{3.29a}$$

$$V_{(\alpha,p),(\beta,q)} \mapsto c^{p+q+q_\alpha+q_\beta+1-d} \, V_{(\alpha,p),(\beta,q)} . \tag{3.29b}$$

So the hierarchy (2.47) is invariant with respect to the transformations

$$\begin{aligned} t^\alpha &\mapsto c^{1-q_\alpha} t^\alpha \\ T^{\alpha,p} &\mapsto c^{1-q_\alpha-p} \, T^{\alpha,p} . \end{aligned} \tag{3.30}$$

The τ-function (2.74) with respect to these transformations has the same weight $3 - d$. This gives the identity

$$\sum (p + q_\alpha - 1) \, T^{\alpha,p} \, \partial_{T^{\alpha,p}} \log \tau_{1,1} = (d - 3) \log \tau_{1,1} . \tag{3.31}$$

Note that the term with the number $(\alpha, p) = (1, 1)$ cancels in the l.h.s of (3.31). That means that the τ_0-function (2.60) satisfies the same equation (3.31). I recall that the τ_0-function is a homogeneous function of the degree 2 (see (2.63)). This can be used (as in [19]) for construction of self-similar solutions of the hierarchy (2.47) with other similarity indices. Let us fix (α, p). The τ-function $\tau_{\alpha,p}(T)$ is defined (formally[5]) by the shift

$$\tau_{\alpha,p}(T) = \tau_0(T^{1,0}, \ldots, T^{\alpha,p} + 1, \ldots) = \tau_{1,1}(T^{1,0}, \ldots, T^{1,1} - 1, \ldots, T^{\alpha,p} + 1, \ldots) . \tag{3.32a}$$

The corresponding solution $t = t(T)$ of (2.47) is determined by the variational problem

$$\nabla(\Phi_T(t) - h_{\alpha,p}) = 0 \tag{3.32b}$$

cf. (2.57), (2.73)). This is the self-similar solution of (2.47) w.r.t. the transformations (cf.[10])

$$t^\beta \mapsto c^{1-q_\beta} t^\beta \tag{3.33a}$$

$$T^{\beta,r} \mapsto c^{q_\alpha - q_\beta + p - r} T^{\beta,r} . \tag{3.33b}$$

The shifted τ-function satisfies the equation

$$\sum_{\beta,r}(q_\beta - q_\alpha - p + r) T^{\beta,r} \partial_{T^{\beta,r}} \log \tau_{\alpha,p} = [d - 1 + 2(p + q_\alpha)] \log \tau_{\alpha,p} . \tag{3.33c}$$

[5] The shift (3.32a) might not be well-defined due to the gradient catastrophe for the system of hydrodynamic type (2.47) (see [20]) for discussion of the role of self-similar solutions of the Whitham hierarchy in investigation of dispersionless shock-waves).

Therefore the self-similar solutions of the hierarchy of hydrodynamic type (2.47) are in $1-1$ correspondence with the multi-critical topological models (see [5]). In the multi-critical point (3.33) one has in the notation of [10]

$$\gamma_{\text{string}} = (d-1)/(p+q_\alpha) \ . \tag{3.33d}$$

We end this section with the discussion of bi-Hamiltonian structure of the hierarchy (2.47) for self-similar potential deformations of decomposable Fröbenius algebras. In this case another flat diagonal Egoroff metric is determined on the coupling space M. Let

$$d\tilde{s}^2 = \sum_{i=1}^{N} \frac{g_{ii}(u)}{u^i} (du^i)^2 \ . \tag{3.34}$$

Lemma 1. For a flat Egoroff metric $ds^2 = \sum_{i=1}^{N} g_{ii}(u)(du^i)^2$ with scaling invariant rotation coefficients $\gamma_{ij}(cu) = c^{-1}\gamma_{ij}(u)$ the metric

$$ds^2_{a,b} = \sum_{i=1}^{N} \frac{g_{ii}(u)}{a+bu^i} (du^i)^2 \tag{3.35}$$

also is a flat one.

Proof. The rotation coefficients of the metric (3.35) with respect to the co-ordinates

$$\tilde{u}^i = \log(a+bu^i) \tag{3.36}$$

equal

$$\tilde{\gamma}_{ij} = \frac{1}{b} \exp\left(\frac{\tilde{u}^i + \tilde{u}^j}{2}\right) \gamma_{ij}\left(\frac{e^{\tilde{u}^1} - a}{b}, \ldots, \frac{e^{\tilde{u}^N} - a}{b}\right) \ . \tag{3.37}$$

Since $\gamma_{ij}(u) = \gamma_{ji}(u)$ enjoy the same system (1.24) the coefficients $\tilde{\gamma}_{ij}(\tilde{u})$ enjoy the same system with respect to the $\tilde{u}$-variables. Lemma is proved.

Proposition 3.3 The Poisson brackets $\{\,,\,\}_{ds^2}$ and $\{\,,\,\}_{d\tilde{s}^2}$ are compatible (i.e. any linear combination of them again is a Poisson bracket). Functionals from the canonical Lagrangian subspace $\mathcal{H}$ (2.46) commute with respect to $\{\,,\}_{d\tilde{s}^2}$. For any homogeneous function $h(u) \in \mathcal{H}$, $h(cu) = c^q h(u)$, the following identity holds

$$\left\{\cdot, \int h\,dx\right\}_{d\tilde{s}^2} = \left(\frac{d-1}{2} + q\right)\left\{\cdot, \int \partial^{-1} h\,dx\right\}_{ds^2} . \qquad (3.38)$$

Proof. Let g_{ij} any $\tilde{g}_{ij}$ be any two flat metrics with the corresponding Levi-Civita connections Γ^i_{jk} and $\tilde{\Gamma}^i_{jk}$. They determine two Poisson brackets $\{\,,\}_{d\tilde{s}^2}$ by the formula (2.1).

