ISTITUTO NAZIONALE DI FISICA NUCLEARE

Sezione di Trieste

INFN/AE-89/08

20 Settembre 1989

P. Furlan, R.R. Paunov and I.T. Todorov:

EXTENDED U(1) CONFORMAL FIELD THEORIES IN TWO DIMENSIONS

Servizio Documentazione
dei Laboratori Nazionali di Frascati

EXTENDED U(1) CONFORMAL FIELD THEORIES IN TWO DIMENSIONS *

P. Furlan
Dipartimento di Fisica Teorica, University of Trieste, Italy
and
Istituto Nazionale di Fisica Nucleare (I.N.F.N.), Sezione di Trieste, Italy
R.R. Paunov **
Istituto Nazionale di Fisica Nucleare (I.N.F.N.), Sezione di Trieste, Italy
I.T. Todorov **
Institut des Hautes Etudes Scientifiques, F-91440 Bures-sur-Yvette

ABSTRACT

A systematic approach is developed for studying local chiral algebras generated by a pair of oppositely charged fields $\psi(z,\pm g)$ such that the operator product expansion (OPE) of $\psi(z_1, g)\,\psi(z_2,-g)$ involves a U(1) current. The main tool in the study is the factorization property of the charged fields (exhibited in [PT 2, 3]) for Virasoro central charge c>1 into U(1)-vertex operators tensored with Zamolodchikov Fateev [ZF1] (generalized) Z_k-parafermions. The case $\Delta_2 = 4(\Delta_1 - 1)$, where $\Delta_\nu = \Delta_{k-\nu}$ $(\Delta_0 = 0)$ are the conformal dimensions of the parafermionic currents, is studied in detail and the corresponding quantum field theoretic (QFT) representations of the chiral algebra are displayed.

---

* To be submitted for publication.

** Permanent address: Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Sofia 1784, Bulgaria.

CONTENTS

1. Introduction. The family of chiral algebras $\mathcal{A} = \mathcal{A}(s, g^2; k, c)$.

The construction of a (2-dimensional) conformal quantum field theory (CQFT) can be viewed as a three step process. One first defines the chiral (left and right movers') algebra $\mathcal{A}$ of local (integer spin) currents. Secondly, one constructs the positive energy QFT representations of $\mathcal{A}$ generated by all (multivalued) $\mathcal{A}$-primary fields that are relatively local to the currents. Finally, one has to combine the "right moving" primary fields $\varphi_i(z)$ with their "left moving" counterparts $\overline{\varphi}_j(\overline{z})$ into (a complete set of) local 2-dimensional fields

$$\phi(z, \overline{z}) = \sum_{ij} a_{ij}\, \varphi_i(z)\, \overline{\varphi}_j(\overline{z})$$

that have single valued euclidean Green's functions.

The right movers' algebra $\mathcal{A}$ is a local extension of the Virasoro algebra Vir, generated by the chiral component $T(z)=1/2\,(T^o{}_o+T^1{}_o)$ of the stress energy tensor. It is characterized by the property that all correlation functions of the basic fields are (single valued) rational functions with (possible) poles for coinciding arguments only.

An axiomatic description of such local chiral algebras has been given by Frenkel et al. [FL] (see also [G]) in terms of a lattice of integer spin vertex operators. (For a somewhat different but also mathematically rigorous approach - see [FFK], [FrK].) A large class of local algebras including all Kac-Moody current algebras (or "Wess-Zumino-Witten models") can be obtained by classifying all (maximal) local extensions of an abelian U(1)x...xU(1) - current algebra. In this paper we continue the program advanced in [BMT] (and [BMPT]) and further pursued in [PT 1-3] of describing the theories involving just a single U(1) extended conformal current algebra.

The work of Buchholz, Mack and one of the present authors [BMT] (see also

[PT1]) contains an essentially complete treatment of what could be termed "level 1 extended current algebras" from the point of view of algebraic QFT of Doplicher, Haag and Roberts ([DHR], [DR]). The case of higher level algebras was first approached in [PT 2, 3]. We shall sum up these first steps in the context of the more general program followed here.

We are searching for a family of algebras

$\mathcal{A} = \mathcal{A}(s, g^2; k, c)$

including a "level k" U(1) current algebra $\hat{U}_k$ defined by the Weyl commutation relations (cf. [BMT])

$$e^{i J(u)} e^{i J(v)} = \exp\left\{- \frac{k}{2} A(u, v) + i J(u+v)\right\} \tag{1.1}$$

where J(u) is the smeared U(1) current

$$J(u) = \oint_{S^1} J(z)\, u(z) \frac{dz}{2\pi i} \qquad u(z) \text{ real for } z\ (=e^{i\vartheta}) \in S^1 , \tag{1.2}$$

k is a positive integer and A(u, v) is the 2-cocycle

$$A(u, v) = \frac{1}{k}\left[J(u), J(v)\right] = \oint u'(z) v(z) \frac{dz}{2\pi i} \left(= - A(v,u)\right), \tag{1.3}$$

and $\mathcal{A}$ is generated by a pair of oppositely charged fields (with respect to the charge operator

$$J_0 = \oint J(z) \frac{dz}{2\pi i} \tag{1.4}$$

that generates the centre of $\hat{U}_k$) $\psi(z,\pm g)$ satisfying the following requirements.

(i) The fields ψ satisfy the standard Ward identities with respect to the current J. These can be written as gauge transformations involving the unitary generators of $\hat{U}_k$,

$$e^{iJ(u)} \psi(z, g)\, e^{-i J(u)} = e^{igu(z)} \psi(z, g), \tag{1.5}$$

or as a small distance relation

$$\lim_{z'\to z} (z'-z)\, J(z')\, \psi(z, g) = g\ \psi(z, g). \qquad (1.6)$$

(ii) The fields $\psi(z,\pm g)$ are primary conformal fields carrying the same (integer or half-odd-integer) spin

$$s = \left(\frac{1}{2}\right),\ 1,\ \frac{3}{2},\ 2,\ \ldots \qquad (1.7)$$

with respect to the local stress energy tensor T(z). Their infinitesimal reparametrization properties are expressed in terms of the small distance OPE

$$T(z_1)\, \psi(z_2,g) = s\, z_{12}^{-2}\ \psi(z_2, g) + z_{12}^{-1}\ \psi'(z_2, g) + O(1), \quad z_{12} = z_1 - z_2. \qquad (1.8)$$

T(z) is normalized, as usual, in terms of the Virasoro central charge c:

$$2\, z_{12}^{4} \langle T(z_1)\, T(z_2) \rangle = c. \qquad (1.9)$$

Remark 1.1 The case s=1/2 corresponds to a free complex spin 1/2 (Weyl) field that fits a point of the level 1 series (for k=1=c). For c>1 (k>1) the theory is just the tensor product of free spin 1/2 theory with a c-1 theory (see [GS1]). We shall consider an example of this type in sec. 4. In the rest of the paper we shall assume that $s \geq 1$.

Remark 1.2 If $s \in \mathbf{N} + 1/2$ then the algebra $\mathcal{A}$ has an obvious $\mathbf{Z}_2$ grading. Its even subalgebra is an element of our family with the same values of k and c but with a double charge:

$$\mathcal{A}\ (4s - 2n, 4g^2; k, c) \subset \mathcal{A}\,(s, g^2; k, c). \qquad (1.10)$$

Here n=0 for k=1=c; we shall prove that, in general, n is a non-negative integer, which can be used as an additional label for our chiral algebras.

(iii) All vacuum expectation values of the ψ's are rational (hence, single valued) functions of the z's.

Corollary. Requirement (iii) and the analysis in [BMT] implies that g^2 is a (positive) integer.

(iv) The fields $\psi(z,\pm g)$ can be normalized in such a way that we have the small distance OPE

$$z_{12}^{2s}\,\psi(z_1,g)\;\psi(z_2,-g) = k + g\int_{z_2}^{z_1} J(\zeta)\,d\zeta + \frac{g^2}{c} R_2(z_1,z_2)\,z_{12}^2 \qquad (1.11)$$

where the remainder R_2 has a finite limit, $R_2(z,z)$, for coinciding arguments and is orthogonal to the first two terms in the expansion in the sense that $\langle R_2\rangle=0=\langle R_2(z_1,z_2)\,J(z_3)\rangle$.

<u>Remark 1.3</u> It is sufficient to assume that the coefficient to the integral of the current is g; then the c-number term in the above OPE is proven to coincide with the normalization k of the current 2-point function as a consequence of the Ward identity (i). Indeed it follows from (1.6) that

$$\langle\psi(z_1,g)\;\psi(z_2,-g)\;J(z_3)\rangle = \frac{g\,z_{12}}{z_{13}\,z_{23}}\langle\psi(z_1,g)\,\psi(z_2,-g)\rangle. \qquad (1.12)$$

If we insert (1.11) into the left hand side of (1.12), use $\langle J(z)\rangle=0$ and take care of the expression for the current 2-point function

$$z_{12}^2\langle J(z_1)\,J(z_2)\rangle = k \qquad (1.13)$$

that follows from (1.3), we deduce the Möbius (or SU(1,1)-) invariant expression for the 3-point function

$$\langle\psi(z_1,g)\;\psi(z_2,-g)\;J(z_3)\rangle = \frac{gk\;z_{12}^{1-2s}}{z_{13}\,z_{23}} \qquad (1.14)$$

which yields the first term in (1.11).

(v) The charge squared, g^2, is an integer valued function of s and k with the property that the difference

$$k\,\Delta = ks - \frac{1}{2}g^2 \qquad (1.15)$$

is a non-negative integer that vanishes for k=1. We also have

$$c = 1 \quad \text{for} \quad k = 1\ . \qquad (1.16)$$

Remark 1.4 The non-negativity of Δ will appear as a consequence of Wightman positivity (see Proposition 1.1 below; we assume without spelling them out the standard axioms of a general loal QFT: Hilbert space structure, energy positivity, local commutativity of observable fields, uniqueness of the vacuum ...). The vanishing of Δ and c-1 for k=1 is equivalent to the assumption that the stress energy tensor T is a function of the current (an assumption that implies the Sugawara formula).

The following fundamental proposition shows that the U(1)- current algebra can, in some sense, be factored out (it thus may be regarded as an extension of the factorization property for spin 1/2 fields referred to in Remark 1.1).

Proposition 1.1 Under the above assumptions the stress energy tensor of the theory splits into a sum of two commuting pieces

$$T(z) = \frac{1}{2k} : J^2(z) : + T_R(z) \qquad (1.17)$$

where the normal product of two currents is defined by the free field OPE

$$J(z_1)\ J(z_2) = k\, z_{12}^{-2} + : J(z_1)\ J(z_2) : \qquad (1.18)$$

and the residual stress tensor T_R commutes with the current

$$[T_R(z_1),\ J(z_2)] = 0\,. \qquad (1.19)$$

In an unitary theory T_R does not vanish only if the total central charge c exceeds 3/2:

$$c = 1 + c_R,\ c_R = 2\, z_{12}^4 < T_R(z_1)\ T_R(z_2) > \geq \tfrac{1}{2} \ (\text{for } T_R \neq 0). \qquad (1.20)$$

The charged field $\psi(z, g)$ can be represented as the tensor product of a vertex operator V of dimension $s - \Delta = 1/(2k)\, g^2$ and a primary conformal field φ of dimension Δ (>0):

$$\Psi (z, g) = \sqrt{k}\ V (z, \frac{1}{\sqrt{k}} g) \otimes \varphi (z) \tag{1.21a}$$

$$\left[V (z, e), T_R(w)\right] = 0 = \left[\varphi (z), J (w) \right]; \tag{1.21b}$$

here $g/\sqrt{k}$ is the effective charge of V (with respect to the normalized current $J/\sqrt{k}$) and

$$z_{12}^2\, T_R (z_1)\ \varphi (z_2) = \Delta\, \varphi (z_2) + z_{12}\, \varphi'(z_2) + O (z_{12}^2). \tag{1.22}$$

We only give a <u>sketch of the proof</u> which follows a well known argument of Goddard and Olive (see [GO] and references to earlier work cited there). Since J is a spin 1 primary field of Vir its commutator with T and with $1/2k : J^2 :$ is the same. This yields (1.19) which in turn implies that the two terms in the right hand side of (1.17) commute and hence the central charge is additive (Eq. (1.20) takes place). The inequality $c_R \geq 1/2$ follows from the results [FQS] on the unitary discrete series of Vir. Eqs. (1.21) and (1.22) are then corollaries of (1.17).

The following irreducibility assumption restricts the subsequent analysis to a rank 1 current algebra.

(vi) There is no spin 1 contribution to the vacuum sector OPE $\varphi^* (z_1)\, \varphi (z_2)$ and the reduced stress energy tensor T_R (z), defined by (1.17) is the only quasiprimary field of dimension 2 appearing in this OPE:

$$z_{12}^{2\Delta}\, \varphi^* (z_1)\, \varphi (z_2) = 1 + \frac{12\Delta}{c_R\, z_{12}} \int_{z_2}^{z_1} (z_1 - \zeta)\, (\zeta - z_2)\ \ T_R (\zeta)\, d\zeta\, +$$

$$+ \frac{A}{z_{12}^2}\, 30 \int_{z_2}^{z_1} (z_1 - \zeta)^2\, (\zeta - z_2)^2\ W (\zeta)\, d\zeta\ + z_{12}^4\, R_4 (z_1, z_2), \tag{1.23}$$

where W is an hermitean primary field of Vir of spin 3 normalized by $z_{12}^6 < W(z_1)\, W(z_2) > = 1$, and the remainder is orthogonal to the preceding terms.

The weight factors in the integral representation (1.23) are chosen to reproduce the exact 3-point functions of $\varphi^*(z_1)\, \varphi(z_2)$ times the corresponding tensor field. To verify this statement we use the general integral formula

$$\Gamma(\lambda+\mu) \int_{z_2}^{z_1} \frac{(z_1-\zeta)^{\lambda-1} (\zeta-z_2)^{\mu-1}}{\Gamma(\lambda)\, \Gamma(\mu)\, (\zeta-z_3)^{\lambda+\mu}}\, d\zeta = \frac{z_{12}^{\lambda+\mu-1}}{z_{13}^{\mu}\, z_{23}^{\lambda}} \, . \qquad (1.24)$$

The structure of the reduced algebra is specified by the observation that the vacuum expectation value

$$F_{2n} = \left[\frac{\prod_{k,\ell=1}^{n} z_{2k-1,\, 2\ell}}{\prod_{1\le i<j\le n} z_{2i-1,\, 2j-1}\, z_{2i,\, 2j}} \right]^{2\Delta} < \varphi(z_1)\, \varphi^*(z_2) \dots \varphi(z_{2n-1})\, \varphi^*(z_{2n}) > \qquad (1.25)$$

is a single valued meromorphic function of the 2n-3 independent cross-ratios of the complex variables z_k. This observation is a straightforward consequence from our assumption (iii) and from the decomposition property (1.17). We have thus reduced the problem of classifying the local chiral algebras involving an U(1) current to the study of the algebra generated by a field φ of rational dimension Δ (given by (1.15)) and its hermitean conjugate.

2. General properties of the reduced algebra.

2.A. Locality of $\mathcal{A}$ implies parafermionic fusion rules.

Proposition 2.1 The locality assumption (iii) implies that the fusion rules for products of fields φ generate a set of primary fields $\varphi_1=\varphi$, φ_2, ... such that

$$z_{12}^{\Delta_\mu+\Delta_\nu-\Delta_{\mu+\nu}} \varphi_\mu(z_1)\,\varphi_\nu(z_2) =$$

$$= \frac{C_{\mu\nu}\,\Gamma(2\Delta_{\mu+\nu})\,z_{12}^{1-2\Delta_{\mu+\nu}}}{\Gamma(\Delta_\mu-\Delta_\nu+\Delta_{\mu+\nu})\,\Gamma(\Delta_\nu-\Delta_\mu+\Delta_{\mu+\nu})}\int_{z_2}^{z_1}(z_1-\zeta)^{\Delta_\nu+\Delta_{\mu+\nu}-\Delta_\mu-1}(\zeta-z_2)^{\Delta_\mu+\Delta_{\mu+\nu}-\Delta_\nu-1}\varphi_{\mu+\nu}(\zeta)\,d\zeta$$

+ higher order orthogonal terms

$$\approx C_{\mu\nu}\left\{\varphi_{\mu+\nu}(z_2)+\frac{\Delta_\mu-\Delta_\nu+\Delta_{\mu+\nu}}{2\Delta_{\mu+\nu}}\,z_{12}\,\varphi'_{\mu+\nu}(z_2)+\dots\right\}, \qquad (2.1a)$$

$$z_{12}^{\Delta_{\mu+\nu}+\Delta_\nu-\Delta_\mu}\varphi_{\mu+\nu}(z_1)\,\varphi_\nu^*(z_2)\approx C_{\mu\nu}\{\varphi_\mu(z_2)+$$

$$+\frac{\Delta_{\mu+\nu}+\Delta_\mu-\Delta_\nu}{2\Delta_\mu}\,z_{12}\,\varphi'_\mu(z_2)+\dots\}, \qquad (2.1b)$$

where φ_ν are normalized by

$$<\varphi_\mu^*(z_1)\,\varphi_\nu(z_2)> = \delta_{\mu\nu}\,z_{12}^{-2\Delta_\nu}, \qquad (2.2)$$

and $C_{\mu\nu}$ $(=C_{\nu\mu})$ are structure constants which can consistently be chosen real. The conformal weights Δ_ν obey the "braid relation"

$$e^{i\pi(\Delta_\lambda+\Delta_\mu+\Delta_\nu+\Delta_{\lambda+\mu+\nu})} = e^{i\pi(\Delta_{\lambda+\mu}+\Delta_{\lambda+\nu}+\Delta_{\mu+\nu})} \tag{2.3}$$

and are further restricted by the condition

$$(0\leq)\ \Delta_\nu = \nu^2\Delta_1 - 2n_\nu \quad n_\nu = 0, 1, \ldots\ (n_0 = 0 = n_1). \tag{2.4}$$

The dimension Δ_k is an integer.