Lemma 2. The Poisson brackets $\{\,,\}_{ds^2}$ and $\{\,,\}_{d\tilde{s}^2}$ are compatible iff the metric $ag^{ij} + b\tilde{g}^{ij}$ is a flat one for any a, b and the tensor

$$T_s{}^j{}_t = \tilde{\Gamma}_s{}^j{}_t - \Gamma_s{}^j{}_t \qquad (3.39)$$

satisfies the condition

$$g^{it}\tilde{g}^{ks}T_s{}^j{}_t = g^{kt}\,\tilde{g}^{is}T_s{}^j{}_t . \qquad (3.40)$$

Proof. As it was proved in [23], the formula (2.1) determines a Poisson bracket (for non-degenerate matrix g^{ij}) iff the coefficients g^{ij} are the contravariant components of a flat metric and $\Gamma_j{}^i{}_k$ are the Christoffel symbols of the corresponding Levi-Civita connections. For the linear combination $a\{\,,\}_{ds^2} + b\{\,,\}_{d\tilde{s}^2}$ the coefficient before $\delta'(x-y)$ equals $ag^{ij} + b\tilde{g}^{ij}$. So this metric should be a flat one. The equation (3.40) is equivalent to the symmetry of the corresponding connection. Lemma is proved.

In our case one has

$$ag^{ij} + b\tilde{g}^{ij} = g^{ii}(a + bu^i)\delta^{ij} . \qquad (3.41)$$

Flatness of this metric was proved in the Lemma 1.

The tensor (3.39) has the form

$$T_s{}^j{}_t = -\frac{\delta_{st}\,\delta_{sj}}{2u^j} . \qquad (3.42)$$

The equation (3.40) for this tensor can be verified straightforward.

To prove the commutativity of functionals from $\mathcal{H}$ it is sufficient to verify (3.38) since the homogeneous functions $h_{(\alpha,p)}$ span $\mathcal{H}$. We recall that for a flat Egoroff metric

$$d\tilde{s}^2 = \sum_{i=1} \tilde{g}_{ii}(\tilde{u})\,(d\tilde{u}^i)^2$$

the functional $H = \int h\,dx$ belongs to the canonical Lagrangian family iff the vector-function

$$\tilde{\psi}_i^h = \tilde{g}_{ii}^{-1/2}\,\tilde{\partial}_i h\ ,$$

$\tilde{\partial}_i = \partial/\partial\tilde{u}^i$, satisfies the system

$$\tilde{\partial}_j\,\tilde{\psi}_i^h = \tilde{\gamma}_{ij}\,\tilde{\psi}_j^h\ , \qquad i \neq j\ . \tag{3.43}$$

Then

$$\left\{\tilde{u}^i(x)\ ,\quad \int h(\tilde{u})dy\right\}_{d\tilde{s}^2} = \tilde{g}_{ii}^{-1/2}\,\tilde{\partial}\,\tilde{\psi}_i^h\ , \tag{3.44a}$$

$$\tilde{\partial} = \sum_{i=1}^{N} \tilde{\partial}_i\ . \tag{3.44b}$$

In our case in the co-ordinates $\tilde{u}^i = \log u^i$ the components of the Egoroff metric (3.34) are

$$\tilde{g}_{ii} = u^i g_{ii}\ . \tag{3.45a}$$

and the rotation coefficients are

$$\tilde{\gamma}_{ij} = \sqrt{u^i u^j}\,\gamma_{ij}\ . \tag{3.45b}$$

So

$$\tilde{\psi}_i{}^h = \sqrt{u^i}\,\psi_i{}^h \tag{3.45c}$$

where $\psi_i{}^h = g_{ii}^{-1/2}\,\partial_i h$. So the function $\tilde{\psi}_i^h$ satisfies the system (3.43) iff the function $\psi_i{}^h$ satisfies the system

$$\partial_j\,\psi_i{}^h = \gamma_{ij}\,\psi_j{}^h\ .$$

That means that the canonical Lagrangian subspaces $\tilde{\mathcal{H}}$ and $\mathcal{H}$ coincide.

The formula (3.38) follows from (3.44a) since

$$\tilde{\partial} = \sum_{i=1}^{N} = u^i\,\partial_i\ .$$

The proposition is proved.

Corollary. In the non-resonant case

$$\frac{d+1}{2} - q_\alpha + p \neq 0 \qquad \text{for any} \quad p = 0,\,1,\,\ldots \tag{3.46}$$

all the equations of the Hamiltonian hierarchy (2.47) are also Hamiltonian systems with respect to the second Poisson structure $\{\,,\,\}_{d\tilde{s}^2}$.

As it follows from (3.38) the recursion operator for the compatible pair of Poisson structures coincides (up to multiplication by a constant matrix) with the operator ∂^{-1}.

For the example 3 of Sect.4 (below) the nonresonance conditions (3.46) are valid. But for the example 4 there are g "resonant" systems in the hierarchy (2.47). The functionals $\int t^\alpha dX$, $\alpha = n,\,\ldots,\,n+g-1$, belong to the annihilator of the both P.B. $\{\,,\,\}_{ds^2}$ and $\{\ \}_{d\tilde{s}^2}$.

Remark. The difference tensor $T_s{}^j{}_t$ (3.39), (3.42) determine a new multiplication of vector fields (see Sect.1 above) by the formula

$$\tilde{\partial}_i \times \tilde{\partial}_j = \delta_{ij}\,\tilde{\partial}_i\ , \tag{3.47}$$

$\tilde{\partial}_i = \partial/\partial\tilde{u}^i$, $\tilde{u}^i = \log u^i$ (we omit the coefficient $-1/2$). The metrics $ds^2_{a,b}$ (3.35) are invariant for this multiplication for any a, b. So

$$\tilde{c}^{\gamma}_{\alpha\beta} = \sum_{i=1}^{N} \frac{1}{u^i} \frac{\partial u^i}{\partial t^\alpha} \frac{\partial u^i}{\partial t^\beta} \frac{\partial t}{\partial u^i}$$

is a potential deformation. But the normalization condition (1.8b) does not hold, since the unity $\tilde{\partial} = \sum_i \tilde{\partial}_i$ is not covariant constant.

Appendix 2. Inverse spectral transform for the similarity reduction of the WDVV.

Here I outline the solution of the system (3.3) via reduction to a matrix Riemann problem.

Let R_{ij}, $1 \leq i, j \leq N$, be rays in the complex z-plane of the form

$$\operatorname{Re} z(u^i - u^j) = 0\ , \quad \operatorname{Re} z e^{i\varepsilon}(u^i - u^j) < 0 \quad \text{for} \quad \varepsilon > 0\ , \quad z \in R_{ij}\ . \qquad \text{(A.2.1)}$$

Let R be a line via the origin in the z plane not containing any of the rays (A.2.1). It divides all the z-plane into two half planes π_+ and π_- (with respect to the standard orientation of the z-plane). Let R_+ and R_- be resp. the positive and the negative part of R with respect to the above orientation.