<u>Proof.</u> Consider the spin-s_ν fields

$$\psi(z, \nu g, s_\nu) = \sqrt{k}\, V\left(z, \frac{g}{\sqrt{k}}\right) \otimes \varphi_\nu(z) \tag{2.5}$$

which appear in the OPE of ν factors $\psi(z, g)$. Locality implies that the spin s_ν should differ from $\nu^2 s$ by an even integer

$$\nu^2 s - s_\nu = \nu^2\Delta - \Delta_\nu = 2n_\nu\,. \tag{2.6}$$

We shall prove this statement by induction. For $\nu=1$ we have $n_1=0$, so that (2.6) is trivially satisfied. Let it be true for some ν. The fusion rule

$$z_{12}^{\,s+s_\nu-s_{\nu+1}}\ \psi(z_1, g, s)\, \psi(z_2, \nu g, s_\nu) \approx \psi(z_2, (\nu+1) g, s_{\nu+1})$$

and the locality condition

$$\psi(z_2, \nu g, s_\nu)\ \psi(z_1, g, s) = (-1)^{4s\, s_\nu}\ \psi(z_1, g, s)\ \psi(z_2, \nu g, s_\nu)$$

then imply that $s+s_\nu - s_{\nu+1}$ has the same parity as $4ss_\nu$. Using the induction assumption we deduce that

$$s\,(1+\nu^2) - s_{\nu+1} = 4\,s^2\nu^2 \quad \mathrm{mod}\ 2\ .$$

This equation is satisfied by $s_{\nu+1} = (\nu+1)^2 s$ mod 2, since $2s\,(2s+1)\,\nu$ is even (as long as ν and $2s$ are integers).

The local OPEs for $\psi\,(z, \nu g, s_\nu)$ imply the fusion rules (2.1) for φ_ν. To prove that n_ν is non-negative we shall use the following estimate.

Lemma 2.2 The weights Δ_ν satisfy the inequalities

$$\Delta_\nu \le \nu^2\,\Delta \qquad (\Delta = \Delta_1)\ . \tag{2.7}$$

Proof of lemma. Consider the product

$$f_\nu\,(z_0) = \langle\, \varphi^*(z_0)\,\varphi\,(z_1)\,\ldots\,\varphi\,(z_{\nu+1})\ \varphi^*_\nu\,(o)\,\rangle\,(z_{01}\ldots z_{0\,\nu+1})^{2\Delta}\ z_0^{\Delta+\Delta_\nu-\Delta_{\nu+1}}\ . \tag{2.8}$$

The fusion rules (2.1) imply that f_ν is an entire function of z_0. Global conformal invariance yields the asymptotic behaviour $f_\nu \sim z_0{}^{(2\nu+1)\,\Delta+\Delta_\nu-\Delta_{\nu+1}}$.

The Liouville theorem tells us that f_ν vanishes unless

$$\Delta_{\nu+1} \le \Delta_\nu + (2\nu+1)\,\Delta\ . \tag{2.9}$$

The estimate (2.7) then follows by induction:

$\Delta_1 = \Delta$; if $\Delta_\nu \le \nu^2\Delta$ then $\Delta_{\nu+1} \le (\nu+1)^2\,\Delta$.

To complete the proof of Proposition 2.1 we observe that the weights (2.4) satisfy (2.3). We note that Eq. (2.3) is a direct consequence of the fusion rules. Indeed, Eq. (2.1a) implies the "exchange relation" (for, say, $|z_1| > |z_2|$)

$$\varphi_\nu(z_2)\ \varphi_\mu(z_1) = e^{i\pi(\Delta_{\mu+\nu}-\Delta_\mu-\Delta_\nu)}\ \varphi_\mu(z_1)\ \varphi_\nu(z_2). \tag{2.10}$$

Eq. (2.3) then follows by analysing the triple product $\varphi_\lambda(z_1)\ \varphi_\mu(z_2)\ \varphi_\nu(z_3)$ (see [HT]). Finally, Eqs. (1.15) and (2.4) imply

$$\Delta_k = k\,(k\,s - \frac{1}{2}g^2) - 2\,n_k \tag{2.11}$$

so that - according to assumption (v) - Δ_k is an integer.

Corollary 2.3 According to (1.15) Δ can be written in the form

$$\Delta = n - \frac{p}{k} \quad \text{where } p \in Z, \quad |p| < k \ \ (p \neq 0) \tag{2.12}$$

n is a positive integer ($n=[\Delta+1]$ for $p\geq 1$). Then the OPE algebra (2.1) admits an automorphism

$$\gamma\,\varphi_\nu(z) = e^{-2\pi i \frac{p\nu}{k}}\ \varphi_\nu(z). \tag{2.13}$$

If p and k are coprimes then γ generates a Z_k - gauge group of the first kind for the reduced model. (The reduced observable algebra is, by definition, the Z_k -invariant subalgebra of the field algebra. It is generated by φ_k, T_R and by the primary fields appearing in the OPE of $\varphi_\nu^*(z_1)\ \ \varphi_\nu(z_2)$.) More generally, if (p,k) is the largest common divisor of p and k, then γ generates a (gauge) symmetry group $Z_{k/(p,k)}$.

Remark 2.1 The inequality (2.7) is saturated iff the reduced Virasoro central charge c_R (1.20) is a (positive) integer and if φ is (a Klein transform of) a $u(1)^{\otimes c_R}$ - vertex operator. An example of this type, corresponding to $c_R=1$ for which the additional u(1) current does not appear in the OPE $\varphi*\varphi$ (and which is, therefore, not excluded by assumption (vi) of Sec. 1) is considered in Sec. 4.

We shall make at this point the <u>parafermionic hypothesis</u> that $\Delta_k=0$, and, moreover,

$$\varphi^*_\nu = \varphi_{k-\nu} \; (\varphi_0 = 1 = \varphi_k) \qquad \Delta_\nu = \Delta_{k-\nu} \, . \tag{2.14}$$

If we assume, more generally, that φ_k is hermitian, then we would have $\varphi_{2k}=1$ and we would again come to (2.14) with k substituted by 2k. Thus, the parafermionic hypothesis appears to us rather natural and not very restrictive (once we have established Proposition 2.1 and Corollary 2.3).

<u>Corollary 2.4</u> The general solution of the braid relations (2.3) satisfying (2.14) can be written in the form

$$\Delta_\nu = p\nu \left(1 - \frac{\nu}{k}\right) + 2\, m_\nu \qquad m_\nu = m_{k-\nu} \; (m_0 = 0). \tag{2.15}$$

<u>Proof.</u> One first derives (2.15) with $2\, m_\nu$ substituted by M_ν such that all $M_{\nu+2\ell}$ have the same parity. It follows that $M_{2\ell}$ and $M_{k-2\ell}$ are even. For k odd this means that all M_ν's are even and we can write (2.15). If for k even M_1 is odd we can substitute p by p–k sign p so that the transformed M_1 is even:

$M_1+(k-1)$ sign $p=2m_1$ (and the new p again satisfies $|p|<k$ as in (2.12) if the original p does).

2.B. Inequalities for parafermionic conformal weights.

Proposition 2.5. The conformal weights of the $\mathbf{Z}_k$ parafermionic currents satisfy the inequalities

$$\Delta_\nu \leq \nu \frac{k-\nu}{k-1} \Delta \quad (m_\nu \leq \nu \frac{k-\nu}{k-1} m_1) , \tag{2.16}$$

$$(\nu-2)\, \Delta_\nu + \nu\, \Delta \geq \nu\, \Delta_{\nu-1} \, . \tag{2.17}$$

Proof. We consider the (k–ν+1)– and the (ν+1)– point functions

$$(z_{01} \cdots z_{0k-\nu})^{\Delta_\nu+\Delta-\Delta_{\nu+1}} < \varphi_\nu (z_0)\, \varphi (z_1) \ldots \varphi (z_{k-\nu}) > \tag{2.18a}$$

$$(z_{01} \cdots z_{0\nu})^{\Delta_\nu+\Delta-\Delta_{\nu-1}} < \varphi^*_\nu (z_0)\, \varphi (z_1) \ldots \varphi (z_\nu) > \tag{2.18b}$$

and apply the argument in the proof of Lemma 2.2 with the result

$$\Delta_{\nu+1} \leq \frac{k-\nu-2}{k-\nu} \Delta_\nu + \Delta \tag{2.19}$$

and (2.17), respectively. Eq. (2.16) follows from (2.19) by induction.

Corollary 2.6 The parafermionic conformal weights satisfy the triangle inequalities

$$\Delta_\nu - \Delta \leq \Delta_{\nu+1} \leq \Delta_\nu + \Delta \quad . \tag{2.20}$$

Remark 2.2 The inequality (2.16) becomes an equality for $k\Delta_\nu = p\nu\,(k-\nu)$. As a result the integers m_ν in the general formula (2.15) for the conformal weights satisfy the same inequality. We observe that the inequalities (2.16) and (2.17) are valid without any positivity assumption. (This point is discussed in more detail in [PT 4].)

2.C. Four point function. Relations among c_R, Δ and structure constants.

We now proceed to the analysis of the 4-point function of φ and φ^* (which is determined by, and itself determines, the 4-point function of $\psi(z, \pm g)$).

Proposition 2.7 Under the above assumptions the 4-point correlation function:

$$F_4(\eta) = \left(\frac{z_{12}\, z_{23}\, z_{34}\, z_{14}}{z_{13}\, z_{24}}\right)^{2\Delta} < \varphi^*(z_1)\, \varphi(z_2)\, \varphi^*(z_3)\, \varphi(z_4) > , \qquad (2.21)$$

where η is the cross ratio

$$\eta = \frac{z_{12}\, z_{34}}{z_{13}\, z_{24}} \left(= 1 - \frac{z_{14}\, z_{23}}{z_{13}\, z_{24}}\right) , \qquad (2.22)$$

is a polynomial of degree $n_2 = 2\Delta_1 - 1/2\ \Delta_2$ in the crossing symmetric variable

$$\eta\,(1-\eta) = \frac{z_{12}\, z_{23}\, z_{34}\, z_{14}}{z_{13}^{\,2}\, z_{24}^{\,2}} \qquad (2.23)$$

$$F_4(\eta) = 1 + 2\Delta \sum_{\ell=1}^{n_2} (-)^{\ell}\, \frac{b_\ell}{\ell}\, \eta^{\ell}\, (1-\eta)^{\ell} . \qquad (2.24a)$$

The first four coefficients of this polynomial are determined from the OPE (1.23):

$$b_1 = 1, \quad b_2 = \frac{2\Delta}{c_R} + 2\Delta - 3, \quad b_3 = (2-\Delta)(5-2\Delta) - \frac{3\Delta}{c_R}(3-2\Delta) - v^2 \qquad (2.24b)$$

where we have set $A^2 = (2\Delta/3)\, v^2$.

The proof is based on the observation that a conformally invariant correlation function can only have singularities for coinciding arguments; these correspond to the points $\eta = 0$, 1 and ∞ in the case of F_4. In a positive metric theory the powers of the singularities of the 4-point function do not exceed the corresponding powers of the 2-point functions. Hence F_4 has no pole for $\eta=0$. The symmetry of (2.21) under the substitution $z_2 \leftrightarrow z_4$ implies $F_4(\eta) = F_4(1-\eta)$. It then follows that F_4 is a polynomial of degree n_2 in the crossing symmetric product (2.23). The coefficients in the expansion (2.24a) are determined by inserting the OPE (1.23) in the 4-point function (see Appendix A):

$$z_{12}^{2\Delta}\, z_{34}^{2\Delta} < \varphi^*(z_1)\,\varphi(z_2)\,\varphi^*(z_3)\,\varphi(z_4) > \; = \; (1-\eta)^{-2\Delta}\, F_4(\eta) =$$

$$= 1 + \frac{2\Delta^2}{c_R}\,\frac{\eta^2}{1-\eta} + A^2\eta^3 + O(\eta^4)\,, \qquad (2.25)$$

A ($= v\sqrt{2\Delta/3}$) being the structure constant in front of the spin-3 field in the OPE (1.23). The coefficient to the highest power of $\eta(1-\eta)$ in the expansion (2.24) is related to the structure constant C_{11} of Eq. (2.1):

$$C_{11}^2 = \frac{2\Delta\, b_{n_2}}{n_2} \equiv C^2. \qquad (2.26)$$

Proposition 2.7 is particularly informative for small n_2. For $n_2=1$ it fixes both the reduced central charge and the structure constants C^2 and v^2 (see (2.24b)):

$$\Delta_2 = 4\Delta - 2 \Rightarrow c_R = \frac{2\Delta}{3-2\Delta}\,, \quad C^2 = 2\Delta \;. \qquad (2.27)$$

Noting that the coefficient b_3 of the expansion (2.24) vanishes, we find

$$v^2 \left(= \frac{3A^2}{2\Delta}\right) = (2-\Delta)(5-2\Delta) - \frac{3\Delta}{c_R}(3-2\Delta) \tag{2.28}$$

or, inserting the value of c_R from (2.27)

$$2v^2 = (7-4\Delta)\ (2\Delta-1)\,. \tag{2.29}$$

An unitary solution of these conditions is provided by the standard parafermions [ZF1] [GQ] characterized by p=1, m_v=0 (in (2.15)):

$$\Delta_v = v\left(1-\frac{v}{k}\right), \qquad c_R = 2\,\frac{k-1}{k+2} \tag{2.30}$$

$$C^2_{\mu\nu} \equiv \bar{C}^2_{\mu\nu} = \frac{(\mu+\nu)!\,(k-\mu)!\,(k-\nu)!}{k!\,\mu!\,\nu!\,(k-\mu-\nu)!}\,. \tag{2.31}$$

They were studied in the present context in [PT 2, 3, 4] and [T]. Here we shall concentrate on the next simplest case, n_2=2, in which Proposition 2.7 gives two relations between the four parameters Δ, c_R, v and C characterizing the theory. One is given by Eq. (2.28) or

$$c_R = \frac{3\Delta\,(3-2\Delta)}{(2-\Delta)\,(5-2\Delta) - v^2} \qquad \text{for } \Delta_2 = 4\,(\Delta-1) \tag{2.32}$$

since the coefficient b_3 in the expansion (2.24) vanishes in this case, too. The second is provided by Eq. (2.26), which, combined with (2.24b), gives

$$C^2 = \Delta\, b_2 = \Delta\left(2\Delta - 3 + \frac{2\Delta}{c_R}\right), \tag{2.33}$$

or

$$c_R = \frac{2\,\Delta^2}{C^2 + (3-2\Delta)\,\Delta}\ . \qquad (2.34)$$

Comparing (2.32) and (2.34) we also find

$$(0\leq)\ v^2 = (2-\Delta)\,(5-2\Delta) - 3\ \frac{3-2\Delta}{2\Delta}\ \left[C^2 + (3-2\Delta)\,\Delta\right]. \qquad (2.35)$$

We shall prove in the next section that all structure constants $C_{\mu\nu}$ of the parafermionic theory can be expressed in terms of C $(=C_{11})$ provided that

$$2\Delta + 2\Delta_\nu - \Delta_{\nu+1} - \Delta_{\nu-1}\ \left(= 2\,(p + 2m_1 + 2m_\nu - m_{\nu+1} - m_{\nu-1})\right) = 4\,. \qquad (2.36)$$

(Eq. (2.36) is an extension of the requirement $\Delta_2=4(\Delta-1)$ to which it reduces for $\nu=1$.)

3. Parafermionic theories with $\Delta_2 = 4\,(\Delta-1)$.

In the case $n_2=2$ we can restrict the set of possible parafermionic models to be taken into account by the following lemma.

Lemma 3.1. If for $n_2=2$ we impose on the integers m_ν' s appearing in Eq. (2.15) the condition

$$m_1 = m_2 = \ldots = m_{k-1} \equiv m \qquad (3.1)$$

(in agreement with the constraint $m_v = m_{k-v}$), then positivity and uniqueness of the vacuum imply that

$$m = 0 \qquad (\Rightarrow p = 2) \qquad \Rightarrow k \geq 3 \tag{3.2a}$$

$$m = 1 \qquad (\Rightarrow p = -1) \qquad \Rightarrow k \geq 3 \tag{3.2b}$$

$$m \geq 2 \qquad (\Rightarrow p = 2\text{-}3m) \qquad \Rightarrow k = 3 \qquad (\Rightarrow \Delta = \frac{4}{3}) \ . \tag{3.2c}$$

<u>Proof.</u> Inserting Eqs. (2.32) and (3.1) into Eq. (2.15) we have that $p = 2\text{-}3m$; then positivity of Δ_2 gives the condition

$$\frac{3m - 2}{k} > m - 1 \ ,$$

from which Eqs. (3.2) follow.

The case (3.2a) appears as an "extremal solution" of (2.3) for which the inequality (2.16) becomes an equality

$$\Delta_v = 2v \ (1 - \frac{v}{k}) \ ; \tag{3.3}$$

it corresponds to the class of (non-standard) Z_k-parafermions briefly considered in Appendix A to [ZF1].

We shall also study in this section the case (3.2b) for which we shall only find a finite set of unitary models.

Finally we shall devote the whole section 6 to analyze the special case $k = 3$, $\Delta = 4/3$, selected by Eq. (3.2c) (and also appearing in the families (3.2a) and (3.2b)).

3.A. The case $m_\nu=0$. Solution of the associativity problem for $C_{\mu\nu}$.