The datum of the inverse problem for solving (3.3) is a Stokes matrix S. It is a $N \times N$ matrix $S = (s_{ij})$ with the properties

$$s_{ii} = 1, \quad i = 1, \ldots, N, \qquad s_{ij} = 0 \quad \text{if} \quad R_{ij} \subset \pi_-\ . \qquad \text{(A.2.2)}$$

There are $N(N-1)/2$ independent parameters in S.

The formulation of the Riemann problem is as follows. Let $\Psi_+(u,z)$, $\Psi_-(u,z)$ be two $N \times N$-matrix-value functions analytic in z in the half-planes π_+ and π_- resp. satisfying the following boundary conditions:

$$\begin{aligned} \Psi_+|_{R_+} &= \Psi_-|_{R_+} G \\ \Psi_+|_{R_-} &= \Psi_-|_{R_-} G^T \end{aligned} \qquad \text{(A.2.3a)}$$

(G^T means the transposed matrix) with the asymptotics with $z \to \infty$ of the form

$$\Psi_\pm(u,z)e^{-zU} = I + O(z^{-1}) \ , \qquad z \to \infty \ . \tag{A.2.3b}$$

(I is the unity). The solution of this Riemann problem can be reduced to solution of linear integral equations in the standard way (see, e.g. [31]).

Proposition. If $\Psi_\pm(u,z)$ is a solution of the Riemann problem (A.2.3) then the matrix

$$\Gamma(u) = (\gamma_{ij}(u)) = \lim_{z\to\infty} z\left[\Psi_\pm(u,z)\, e^{-zU} - I\right] \tag{A.2.4}$$

satisfies (3.3).

The proof is standard for the isomonodromy deformation theory.

Note that the eigenvalues of the matrix $[U,\Gamma]$ can be calculated as

$$\text{eigen} \ \ [U,\Gamma] = \frac{1}{2\pi i} \ \text{eigen} \ \ \log S\, S^{T^{-1}} \ . \tag{A.2.5}$$

This is the analogue of the cyclic relations [31].

4. Main examples

Any solutions $\gamma_{ij}(u)$ of the system (1.24) determines a N-parametric family of solutions of the WDVV equations. The system (1.24) can be solved using the standard machinery of the "inverse spectral transform" (see, e.g. [28,29] for localized solutions, [29] for algebraic geometry solutions, [26] for self-similar solutions). There are also solutions in elementary functions.

Example 1. $N = 2$. The system (1.24) is linear. The general self-similar solution has the form

$$\gamma_{12}(u) = \gamma_{21}(u) = \frac{i\mu}{u^1 - u^2} \tag{4.1}$$

for some real μ. The basis $\psi_{i\alpha}(u)$ of solutions of the system (1.28) for $z = 0$ has the form

$$\psi_1 = \frac{a}{\sqrt{2}} \begin{pmatrix} r^\mu \\ ir^\mu \end{pmatrix}, \qquad \psi_2 = \frac{a^{-1}}{\sqrt{2}} \begin{pmatrix} r^{-\mu} \\ -ir^{-\mu} \end{pmatrix}, \qquad r = u^1 - u^2 \tag{4.2}$$

for any $a \neq 0$. Let $\mu \neq -\frac{1}{2}$. Then the flat co-ordinates have the form

$$t^1 = \frac{u^1 + u^2}{2}, \qquad t^2 = \frac{a^2 r^{2\mu+1}}{2(2\mu+1)}. \tag{4.3}$$

The metric ds^2 has the form

$$ds^2 = \frac{a^2}{2}\left[(u^1 - u^2)^{2\mu}(du^1)^2 - (u^1 - u^2)^{2\mu}(du^2)^2\right] = 2\, dt^1\, dt^2 . \tag{4.4}$$

Here

$$q_1 = 0, \qquad q_2 = d = -2\mu . \tag{4.5}$$

For $\mu \neq \pm 1/2$, $-3/2$ the primary free energy has the form

$$F = \frac{1}{2}(t^1)^2 + a^{-4}\frac{(1+2\mu)^3}{2(1-2\mu)(2\mu+3)}\left[2a^{-2}(2\mu+1)\right]^{-\frac{4\mu}{2\mu+1}}(t^2)^{\frac{2\mu+3}{1+2\mu}} . \tag{4.6}$$

For $\mu = -1/2$ $(d = 1)$ the flat co-ordinates are

$$t^1 = \frac{u^1 + u^2}{2}, \qquad t^2 = \frac{a^2}{2}\log(u^1 - u^2) . \tag{4.7}$$

The potential $F(t)$ is not a homogeneous function:

$$F(t) = \frac{1}{2}(t^1)^2 t^2 + 2^{-6} a^2 \exp 4a^{-2} t^2 \qquad (4.8)$$

(the free energy of the CP^1-model [5]). For $\mu = \frac{1}{2}$, $-3/2$ the formula (4.3) still is valid. And logarithms should be involved into the formula (4.6). The deformed chiral algebra of two elements $e_1 = e$, e_2 has the form

$$e_1 e_1 = e_1,\ e_1 e_2 = e_2\,, \qquad e_2 e_2 = a^{-4}[2a^{-2}(1-d)t^2]^{-\frac{2d}{1-d}}\, e_1, \quad d \neq 1 \qquad (4.9)$$

and

$$e_2 e_2 = a^{-4}(\exp 4a^{-2} t^2)\, e_1\,, \qquad d = 1\,. \qquad (4.10)$$

In fact, all the formulae were obvious *a priori.* Only their dependence on the "inverse data" is not obvious. And this dependence is important for calculation of the τ-function.