First of all, we shall solve the associativity condition

$$\varphi_\lambda\,(\varphi_\mu\,\varphi_\nu) = (\varphi_\lambda\,\varphi_\mu)\,\varphi_\nu \Rightarrow C_{\lambda\mu}\,C_{\lambda+\mu\,\nu} = C_{\lambda\,\mu+\nu}\,C_{\mu\nu} \qquad (3.4)$$

for the structure constants in Eq. (2.1) when the conformal weights are given by (3.1). Eq. (3.4) tells us that all $C_{\mu\nu}$ can be expressed (inductively) in terms of $C_{1\nu}$. We shall derive a recurrence relation for $C_{1\nu}$ by studying the 4-point function

$$F_\nu\,(\eta) = z_{14}^{2\Delta}\;\; z_{23}^{2\Delta_\nu}\,\eta^{\Delta+\Delta_\nu-\Delta_{\nu-1}-2} < \varphi^*(z_1)\,\varphi_\nu(z_2)\,\varphi_\nu{}^*(z_3)\,\varphi\,(z_4) > . \qquad (3.5)$$

Under the assumption (2.36) F_ν is a polynomial of degree two in the variables η and η^{-1}. In fact, Eq. (3.1) gives the general solution of (2.36) consistent with the symmetry property $m_\nu=m_{k-\nu}$. Assuming (3.1) we can write

$$\Delta + \Delta_\nu - \Delta_{\nu-1} - 2 = 2 - 4\,\frac{\nu}{k} = 2 + \Delta_{\nu+1} - \Delta_\nu - \Delta$$

$$\Delta + \Delta_{\nu-1} - \Delta_\nu = 4\,\frac{\nu-1}{k}\,, \qquad \frac{\Delta_\nu - \Delta}{\Delta_{\nu-1}} = \frac{k-\nu-1}{k-\nu+1}\,.$$

Proposition 3.2. For Δ_ν given by (3.1) the function (3.5) can be written in the form

$$F_\nu\,(\eta) = C_{1\nu}^2\;(\eta^2 - \frac{4\nu}{\nu+1}\,\eta) + a_0 + C_{1\nu-1}^2\;\left(\eta^{-2} - 4\,\frac{k-\nu}{k-\nu+1}\,\eta^{-1}\right) \qquad (3.6a)$$

where

$$a_0 = 1 + C^2_{1\nu}\,\frac{3\nu-1}{\nu+1} + C^2_{1\nu-1}\,\frac{3(k-\nu)-1}{k-\nu+1}. \tag{3.6b}$$

Eq. (3.6a) together with the small OPE (using just the absence of a spin 1 contribution in the vacuum sector) yields the recurrence relation

$$\frac{\nu-1}{\nu+1}\,C^2_{1\nu} = \frac{k-\nu-1}{k-\nu+1}\,C^2_{1\nu-1} + \frac{2\nu}{k} - 1, \qquad \nu=1,\dots,k-1. \tag{3.7}$$

The general solution of (3.4) (3.7) and (2.34) can be expressed in terms of a single parameter λ:

$$C^2_{\mu\nu} = \frac{\Gamma(\mu+\nu+\lambda)\,\Gamma(k+\lambda-\mu)\,\Gamma(k+\lambda-\nu)\,\Gamma(\lambda)}{\Gamma(k+\lambda-\mu-\nu)\,\Gamma(\lambda+\mu)\,\Gamma(\lambda+\nu)\,\Gamma(k+\lambda)}\,\bar{C}^2_{\mu\nu} \tag{3.8}$$

where $\bar{C}^2_{\mu\nu}$ is given by (2.31) and

$$c_R = 4\lambda\,\frac{(k-1)(k+\lambda-1)}{(k+2\lambda)(k+2\lambda-2)}. \tag{3.9}$$

<u>Proof.</u> F_ν is a single valued meromorphic function of η because of the fusion rules for the parafermionic currents and since $\Delta+\Delta_\nu-\Delta_{\nu-1}$ and $\Delta_{\nu+1}-\Delta-\Delta_\nu$ differ by an integer. Eqs. (2.36) and (2.1) show that the only singularities of F_ν are second (and first) order poles for $\eta\to 0$ and for $\eta\to\infty$. The coefficients to η^ν, $\nu=\pm1,\pm2$ in (3.6a) are determined by the small distance OPEs (2.1). For $\eta\to 0$ we find, using (2.1b) and

$$\left(1+\frac{z_{12}}{z_{24}}\right)^{4\frac{\nu-1}{k}} \left\{1 - 4\,\frac{(\nu-1)\,z_{12}}{k(k-\nu+1)}\left(\frac{k-\nu}{z_{23}}+\frac{1}{z_{24}}\right)\right\} \approx$$

$$\approx 1 - 4\,\frac{(\nu-1)(k-\nu)}{k(k-\nu+1)}\,\eta \qquad \text{for} \quad \frac{z_{12}}{z_{24}} \to 0\ ,$$

the relation

$$F_\nu(\eta) \approx C^2_{1\nu-1}\,\eta^{-2}\left(1-4\frac{k-\nu}{k-\nu+1}\eta\right) \quad \text{for} \quad \eta\to 0. \qquad (3.10)$$

Similarly, for $\eta\to\infty$ we deduce from (2.1a)

$$F_\nu(\eta) \approx C^2_{1\nu}\left(\eta^2-4\frac{\nu}{\nu+1}\eta\right) \qquad (\eta\to\infty)\,. \qquad (3.11)$$

To find the expression (3.6b) for a_0 and to derive the recurrence relation (3.7) we use the absence of a spin 1 contribution in the 4-point function (3.5) for $z_{23}\to 0$. We have

$$z_{14}^{2\Delta}\, z_{23}^{2\Delta_\nu} < \varphi^*(z_1)\,\varphi_\nu(z_2)\,\varphi^*_\nu(z_3)\,\varphi(z_4) > = 1 + O\left((1-\eta)^2\right) \quad \text{for}\ \eta\to 1$$

and hence

$$F_\nu(\eta) \approx 1 - 2\left(1-\frac{2\nu}{k}\right)(1-\eta) \quad \text{for} \quad \eta\to 1\,. \qquad (3.12)$$

Setting $\eta=1$ in (3.6a) and comparing with (3.12) we obtain (3.6b). Comparing the terms proportional to $1-\eta$ in both equations we find (3.7). The form (3.8) of the solution is obtained if we trade the undetermined structure constant C $(=C_{11})$ for the parameter λ writing

$$C^2 = 2\,\frac{\lambda+1}{\lambda}\,\frac{k-1}{k}\,\frac{\lambda+k-2}{\lambda+k-1} = \frac{\lambda+1}{\lambda}\,\frac{\lambda(2-\Delta)+2(\Delta-1)}{\lambda(2-\Delta)+\Delta}\,\Delta\,. \qquad (3.13)$$

Note. The form (3.8) (3.9) for the structure constants and the central charge for the p=2 parafermions was first written down (without derivation) in Appendix A to [ZF1].

3.B. Models with no spin 3 vacuum contribution.

The freedom left by the condition $n_2=2$ (for n_v defined by (2.6)) on the choice of the parameters m_v and p determining the weights (2.15) and of the structure constant C can be used to restrict the number of singular terms in the vacuum expansion (1.23). The following result singles out theories with no spin 3 contribution to this expansion. We note that for s=2 the fields $\psi(z,\pm g)$ generate in this case an infinite algebra with quadratic relations of the type considered by Zamolodchikov [Z].

Proposition 3.3. If the fusion rules for the parafermionic current $\varphi=\varphi_1$ in an unitary CQFT are restricted by the condition

$$\Delta_2 = 4\,(\Delta-1)\,(>0)\ , \tag{3.14}$$

then the structure constant A $(=v\sqrt{2\Delta/3})$ in the OPE (1.23) only vanishes in the following two cases

$$k=3,\ \ \Delta=\frac{4}{3}=\Delta_2\,,\ \ c_R=\frac{6}{7}\ \left(C^2=\frac{100}{27}\right) \tag{3.15}$$

$$k=4,\ \ \Delta=\frac{5}{4}\ (=\Delta_3)\,,\ \ \Delta_2=1\,,\ \ c_R=1\ \left(C^2=\frac{5}{2}\right). \tag{3.16}$$

Proof. For v=0 (=A) Eq. (2.32) gives the value of the (reduced) central charge c_R as a function of Δ:

$$c_R=\frac{3\Delta\,(3-2\Delta)}{(2-\Delta)\,(5-2\Delta)}=1-2\,\frac{(4\Delta-5)\,(\Delta-1)}{(2-\Delta)\,(5-2\Delta)}\,. \tag{3.17}$$

Eqs. (2.16) and (3.14) give $1<\Delta<2$. An analysis of the unitarity condition (based on [FQS]) shows that Eq. (3.15) gives the lowest allowed value of c_R and hence the upper limit for Δ:

$$c_R \geq \frac{6}{7}\,, \qquad 1 < \Delta \leq \frac{4}{3}\,. \tag{3.18}$$

(The 4-point function (2.21) (2.24) has just three non-vanishing terms and the positivity of the square of the structure constant

$$C^2 = \Delta \left\{ \frac{2\,(2-\Delta)\,(5-2\Delta)}{3\,(3-2\Delta)} + 2\,\Delta - 3 \right\} = \frac{\Delta}{3}\,\frac{7-4\Delta}{3-2\Delta}\,(2\,\Delta-1) \tag{3.19}$$

is a consequence of (3.18).)