The solutions $\psi_{i\alpha}(u,z)$ of the linear system(1.26) have the form

$$\begin{pmatrix} \psi_{11} \\ \psi_{21} \end{pmatrix} = \frac{a}{\sqrt{2}} \left(\frac{2}{z}\right)^{\mu} \sqrt{rz} \begin{pmatrix} I_{\mu-\frac{1}{2}}(rz) + I_{\mu+\frac{1}{2}}(rz) \\ i\left[I_{\mu-\frac{1}{2}}(rz) - I_{\mu+\frac{1}{2}}(rz)\right] \end{pmatrix} e^{z(u^1+u^2)}\,, \qquad (4.11)$$

$$\begin{pmatrix} \psi_{12} \\ \psi_{22} \end{pmatrix} = \frac{a^{-1}}{\sqrt{2}} \left(\frac{z}{2}\right)^{\mu} \sqrt{rz} \begin{pmatrix} I_{-\mu-\frac{1}{2}}(rz) + I_{-\mu+\frac{1}{2}}(rz) \\ i\left[I_{-\mu+\frac{1}{2}}(rz) - I_{-\mu-\frac{1}{2}}(rz)\right] \end{pmatrix} e^{z(u^1+u^2)} \qquad (4.12)$$

Here $r = u^1 - u^2$; the modified Bessel function $I_\nu(x)$ (see [35]) is determined as the solution of the equation

$$x^2 I_\nu'' + x I_\nu^p - (x^2 + \nu^2) I_\nu = 0 \qquad (4.13a)$$

of the form

$$I_\nu(x) = \sum_{m=0}^{\infty} \frac{(z/2)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)}\,. \qquad (4.13b)$$

The function $V_{\alpha\beta}(z,w)$ can be calculated using the standard expansions for products of Bessel functions.

Example 2. Let us consider 4-dimensional Fröbenius algebra with a basis $P = e$, Q, R, S and with the multiplication given in the table

$\times$	P	Q	R	S
P	P	Q	R	S
Q	Q	$f'''(t_Q)R$	S	0
R	R	S	0	0
S	S	0	0	0

(4.14)

Here f is an arbitrary function. This deformation is indecomposable. The symmetric scalar product has the form $\eta_{PS} = \eta_{RQ} = 1$, otherwise zero. Let t_P, t_Q, t_R, t_S be the corresponding flat co-ordinates on the coupling space. The flat co-ordinates $x_\alpha(t, z)$ of the perturbed connection (1.14) can be found explicitly:

$$\begin{aligned} x_P &= [t_S + zt_Qt_R + z^2(t_Qf'(t_Q) - 2f(t_Q))]e^{zt_P} \\ x_Q &= [t_R + zf'(t_Q)]e^{zt_P} \\ x_R &= t_Qe^{zt_P} \\ x_S &= z^{-1}[e^{zt_P} - 1] \ . \end{aligned} \tag{4.15}$$

The generating function for the double correlators $V_{\alpha\beta}(t; z, w) = V_{\beta\alpha}(t; w, z)$ have the form

$$\begin{aligned} V_{PP}(t; z, w) = & \ [t_S + (z + w)t_Qt_R + (z^2 - zw + w^2)(t_Qf'(t_Q) - 2f(t_Q)) + \\ & +zw(t_Qf''(t_Q) - f'(t_Q))]e^{(z+w)t_P} \ , \\ V_{PQ}(t; z, w) = & \ [t_R + zt_Qf''(t_Q) + (w - z)f'(t_Q)]e^{(z+w)t_P} \ , \\ V_{PR}(t; z, w) = & \ t_Qe^{(z+w)t_P} \ , \\ V_{PS}(t; z, w) = & \ (z + w)^{-1}[e^{(z+w)t_P} - 1] \ , \\ V_{QQ}(t; z, w) = & \ f''(t_Q)e^{(z+w)t_P} \ , \\ V_{QR}(t; z.w) = & \ (z + w)^{-1}[e^{(z+w)t_P} - 1] \ , \end{aligned} \tag{4.16}$$

other components of the matrix $V_{\alpha\beta}$ vanish. This completes the solution of the model. Particularly, for the case

$$f'''(t) = bt + e^{ct} \tag{4.17}$$

(b, c are some constants) one obtains the correlators of the topological sigma model with a Calabi - Yau target space being considered in [5] (in the case the Calabi - Yau manifold has the smallest possible Hodge numbers $b_{0,0} = b_{1,1} = b_{2,2} = b_{3,3} = 1$).

Example 3 (see [10]). The coupling space for the A_{n-1}-topological minimal model is a set of all polynomials of given degree n of the form

$$M = \{\lambda(p) = p^n + a_{n-2}p^{n-2} + \ldots + a_0 \,|\, a_0, \ldots, a_{n-2} \in \mathbf{C}\} . \tag{4.18}$$

Here $N = n - 1$. For any polynomial $\lambda \in M$ (it is called Landau - Ginzburg potential) the Fröbenius algebra $A = A_\lambda$ is the algebra of truncated polynomials

$$A_\lambda = \frac{\mathbf{C}[p]}{(\lambda'(p) = 0)} \tag{4.19}$$

with the scalar product

$$\langle f(p), g(p)\rangle = -\frac{1}{n} \operatorname*{res}_{p=\infty} \frac{f(p)g(p)}{\lambda'(p)} . \tag{4.20}$$

It is easy to see that (4.19), (4.20) is a Fröbenius algebra for any $\lambda(p)$. This is a potential deformation as it was proved in [10].

The affine structure on M is introduced in the following way. Let $\phi_\alpha = \phi_\alpha(p;\lambda)$, $\alpha = 1, \ldots, n-1$, be the orthogonal basis of A_λ:

$$\langle \phi_\alpha, \phi_\beta\rangle = \eta_{\alpha\beta} \equiv \delta_{\alpha+\beta,n} , \qquad \deg \phi_\alpha = \alpha - 1 . \tag{4.21}$$

For the polynomials ϕ_α in [10] was obtained the formula

$$\phi_\alpha(p;\lambda) = \frac{n}{\alpha}\frac{d}{dp}[\lambda^{\frac{\alpha}{n}}(p)]_+ . \tag{4.22}$$

Here $[\ \ ]_+$ means the polynomial (in p) part of the series $\lambda^{\frac{\alpha}{n}}(p)$. The dependence of $\lambda(p)$ (i.e. of its coefficients) on the flat co-ordinates t^α are determined by the equations

$$\frac{\partial\lambda(p)}{\partial t^\alpha} = -\phi_\alpha(p;\lambda)\ , \qquad \alpha = 1, \ldots, n-1\ . \tag{4.23}$$