Eq. (3.18) tells us that the integer n_3 in Eq. (2.6) (for $\nu=3$) cannot exceed 6 so that

$$\Delta_3 \geq 9\,\Delta - 12\,. \tag{3.20}$$

(If $\Delta_3=9\Delta-14$ then its positivity would contradict the inequality $\Delta\leq 4/3$.) This lower bound for Δ_3 allows to derive the following expression for the mixed 4-point function of φ and φ_2:

$$F(\varphi_2\,\varphi\,\varphi^*\,\varphi_2{}^*;\xi) = z_{14}^{8(\Delta-1)}\; z_{23}^{2\Delta} \left(\frac{z_{13}\,z_{24}}{z_{12}\,z_{34}}\right)^{4\Delta-5} < \varphi_2(z_1)\,\varphi(z_2)\,\varphi^*(z_3)\,\varphi_2{}^*(z_4) > =$$

$$= 1 + (4\Delta-5)\;\xi\;\left\{ 1 + (\Delta-1)\,\frac{11-6\Delta}{3-2\Delta}\,\xi\left(\frac{1}{3}+\xi\right) \right\} + \frac{\Delta}{3}\,\frac{7-4\Delta}{3-2\Delta}\,(2\Delta-1)\,\frac{\xi^2}{1+\xi} \tag{3.21}$$

where ξ is the cross ratio (alternative to (2.22))

$$\xi = \frac{z_{14}\,z_{23}}{z_{12}\,z_{34}} \;\left(= \frac{1-\eta}{\eta}\right). \tag{3.22}$$

(In writing down the last term in (3.21), which comes from the contribution of the stress energy tensor to the OPE of $\varphi(z_2)\varphi^*(z_3)$, we use its symmetry under the substitution $z_2 \leftrightarrow z_3$ or $\xi \leftrightarrow -\xi/(1+\xi)$, noting that $\xi^2/(1+\xi) = \xi - \xi/(1+\xi)$.) For small z_{12} (and z_{34}) we can use the fusion rule

$$\varphi_2(z_1)\,\varphi(z_2)\,z_{12}^{\Delta+\Delta_2-\Delta_3} \to C_{12}\,\varphi_3(z) \tag{3.23}$$

to relate F in that limit with the 2-point function of φ_3. The positivity of $|C_{12}|^2$ and Eqs. (3.18) (3.21) imply

$$(\Delta-1)\ (4\Delta-5) \geq 0\,. \tag{3.24}$$

It then follows from the second equation (3.17) that $c_R \leq 1$. The value $c_R = 1$ is reached for $\Delta = 1$ and $\Delta = 5/4$. In the first case we have a free $\mathbf{Z}_2$ parafermion. It generates a subalgebra of the $\mathbf{Z}_4$ parafermionic algebra for $\Delta_1 (= \Delta) = 5/4$ (since then $\Delta_2 = 4(\Delta_1 - 1) = 1$) and need not be considered separately. For $5/4 < \Delta \leq 4/3$, we have $6/7 \leq c_R < 1$ and thus we are led to study the intersection of the above class of models with the minimal unitary CFT's. It is a straightforward exercise to verify that this intersection consists of a single point, $\Delta = 4/3$, $c_R = 6/7$. Let indeed $\Delta = 1 + q/k$ where q and k are coprimes and $2q < k < 4q$. Comparing then (3.17) with the formula for the central charge in the unitary minimal series [FQS]

$$c_m = 1 - \frac{6}{(m+2)\,(m+3)}\,,\quad m = 1, 2, \ldots \tag{3.25}$$

we find

$$3\,(k-q)\ (3k-2q) = q\,(4q-k)\ (m+2)\,(m+3)\ . \tag{3.26}$$

This equation shows that both k and q must be odd. (If either k or q is even, the other one should be odd since q and k are coprimes; then the right hand side of (3.26) is divisible by 4 while the left hand side is not.) For k odd the product $(m+2)(m+3)$ should be divisible by k, since the allowed conformal weights in the unitary discrete series,

$$\Delta_{r,t} = \frac{[(m+3)r - (m+2)t]^2 - 1}{4(m+2)(m+3)} \tag{3.27}$$

$(1 \le r \le m+1,\ 1 \le t \le m+2)$ should contain among them the weight $1 + q/k$. That is, however, only consistent with (3.26) for $k=3$. This completes the proof of Proposition 3.3.

3.C. Unitary parafermionic theories with p=-1.

The two models (3.15) and (3.16) happen to be the only solution of the form (2.15) for $p=-1$ and $m_1 = \ldots = m_{k-1} = 1$:

$$\Delta_\nu = 2 - \nu + \frac{\nu^2}{k} \qquad \nu = 1, \ldots, k-1. \tag{3.28}$$

We shall demonstrate that there are no other parafermionic models for $p=-1$, $n_2=2$ (consistent with positivity).

Lemma 3.4. If $\Delta_\nu = 2m_\nu - \nu + \nu^2/k\ (\ge 0)$ then Eqs. (2.16) and (3.14) imply $m_\nu = 1$ for $1 \le \nu \le k-1$.

Proof. Eq. (3.14) gives $4m_1 = m_2 + 3$, while according to (2.16) and Remark 2.2 $m_2 < 2\,m_1$. The only positive integer solution of these equations is $m_1 = m_2 = 1$. On the other hand Eq. (2.19) implies $m_{\nu+1} < m_\nu + m_1$ or $m_{\nu+1} \le m_\nu$; since $m_{k-1} = m_1$ it follows that $m_1 = m_2 = \ldots = m_{k-1} = 1$.

Lemma 3.5. The conformal weights (3.28) are consistent with positivity and uniqueness of the vacuum if $k \leq 6$.

Proof. For $k \geq 9$, $\Delta_{[k/2]}$ is negative. For $k = 8$, $\Delta_4=0$ but φ_4 carries a non-zero $\mathbf{Z}_8$-charge and hence $\varphi_4(0) \mid 0 >$ is another zero energy state (orthogonal to the vacuum). For $k = 7$ we have $\Delta_1 = 8/7 = \Delta_6$, $\Delta_3 = 2/7 = \Delta_4$ hence $\Delta_6 > 2\Delta_3$ in contradiction to the triangle inequality $\Delta_{\mu+\nu} < \Delta_\mu + \Delta_\nu$ (derived if we regard φ_3 as a basic parafermionic current).

It thus remains to consider five cases: $k = 2, 3, 4, 5, 6$. For $k= 2$, $\Delta_1=3/2$, $\Delta_2=0$ so that Eq. (3.14) is violated (otherwise this is a perfectly admissible parafermionic and superconformal theory also obtained for $m_1=0$, $p=3$). The cases $k = 3, 4$ are incorporated in Proposition 3.3. The $k = 5$ case is a relabelling of the standard $\mathbf{Z}_5$ parafermionic theory. Indeed, the weights $\Delta_2 = \Delta_3 = 4(\Delta - 1) = 4/5$ correspond to $\bar{\Delta}_1 = \bar{\Delta}_4$ of the standard $\mathbf{Z}_5$ parafermions (2.30), and $\bar{\Delta}_2 (= \bar{\Delta}_3) = 4\bar{\Delta}_1 - 2 = 6/5 = \Delta_1$ $(= \Delta_4)$. Finally, the case $k = 6$ provides a non-trivial example of a reducible parafermionic theory. Indeed, Eq. (3.28) gives in that case

$$\Delta_2 = \frac{2}{3} \, (=\Delta_4), \ \Delta_3 = \frac{1}{2}, \ \Delta_1 \, (=\Delta_5) = \Delta_2 + \Delta_3 = \frac{7}{6}. \tag{3.29}$$

The theory is the tensor product of the $c=1/2$ Ising model and the three states $c=4/5$ Potts model (both standard, $\mathbf{Z}_2$ - and $\mathbf{Z}_3$ - parafermionic theories).

4. c=1 Z_4- parafermions as generalized Klein transformed U(1)-vertex operators.

The analysis of the preceding section of the parafermi theories satisfying (3.14) leaves us with a single example, the $\mathbf{Z}_4$ - theory with $\Delta = 5/4$ that requires $m_\nu \neq 0$. It is instructive to study this example in more detail, looking at it as a member of the $\mathbf{Z}_4$ $c_R=1$ family labeled by the integer $n_2 = n = (0), 1, 2, 3, \ldots$. It is obtained by setting

$$\Delta = \Delta_1(n) = \frac{2n+1}{4} = \Delta_3(n)\,,\ \Delta_2(n) = 1\ \ (=c_R) \tag{4.1}$$

and corresponds to the solution (2.15) with

$$p = 3 - 2n\,,\ m_\nu = n - 1\,,\ \nu = 1, 2, 3. \tag{4.2}$$

Regarding φ_2 as an u(1)-current one is tempted to interpret $\varphi = \varphi_1$ $(=\varphi_3{}^*)$ as a vertex operator V_1 presented as exponential of an integral of φ_2 . Such a construction would violate, however, the $\mathbf{Z}_4$ - parafermionic fusion rule. We would have instead

$$V_1(z_1)\ V_1(z_2) \approx z_{12}^{\frac{2n+1}{2}}\ V_2(z) \qquad (z = \sqrt{z_1 z_2}) \tag{4.3a}$$

$$V_1(z_1)\ V_2(z_2) \approx z_{12}^{2n+1}\ V_3(z_2) \tag{4.3b}$$

where

$$V_\nu (z) \equiv V \left(z, \nu \sqrt{\frac{2n+1}{2}}\right), \quad \nu = 1, 2, 3, \tag{4.4}$$