It was observed in [13] that the equations (4.23) and their solution in [10] have a natural interpretation in the theory of the dispersionless Gelfand - Dikii (GD) hierarchy. Also a notion of τ-function of this dispersionless hierarchy was proposed in [13] to obtain a formula for the primary partition function of the model. I recall that the GD (or generalized KdV) hierarchy has the form

$$\partial_{\tau^q} L = [L,\ (L^{\frac{q}{n}})_+] \tag{4.24a}$$

where

$$L = \partial^n + a_{n-2}(x)\partial^{n-2} + \ldots + a_0(x) \tag{4.24b}$$

is an ordinary differential operator, $\partial = \partial/\partial x$, and $(\quad)_+$ means the differential part of the pseudo-differential operator $L^{q/n}$. The dispersionless approximation can be obtained from (4.24) by the substitution

$$a_i = a_i(X, T^1, \ldots), \qquad X = \varepsilon x, \qquad T^q = \varepsilon\tau^q, \qquad \varepsilon \to 0 \tag{4.25}$$

and taking the leading term in ε. The FFM representation of the dispersionless GD hierarchy has the form

$$\partial_{T^q}\, dp|_{\lambda=\text{const.}} = \partial_X\, d\Phi_q|_{\lambda=\text{const.}}\ , \tag{4.26a}$$

$$\Phi_q = [\lambda^{q/n}(p)]_+\ . \tag{4.26b}$$

It can be rewritten via Wronskians [13]

$$\partial_{T^q}\lambda(p) = \partial_p\lambda(\partial_X\Phi_q)_{p=\text{const.}} - \partial_p\Phi_q(\partial_X\lambda)_{p=\text{const.}}\ . \tag{4.27}$$

The standard calculation of [18] gives that the diagonal co-ordinates (Riemann invariants) for the hierarchy (4.26) are the critical values $u^1, \ldots, u^{n-1}$ of $\lambda(p)$

$$u^i = \lambda(p_i), \qquad i = 1, \ldots, N = n-1, \tag{4.28a}$$

where

$$\lambda'(p_i) = 0. \tag{4.28b}$$

The characteristic speeds are

$$v_{q,i}(u) = \left.\frac{d\Phi_q}{dp}\right|_{p=p_i}, \tag{4.29a}$$

$$\partial_{T^q} u^i = v_{q,i}(u)\, \partial_X u^i, \qquad i = 1, \ldots, N. \tag{4.29b}$$

It can be shown that the hierarchy (4.29) coincides with (2.47) for the model (up to normalization of the variables T^q). The algebra A_λ is decomposable if the polynomial $\lambda'(p)$ has simple roots. So $u^1, \ldots, u^N$ are the canonical diagonal co-ordinates. The metric ds^2 in these co-ordinates equals

$$ds^2 = \sum_{i=1}^{N} \frac{(du^i)^2}{\lambda''(p_i)}. \tag{4.30}$$

Let us prove now that the functions

$$h_{\alpha,q} = -\frac{1}{\left(\frac{\alpha}{n}\right)_q} \operatorname*{res}_{p=\infty} \lambda^{\frac{\alpha}{n}+q}\, dp, \qquad \alpha = 1, \ldots, n-1; \quad q = 0, 1, \ldots, \tag{4.31}$$

$$(a)_q \equiv a(a+1)\ldots(a+q-1) \tag{4.32}$$

on the space (4.18) are the basic conservation laws (2.43) of the hierarchy (4.29). Indeed, from (4.26) it is obvious that (4.31) are densities of conservation laws.

One needs to verify only the recursion relations

$$\partial\, h_{\alpha,q} = h_{\alpha,q-1}\,, \qquad q \geq 1\,, \tag{4.33a}$$

$$\partial\, h_{\alpha,0} = \text{const.}\,, \tag{4.33b}$$

$$\partial = \textstyle\sum_{i=1}^{N} \partial_i\,, \qquad \partial_i = \partial/\partial u^i\,. \tag{4.33c}$$

To do it let us observe that the translation along λ of the form

$$\lambda \mapsto \lambda + \varepsilon\,; \qquad p \mapsto p; \qquad a_i \mapsto a_i\,, \quad i \neq 0, \quad a_0 \mapsto a_0 + \varepsilon \tag{4.34}$$

is equivalent to the translation along $u^1 + \ldots + u^N$. So

$$\begin{aligned} \partial h_{\alpha,q} &= \frac{d}{d\varepsilon} h_{\alpha,q}(u^1+\varepsilon,\ldots,u^N+\varepsilon)_{\varepsilon=0} = \\ &= \frac{1}{\left(\frac{\alpha}{n}\right)_q} \operatorname*{res}_{p=\infty} \frac{d}{d\varepsilon}\left((\lambda+\varepsilon)^{\frac{\alpha}{n}+q}\, dp(\lambda+\varepsilon)\right)_{\varepsilon=0} = \frac{1}{\left(\frac{\alpha}{n}\right)_{q-1}} \operatorname*{res}_{p=\infty} \lambda^{\frac{\alpha}{n}+q-1}\, dp\,. \end{aligned} \tag{4.35}$$

This gives (4.33a) for $q \geq 1$ and

$$\partial\, h_{\alpha,0} = \operatorname*{res}_{p=\infty} \lambda^{\frac{\alpha}{n}-1}\, dp = -\delta_{\alpha,n-1}\,. \tag{4.36}$$

The formulae (4.33) are proved. Particularly one obtains the flat co-ordinates on M

$$t^\alpha = -n \operatorname*{res}_{p=\infty} \frac{\lambda^{\frac{n-\alpha}{n}}}{n-\alpha}\, dp\,, \qquad \alpha = 1,\ldots, N = n-1\,. \tag{4.37}$$

The generating functions $x_\alpha(t,z) = \sum h_{\alpha,q}(t) z^q$ can be written in the form

$$x_\alpha(t,z) = -\frac{n}{\alpha} \operatorname*{res}_{p=\infty} {}_1F_1\left(1;\, 1+\frac{\alpha}{n};\, z\lambda\right) dp\,, \qquad \alpha = 1,\ldots, N\,. \tag{4.38}$$

Here ${}_1F_1(a;\, c;\, z)$ is the Kummer (or confluent hypergeometric) function [35]

$${}_1F_1(a;\, c;\, z) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m} \frac{z^m}{m!}\,. \tag{4.39}$$

The scaling dimensions of the variables $t^1, \ldots, t^N$ equal $1, \frac{n-1}{n}, \ldots, \frac{2}{n}$ resp., and $d = \frac{n-2}{n}$. So the eigenvalues of the matrix $[U, \Gamma]$ (see Sect.3 above) are

$$\mu_\alpha = -\frac{1}{2} + \frac{\alpha}{n}, \qquad \alpha = 1, \ldots, n-1. \tag{4.40}$$

I recall that these are the local monodromy indices of the system (3.5) near the point $z = 0$. It would be interesting to calculate all the monodromy data of the solutions (4.38) in z-plane. Note that the nonresonance conditions (3.28) are valid in this case. So all the functions $h_{\alpha,q}$ can be expressed in algebraic way via the co-ordinates t^α and u^i and the rotation coefficients.

The generating function of two-point correlators have the form

$$\begin{aligned} V_{\alpha\beta}(t; z, w) &= \eta^{\mu\nu}(z+w)^{-1}\left\{\left[\operatorname*{res}_{p=\infty} \lambda^{\frac{\alpha}{n}-1}\, {}_1F_1\left(1; \frac{\alpha}{n}; z\lambda\right) \phi_\mu(p)dp\right]\right. \\ &\left.\left[\operatorname*{res}_{p=\infty} \lambda^{\frac{\beta}{n}-1}\, {}_1F_1\left(1; \frac{\beta}{n}; w\lambda\right) \phi_\nu(p)dp\right] - \eta_{\alpha\beta}\right\}. \end{aligned} \tag{4.41}$$

Example 4. A non-zero genus generalization of the LG machinery of the previous example was constructed in [15], [16]. An appropriate moduli space of algebraic curves is taken as the coupling space. More precisely, let $M_{g,n}$ be the moduli space of dimension $N = 2g + n - 1$ of smooth algebraic curves C of genus g with a marked point $Q_\infty \in C$ and with a marked meromorphic function λ on C of degree n with a pole only in Q_∞. If $P_1, \ldots, P_N$ are the branch points of C,

$$d\lambda|_{P_i} = 0 \tag{4.42}$$

then local co-ordinates on $M_{g,n}$ can be constructed as

$$u^i = \lambda(P_i), \qquad i = 1, \ldots, N = 2g + n - 1. \tag{4.43}$$

The one-dimensional affine group $\lambda \mapsto \alpha\lambda + \beta$ acts on $M_{g,n}$ as

$$u^i \mapsto \alpha u^i + \beta, \qquad i = 1, \ldots, N. \tag{4.44}$$

For $g = 0$ the space $M_{0,n}$ coincides with the space (4.18) of polynomials of degree n.

For $g > 0$ the coupling space M of the model is the covering of $M_{g,n}$ being obtained by fixation of a symplectic basis $a_1, \ldots, a_g, b_1, \ldots, b_g \in H_1(C, \mathbf{Z})$ and of a local parameter k^{-1} in a neighbourhood of Q_∞ such that

$$k^n = \lambda \ , \qquad \lambda \to \infty \tag{4.45}$$

Sometimes I will denote k as $\lambda^{1/n}$ for simplicity of notations.

Let us fix an Abelian differential dp on C. such that

$$dp = d(k + O(1)) \ , \qquad \lambda \to \infty \ , \tag{4.46a}$$

$$\oint_{a_s} dp = 0 \ , \qquad s = 1, \ldots, g \ . \tag{4.46b}$$

The flat Egoroff ∂-invariant metric on M in the local co-ordinates (4.43) has the form [15]

$$ds^2 = \sum_{i=1}^{N} g_{ii}(u)(du^i)^2 \ , \tag{4.47a}$$

$$g_{ii}(u) = -\frac{1}{n} \operatorname*{res}_{P_i} \frac{(dp)^2}{d\lambda} \tag{4.47b}$$

(for $n = 2$ this formula was obtained in [26]). The flat co-ordinates $t^1, \ldots, t^N$ for ds^2 have the form [15,16]

$$t^\alpha = -n \operatorname*{res}_{Q_\infty} \frac{\lambda^{\frac{n-\alpha}{n}}}{n-\alpha} dp \ , \qquad \alpha = 1, \ldots, n-1 \ , \tag{4.48a}$$

$$t^{n-1+\alpha} = \frac{1}{2\pi i} \oint_{a_\alpha} p d\lambda \ , \qquad \alpha = 1, \ldots, g \ , \tag{4.48b}$$

$$t^{g+n-1+\alpha} = \oint_{b_\alpha} dp , \qquad \alpha = 1, \ldots, g . \tag{4.48c}$$

The metric ds^2 in the co-ordinates (4.48) has the form

$$\langle dt^\alpha , dt^\beta \rangle = \delta^{\alpha+\beta} \quad \text{for} \quad 1 \leq \alpha, \beta \leq n-1 , \tag{4.49a}$$

$$\langle dt^{n-1+\alpha} , dt^{g+n-1+\beta} \rangle = \delta^{\alpha\beta} , \qquad 1 \leq \alpha, \beta \leq g \tag{4.49b}$$

otherwise zero.

The amazing point is in *global* definition on M of the flat co-ordinates $t^1, \ldots, t^N$. So M is an unramified covering over some domain in $\mathbf{C}^N$.

The co-ordinates $t^1, \ldots, t^N$ have a simple interpretation in the Hamiltonian theory of GD hierarchy [20]. Let us consider the first Hamiltonian structure of the hierarchy [36]. The annihilator of it is generated by $(n-1)$ local functionals of L. The values of these functionals on the family M of g-gap solutions equal $t^1, \ldots, t^{n-1}$. Furthermore, $t^n, \ldots, t^{g+n-1}$ are the action variables on M and $t^{g+n}, \ldots, t^N$ are the components of the wave number vector. The same interpretation of $t^1, \ldots, t^{n-1}$ is true for the Example 3 above (the case $M = M_{0,n}$). The action variables and wave numbers for the "0-gap" solutions (constants) are not defined.

Let us construct the hierarchy (2.47). Let

$$p = \int_{P_0}^{P} dp \tag{4.50}$$

be an Abelian integral (a multivalued function) on C. Here the base point P_0 is chosen in such a way that $\lambda(P_0) \equiv 0$ (in some domain on M).