(V_3 carries a triple U(1)-charge, hence it is not the conjugate of V_1).
The parafermionic fields can however be constructed in terms of the vertex operator V_1 and its conjugate and an unitary involutive charge conjugation operator C defined by

$$C\ V_1 (z)\, C = V_1{}^* (z), \qquad C^2=1, \qquad (C-1)\ |0> = 0, \tag{4.5}$$

$$\left(V_1{}^* (z) = V \left(z, -\sqrt{\frac{2n+1}{2}}\right)\right).$$

Let $\hat{U}$ be the chiral algebra generated by an u(1) - current $j(z)$ with 2-point function normalized to 1:

$$z_{12}^2 \ <j (z_1)\, j (z_2)> \ = \ 1 \, . \tag{4.6}$$

(For $n=1, 2, \ldots,$ the hermitian current j will be identified with φ_2 .) The action of C on the current being

$$C\, j (z)\, C = - j (z) \tag{4.7}$$

C defines a Z_2 grading in $\hat{U}$:

$$\hat{U} = \hat{U}_+ \oplus \hat{U}_- \qquad C\ \hat{U}_\pm C = \pm\ \hat{U}_\pm \, . \tag{4.8}$$

$\hat{U}_+$ is the subalgebra of charge conjugation invariant elements of $\hat{U}$. It involves the Sugawara stress-tensor $T^{(j)} (z) = 1/2 : j^2(z):$ and an infinite number of (Virasoro) primary fields (composites of the current) of weights $4m^2$, $m=0,1,2, \ldots$ [FST].

Given the vertex operator V_1 (4.4) and its conjugate, we shall define the basic Z_4 - parafermionic currents by

$$\varphi(z) = \frac{1}{\sqrt{2}} \left(V_1(z) + i\, V_1{}^*(z) \right) C \qquad (4.9a)$$

$$\varphi^*(z) = \frac{1}{\sqrt{2}} \left(V_1(z) + i\, V_1{}^*(z) \right) C. \qquad (4.9b)$$

They have the following small distance OPEs:

$$z_{12}^{2\Delta}\ \varphi^*(z_1)\,\varphi(z_2) = 1 + \frac{i}{2}\ z_{12}^{4\Delta}\left(V_2^*(z) - V_2(z)\right) + \ldots \qquad (4.10)$$

$$z_{12}^{2\Delta-1}\ \varphi(z_1)\,\varphi(z_2) \approx \sqrt{n+\frac{1}{2}}\ j(z) + \frac{i}{2}\, z_{12}^{2\Delta-1}\left(V_2(z) + V_2^*(z)\right) \ (z = \sqrt{z_1 z_2}). \qquad (4.11)$$

The nature of these fusion rules changes for $4\Delta=1$ $(n=0)$. In that case $(i/\sqrt{2})\ (V_2{}^*\text{-}V_2)$ is a (normalized, hermitian) u(1)-current and thus Eq. (4.10) violates our irreducibility assumption (vi) of Sec. 1. In fact, for n=0, we have nothing but the OPEs of the level one primary fields for the $\hat{su}(2)$ current algebra. One can view the tensor product of two $\hat{su}(2)_1$ representations, corresponding to total central charge c=2 as a chiral algebra generated by a pair of complex coniugate spin 1/2 fields $\psi^{(*)}$:

$$\psi(z) = V_1(z) \otimes \varphi(z) \qquad (4.12)$$

and an u(1)-current that commutes with $\psi^{(*)}$. This yields an alternative (to (4.12)) tensor product realization for the resulting CQFT, in accord with the Goddard - Schwimmer theorem [GS1] quoted in Remark 1.1.

5. QFT representations of $\mathcal{A}$.

We shall study the local QFT representations of the bosonic chiral algebras $\mathcal{A} = \mathcal{A}\,(s, g^2; k, c)$ (with positive integer s), corresponding to the value $n_2=2$ in (2.6), -i,e, satisfying the inclusion

$$\mathcal{A}\,(4\,(s-1), 4g^2; k, c) \subset \mathcal{A}\,(s, g^2; k, c) \qquad s = 2, 3, \ldots \tag{5.1}$$

(which fixes the fusion rule for two equal charge fields $\psi\,(z, \; g, s)$). This class includes, for $s = 4\,n = 4\,s_1 - 2$, $g^2 = 4\,g_1^2$, $g_1^2 = 2\,(s_1-1)\,k + 2$, $c = 3k/k+2$, $n = 1, 2, \ldots$ the representations of the graded chiral algebra (generated by fermionic charged fields)

$$\mathcal{A}\,\left(s_1 = n + \frac{1}{2},\; g_1^2 = 2\,(s_1-1)\,k + 2;\, k, \frac{3k}{k+2}\right) \left(\supset \mathcal{A}\,(4s_1-2, 4g_1^2; k, \frac{3k}{k+2})\right)$$

$$s_1 = \frac{3}{2}, \frac{5}{2}, \ldots, \tag{5.2}$$

described in [PT2]. (The inclusion of algebras of half integer spin s satisfying (5.1) - rather than (5.2) - would require some additional work to display the Ramond sector.)

5.A. Representations of A in terms of tensor product representations of $\mathcal{A}_J \otimes \mathcal{A}_{PF}$.

We have constructed the algebra $\mathcal{A}$ as a subalgebra of single valued (integer spin) "currents" of the tensor product of two algebras each of which gives room for multi-valued fields:

$$\mathcal{A} \subset \tilde{\mathcal{A}}_J \left(\frac{g^2}{k}\right) \otimes \tilde{\mathcal{A}}_{PF} \left(\{\Delta_v = \Delta_{k-v}\},\ c_R\right). \tag{5.3}$$

On the other hand $\mathcal{A}$ contains the tensor product of the subalgebras of $\tilde{\mathcal{A}}_J$ and $\tilde{\mathcal{A}}_{PF}$ of single-valued (and Z_k - invariant) fields:

$$\mathcal{A} \supset \mathcal{A}_J \otimes \mathcal{A}_{PF}, \tag{5.4a}$$

$$\mathcal{A}_J = \{ V \in \tilde{\mathcal{A}}_J,\ e(L_o)\ V\, e(-L_o) = V \},\quad e(L) \equiv e^{2\pi i L}$$

$$\mathcal{A}_{PF} = \{ \phi \in \tilde{\mathcal{A}}_{PF};\ e(L_o)\ \phi\, e(-L_o) = \phi,\ \ \gamma\ \phi = \phi \} \tag{5.4b}$$

(γ being the Z_k- automorphism (2.13)). Every irreducible single valued positive energy representation of $\mathcal{A}$ splits into a direct sum of tensor products of such representations of $\mathcal{A}_J$ and $\mathcal{A}_{PF}$, which we proceed to describe.

It follows from (1.15) that g^2 is an even integer:

$$\frac{1}{2} g^2 = k\,(s - p - 2\,m_1) + p = N\,(k, s)\,. \tag{5.5}$$

If we specialize to the cases studied in Sec. 3 (see in particular, Eqs. (3.3) and (3.28)) we would have $k + p < k\,(p+2m_1) < 2k + p$ so that

$$k\,(s-2) < N < k\,(s-1)\,,\quad s = 2, 3, \ldots, \tag{5.6}$$

or, more specifically,

$$N\,(=N_a) = k\,(s-2) + 2 \qquad \text{for } k\,\Delta = 2\,(k-1) \tag{5.7a}$$

$$N\,(=N_b) = k\,(s-1) - 1 \qquad \text{for } k\,\Delta = k + 1 \tag{5.7b}$$

(the two expressions coincide for $k = 3$ only). If k is square free, then the algebra $\mathcal{A}_J$ is generated by vertex operators of charge $\pm k g$. A minimal energy state $| e \rangle = V(0, e/\sqrt{k}) \, | 0 \rangle$ of charge e is characterized by

$$(J_o - e) | e \rangle = 0 = (L_o - \frac{e^2}{2k}) | e \rangle = J_n | e \rangle = L_n | e \rangle , \quad n = 1, 2, \dots \; . \qquad (5.8)$$

If V is a vertex operator carrying charge $k g$ then we would have (using the notation of (1.21))

$$V(z, \sqrt{k}\, g) | e \rangle = z^{ge} \{ 1 + O(z) \} | e + k g \rangle . \qquad (5.9)$$

$| e \rangle$ is a lowest weight vector for a single valued (irreducible, positive energy) representation of $\mathcal{A}$ if

$$g e \in \mathbf{Z} , \qquad 1 - k N \le g e \le k N \; . \qquad (5.10)$$

If $k = \ell^2 q$ then $\mathcal{A}_J$ is generated by vertex operators of charge $\pm \ell q g$ and condition (5.10) subsituted by

$$g e \in \ell \, \mathbf{Z} , \qquad 1 - q N \le \frac{g e}{\ell} \le q N \quad (\text{for } k = q \ell^2) . \qquad (5.11)$$

For the sake of simplicity we shall treat in what follows the case of a square free k (dealing thus with (5.10)).