The multivalued differential $pd\lambda$ on C has the form

$$pd\lambda = k = d\lambda - \left(\sum_{\alpha=1}^{n=1} t^\alpha k^{\alpha-1} + O(k^{n-1}) \right) dk , \quad k = \lambda^{1/n} \to \infty , \tag{4.51a}$$

$$\oint_{a_s} p d\lambda = 2\pi i\, t^{n+s-1} , \qquad s = 1, \ldots, g , \tag{4.51b}$$

$$\Delta_{a_s}(pd\lambda) = 0 , \qquad \Delta_{b_s}(pd\lambda) = t^{g+n-1+s}\, d\lambda,\ s = 1, \ldots, g \tag{4.51c}$$

Here Δ_{a_s}, Δ_{b_s} are the increments along a- and b-cycles:

$$\Delta_{a_s}(f(P)) = f(P + a_s) - f(P) , \qquad \Delta_{b_s}(f(P)) = f(P + b_s) - f(P) \tag{4.52}$$

for any function (or differential) on C.

The primary differentials (may be, multivalued) ϕ_α on C are defined by the formula

$$\phi_\alpha = \partial_\alpha (pd\lambda)_{\lambda=\text{const.}} = -\partial_\alpha (\lambda dp)_{p=\text{const.}} , \quad \alpha = 1, \ldots, N , \tag{4.53}$$

$\partial_\alpha = \partial/\partial t^\alpha$. So the function $\lambda = \lambda(p)$, $p = \int dp$, is the LG potential of the model. More explicitly,

$$\phi_\alpha \equiv dp^{(\alpha)} = (-k^{\alpha-1} + O(k^{-2})dk , \quad k = \lambda^{1/n} \to \infty , \tag{4.54a}$$

$$\oint_{a_s} dp^{(\alpha)} = 0 , \qquad s = 1, \ldots, g \tag{4.54b}$$

(the normalized Abelian differential of the second kind, $\phi_1 = dp^{(1)} = -dp$);

$$\phi_{n-1+\alpha} \equiv \omega_\alpha , \qquad \alpha = 1, \ldots, g \tag{4.55a}$$

are the normalized holomorphic differentials on C,

$$\oint_{a_s} \omega_\alpha = 2\pi i\, \delta_{\alpha s} ; \tag{4.55b}$$

$$\phi_{g+n-1+\alpha} \equiv \sigma_\alpha \ , \qquad \alpha = 1, \ldots, g \tag{4.56a}$$

is a holomorphic (modulo $d\lambda$) everywhere on C multivalued differential with the increments

$$\Delta_{a_\beta} \sigma_\alpha = 0 \ , \qquad \Delta_{b_\beta} \sigma_\alpha = \delta_{\alpha\beta} \, d\lambda \ . \tag{4.56b}$$

The primary part (2.20) of the hierarchy (2.47) here has the FFM form

$$\partial_{T^{\alpha,0}} dp = -\partial_X \phi_\alpha \ , \qquad \alpha = 1, \ldots, N \ . \tag{4.57}$$

Equivalently, in the diagonal variables $u^1, \ldots, u^N$

$$\partial_{T^{\alpha,0}} u^i = - \left. \frac{\phi_\alpha}{dp} \right|_{P_i} \partial_X u^i \ . \tag{4.58}$$

The equations (4.57) for $\alpha = 1, \ldots, n-1$ (together with the corresponding part of the hierarchy (2.47)) can be obtained from the GD hierarchy (4.24) by the averaging procedure [17-20], [23] over the family M of g-gap solutions of GD. This part for $g \geq 0$ was used in [14] to construct solutions of multicut loop equations [21]. But for $g > 0$ without the extension (4.57) for $\alpha = n, \ldots, N$ it is impossible to construct closed primary operator algebra on the base of the averaged GD hierarchy.

For the particular solution (2.77) of (4.57) where $T^{\alpha,0} \equiv T^\alpha$, and $\partial_X u^i \equiv 1, \quad i = 1, \ldots, N$, one obtains

$$\partial_{T^{\alpha,0}} u^i \equiv \partial_\alpha u^i = - \left. \frac{\phi_\alpha}{dp} \right|_{P_i} \ , \qquad i = 1, \ldots,, N \ . \tag{4.59}$$

This coincides with (4.53).

The formulae (4.58), (1.34) immediately gives the residue representation of the primary double correlators:

$$\eta_{\alpha\beta} = \sum_{i=1}^{N} \operatorname{res}_{P_i} \frac{\phi_\alpha \phi_\beta}{d\lambda} , \tag{4.60a}$$

$$c_{\alpha\beta\gamma}(t) = -\sum_{i=1}^{N} \operatorname{res}_{P_i} \frac{\phi_\alpha \phi_\beta \phi_\gamma}{d\lambda \, dp} . \tag{4.60b}$$

The primary operator algebra (4.60) can be represented via relations among some quadratic differentials

$$\phi_\alpha \phi_\beta = c_\alpha{}^\gamma{}_\beta \, \phi_\alpha \, dp \quad (\text{modulo } d\lambda - \text{divisible differentials}) . \tag{4.61}$$

Let us call this model as $M_{g,n}$-model.

The charges q_α of the primary fields ϕ^α equal

$$q_\alpha = \frac{\alpha - 1}{n} , \qquad \alpha = 1, \ldots, n-1 , \tag{4.62a}$$

$$q_{n-1+\alpha} = -\frac{1}{n} , \qquad \alpha = 1, \ldots, g , \tag{4.62b}$$

$$q_{g+n-1+\alpha} = \frac{n-1}{n} , \qquad \alpha = 1, \ldots, g \tag{4.62c}$$

The dimension of the $M_{g,n}$-model equals

$$d = \frac{n-2}{n} . \tag{4.63}$$

There are resonances of the form (3.28) (for $p = 1$). So the recurrence procedure (3.27) for the vectors $\psi_{i(\alpha,1)}$ is not well-defined.

Note that one can choose any of the primary differentials ϕ_α to construct another ∂-invariant Egoroff metric

$$ds_\alpha^2 = \sum g_{ii_\alpha} (du^i)^2 , \tag{4.64a}$$

$$g_{ii_\alpha} = \operatorname*{res}_{P_i} \frac{(\phi_\alpha)^2}{d\lambda} \tag{4.64b}$$

with the same rotation coefficients. This possibility is a reflection of arbitrariness in the choice of the solution ψ_{i1} of the system (1.26). For the metric (4.64) one obtains another global affine co-ordinate system on M. The formulae (4.62), (4.63) for the scaling dimensions also will change.

Let us construct now the generating functions $x_\alpha(t,z)$ and $V_{\alpha\beta}(t;z,w)$. For the generating function $x_\alpha(t,z)$ of the complete family of the conservation laws (2.43) of (4.57) the following formulae are valid:

$$x_\alpha(t,z) = -\frac{n}{\alpha} \operatorname*{res}_{Q_\infty} \lambda^{\frac{\alpha}{n}}\ {}_1F_1\left(1;\, 1+\frac{\alpha}{n};\, z\lambda\right) dp\ , \qquad \alpha = 1, \dots, n-1 \tag{4.65a}$$

$$x_{\alpha'}(t,z) = \oint_{b_\alpha} e^{z\lambda}\, dp\ , \qquad \alpha' = n-1+\alpha\ , \qquad 1 \le \alpha \le g \tag{4.65b}$$

$$x_{\alpha''}(t,z) = \frac{1}{2\pi i} \oint_{a_\alpha} p e^{\lambda z}\, d\lambda\ , \qquad \alpha'' = g+n-1+\alpha\ , \qquad 1 \le \alpha \le g\ . \tag{4.65c}$$

The proof is similar to the previous example.

The generating function $V_{\alpha\beta}(t;z,w)$ for double correlators in the $M_{g,n}$-model coupled to gravity can be calculated now via the formula (2.62), where

$$\partial_\mu x_\alpha(t,z) = \operatorname*{res}_{Q_\infty} \lambda^{\frac{\alpha-n}{n}}\ {}_1F_1\left[1;\, \frac{\alpha}{n};\, z\lambda\right] \phi_\mu(\lambda)\ , \qquad 1 \le \alpha \le n-1 \tag{4.66a}$$

$$\partial_\mu x_{\alpha'}(t,z) = \eta_{\mu\alpha'} - z \oint_{b_\alpha} e^{z\lambda} \phi_\mu(\lambda)\ , \qquad \alpha' = n-1+\alpha\ , \qquad 1 \le \alpha \le g \tag{4.66b}$$

$$\partial_\mu x_{\alpha''}(t,z) = \frac{1}{2\pi i} \oint_{a_\alpha} e^{z\lambda}\, \phi_\mu(\lambda)\ , \qquad \alpha'' = g+n-1+\alpha\ , \qquad 1 \le \alpha \le g\ .$$

(4.66c)

(In [15,16] formulae of another type for the coefficients $V_{(\alpha,p),(\beta,q)}$ were obtained.) Particularly, one obtains [15,16]

$$V_{(\alpha',0),(\beta',0)} \equiv \langle \phi_{\alpha'} \phi_{\beta'} \rangle = -\tau_{\alpha\beta} , \qquad \alpha' = n-1+\alpha,\ \beta' = n-1+\beta , \quad 1 \le \alpha , \quad \beta \le g \tag{4.67a}$$

where

$$\tau_{\alpha\beta} = \oint_{b_\beta} \omega_\alpha \tag{4.67b}$$

is the period matrix of holomorphic differentials on the algebraic curve C. Thus the problem of specification of the solution of the WDVV being described in this example (say, from the point of view of the isomonodromy deformation theory for (3.5)) seems to be very important for solving the Schottky problem of specification of the period matrices of holomorphic differentials on Riemann surfaces [37].

In [15] also more general models of this type were considered where M is a covering over the moduli space $M_{g;n_1,\dots,n_m}$ of algebraic curves of given genus g with m marked points and with marked meromorphic function λ with poles only in these marked points of given orders $n_1, \dots, n_m$. I will not consider this example here.

In [15,16] the $M_{g,n}$-model was called as minimal model of nonzero genus. To avoid an abusement I stress again that $M_{g,n}$ gives a tree-level primary correlators for a model of topological field theory coinciding with the A_{n-1}-minimal model for $g = 0$.

It seems plausible, nevertheless, that $M_{g,n}$-models for $g > 0$ can be obtained from minimal models as a result of phase transitions. I'll outline here this point for the simplest example of $n = 2$. Let us consider the $k = 3$ model of "pure gravity" [2]. Here one has only one primary field $\phi_1 = P$. The dilaton operators for this multicritical point coincides with $\sigma_3(\phi_1)$. Let us consider the dependence of the primary correlator $t = \langle P P \rangle$ on the couplings $T^{1,0} = X$ and $T^{1,1} = T$, other couplings vanish. Let the couplings X, T be real. The dependence is specified by the string equation

$$X + tT = \frac{t^3}{3} . \qquad (4.68)$$

The function $t = t(X,T)$ is smooth for $T < 0$. For $T = 0$ it has the form $t = (3X)^{1/3}$. So triple correlators have a singularity in this point. After formal extension of $t(X,T)$ for positive T one would obtain a three-valued function $t = (t_1(X) < t_2(X) < t_3(X))$ (for fixed $T > 0$) in some domain $X_- < X < X_+$.

Instead of this multi-valued correlators I propose to use the prescription of the dispersive hydrodynamics [32,28,20] having been elaborated in description of the dispersive analogue of shock-waves. The idea is to consider the $M_{1,2}$-model to describe behaviour of the correlator inside some interval $X'_-(T) < X < X'_+(T)$, $X'_\pm(T) = A_\pm T^{3/2}$ for some constants $A_\pm$. Matching with solutions of (4.68) in the edge points of the interval (and also the position of the edge points) is specified by the assumptions of C^1-smoothness for $T \neq 0$ of the correlator $\langle P P \rangle$. This is provided by an appropriate squeezing of the elliptic curve (being the point of $M_{1,2}$). In the left edge X'_- the a-cycle of the curve should be pinched, in the right edge X'_+ the b-cycle should be pinched.

For other multicritical points of minimal models coupled to gravity other $M_{g,n}$ models (with any g) can be obtained as a result of phase transitions.

The models $M_{g;n_1,\dots,n_m}$ could be obtained by "fusion" of a number of $M_{g,n}$-models. I hope to describe this picture of phase transitions of $M_{g,n}$-models in the next publication.

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