The representations of $\mathcal{A}_{PF}$ will be labeled by the $\mathbf{Z}_k$ - charge ν. We say that the subspace $\mathcal{H}^{(\nu)}$ carries $\mathbf{Z}_k$ - charge ν if the monodromy of the parafermionic current φ_μ of weight (2.15) on $\mathcal{H}^{(\nu)}$ is given by

$$\varphi_\mu (e^{2\pi i} z) \, \mathcal{H}^{(\nu)} = e\left(- 2 p \frac{\mu\nu}{k}\right) \mathcal{H}^{(\nu)} \quad (e(\alpha) \equiv e^{2\pi i \alpha}) . \qquad (5.12)$$

(Clearly, a Z_k - invariant field - which can be viewed as a φ_μ for $\mu = 0 \pmod k$ - is single valued when applied to $\mathcal{H}^{(\nu)}$.)

In order to construct local QFT representations of $\mathcal{A}$ out of representations of $\mathcal{A}_J \otimes \mathcal{A}_{PF}$ we consider tensor product spaces $\mathcal{H}_e \otimes \mathcal{H}^{(\nu)}$, where $\mathcal{H}_e$ is the Hilbert space closure of $\mathcal{A}_J \,|e>$ ($\mathcal{H}_e = \overline{\mathcal{A}_J \,|\, e>}$) characterized by

$$V\left(e^{2\pi i}z, \frac{g}{\sqrt{k}}\right) \mathcal{H}_e = e\left(\frac{eg}{k}\right) \mathcal{H}_e \,. \tag{5.13}$$

For each ν we select $e = e_\nu$ in such a way that the integer spin field $\psi(z, g, s)$ (1.21) is single valued on $\mathcal{H}_e \otimes \mathcal{H}^{(\nu)}$. Eqs. (5.12) (5.13) then give the condition

$$\frac{1}{k}\,(e\,g - 2\,p\,\nu) \in Z\,. \tag{5.14}$$

If we write the representation space $\mathcal{H}$ of $\mathcal{A}$ in the form

$$\mathcal{H} = \bigoplus_\nu \mathcal{H}_{e_\nu} \otimes \mathcal{H}^{(\nu)} \tag{5.15}$$

then Eq. (1.21) gives

$$\psi(z, \pm g, s)\, \mathcal{H} \subset \bigoplus_\nu \mathcal{H}_{e_\nu \pm g} \otimes \mathcal{H}^{(\nu\pm 1)}. \tag{5.16}$$

Hence $\mathcal{A}\,\mathcal{H} \subset \mathcal{H}$ provided that

$$e_\nu = e_o + \nu\, g\,. \tag{5.17}$$

Noting that the definition of $\mathcal{H}_e$ (5.13) implies

$$\mathcal{H}_e = \mathcal{H}_{e+k\,g} \tag{5.13'}$$

we deduce that the sum in (5.15) should run on some interval of integers of length k :

$$\nu \in Z_k \equiv Z \; / \; k\, \mathbf{Z} \, . \tag{5.15'}$$

Furthermore, Eq. (5.14) is equivalent to

$$\alpha \equiv \frac{e_0\, g}{k} \in Z . \tag{5.18}$$

(Clearly, (5.18) is a special case of (5.14) obtained for $\nu=0$. Conversely, (5.5) (5.17) and (5.18) give $g\, e_\nu - 2\, p\, \nu = g\, e_0 + \nu\, (g^2\text{-}2p) \in k\, \mathbf{Z}$, thus implying (5.14).) The integer α gives an invariant characteristic of the representation and we shall therefore attach the index α to the representation space $\mathcal{H} = \mathcal{H}_\alpha$.

5.B. The $\mathbf{Z}_{2N}$ gauge symmetry.

The unitary operator $e\,(J_0/g)$ is an idempotent:

$$e\,(\frac{J_0}{g})\; \mathcal{H}_\alpha = e\,(\frac{k\alpha}{2N})\; \mathcal{H}_\alpha \, , \tag{5.19a}$$

$$[\, e\,(\frac{J_0}{g})\,]^{2N} = e\,(g\, J_0) = 1 \quad \text{in } \mathcal{H}_\alpha \tag{5.19b}$$

(since the eigenvalues e of J_0 in $\mathcal{H}_\alpha$ satisfy (5.10)). Consider the field algebra $\mathcal{F}$ compounded by $\mathcal{A}$ and by the vertex operators

$$\psi\left(z, \pm\frac{k}{g}, \frac{k}{4N}\right) = \sqrt{k}\, V\left(z, \pm\frac{\sqrt{k}}{g}\right) \tag{5.20}$$

which intertwine between the vacuum sector (i.e. the space $\mathcal{H}_o = \overline{\mathcal{A} \mid 0 >}$, where the bar denotes Hilbert space closure) and $\mathcal{H}_{\pm 1}$. Let $|v>$ be a lowest weight state (with respect to $\mathcal{A}_{PF}$) in $\mathcal{H}^{(v)}$. The fusion rule

$$\psi(z, g, s) \mid e_v > \otimes \mid v > \approx \sqrt{k}\; z^{\alpha_v} \mid e_v + g > \otimes \mid v + 1 > \tag{5.21}$$

where

$$\alpha_v = \frac{1}{k}(g\, e_v - 2\, p\, v) = \alpha + \frac{2v}{k}(N - p)\,, \tag{5.22}$$

tells us that $\mathcal{F}$ also contains the multivalued fields $\psi(z, g \pm k/g,\; s \pm 1 + k/4N)$ which carry a non-zero Z_k - charge. The product of $(g^2 =)\; 2N$ fields of the type (5.20) (and of any element of $\mathcal{A}$) is an element of $\mathcal{A}$. The operator $e\,(J_0/g)$ gives rise to an automorphism σ of the field algebra $\mathcal{F}$ that defines a Z_{2N} gauge symmetry of the first kind

$$\sigma\,\phi = e\left(\frac{J_0}{g}\right)\, \phi\; e\left(-\frac{J_0}{g}\right),\quad \sigma^{2N} = \text{identity}. \tag{5.23}$$

Remark 5.1 Eq. (5.22) indicates that for $p>0$ the value $N = p$ is distinguished by the property $\alpha_v = \alpha$. For $m_1 = 0$ this distinguished chiral algebra is

$$\mathcal{A}_p = \mathcal{A}\;(s = p,\, g^2 = 2p\,;\, k, c) \quad \left(\Delta = p\,\frac{k-1}{k}\right). \tag{5.24}$$

The term "gauge symmetry" is justified since σ leaves every element of the algebra of observables $\mathcal{A}$ invariant. If k is odd (for a generic p, if k and $2p$ are coprimes), then the QFT representations of $\mathcal{A}$ split according to the eigenvalue of σ into 2N sectors, which can be labeled by α. The field algebra $\mathcal{F}$ regarded as a vector space splits into a direct sum of 2N components $\mathcal{F}_\alpha$ obeying the additive fusion rule

$$\mathcal{F}_\alpha \times \mathcal{F}_\beta \subset \mathcal{F}_{\alpha+\beta}\,, \quad \mathcal{F} = \bigoplus_{\alpha\in Z_{2N}} \mathcal{F}_\alpha\,. \tag{5.25}$$

5.C. Characters and string functions. Modular properties.

We shall now exhibit some general features of the QFT representations of $\mathcal{A}$, leaving for Sec. 6 the study of a specific example: the 1-parameter family of theories given by Proposition 3.1 for $k = 3$, $4\lambda \in N$.

Let A be some (unspecified) label for the irreducible positive energy representation of the $\mathbf{Z}_k$ - current algebra $\mathcal{F}_{PF}$. (For the standard parafermions (2.30) A is to be identified with the quantum number ℓ (= 0, 1, ..., k) - twice the isospin in the su(2) / u(1) coset construction. The monodromy index μ - cf. (5.12) - is then conventionally ([ZF1] [GQ]) replaced by $m = 2\mu$ which has the same parity as ℓ. The minimal conformal weight in the subspace $\mathcal{H}_\ell{}^m$ is

$$\Delta_\ell^m = \frac{\ell\,(\ell+2)}{4\,(k+2)} - \frac{m^2}{4\,k} + \left(\frac{m-\ell}{2}\right)_+ \qquad \text{for } \frac{m+\ell}{2} = 0, 1, 2, \ldots, k \tag{5.26a}$$

where $(\lambda)_+ = 1/2\,(\lambda + |\lambda|)$. If k is odd one can use only integer labels $\mu = m/2$ by using the symmetry property

$$\Delta_\ell^m = \Delta_{k-\ell}^{k+m} \quad \text{for } m \le \ell\,, \quad \Delta_\ell^m = \Delta_{k-\ell}^{m-k} \quad \text{for } m \ge \ell \tag{5.26b}$$

- see Eqs. (3.4) and (3.5) of [PT2].) The representation space $\mathcal{H}_A$ splits into a direct sum of eigenspaces $\mathcal{H}_A{}^v$ on which the monodromy of φ_μ is given by (5.12). We define, following [GQ], the generalized string function $c_v{}^A$ by

$$\eta(\tau)\, c_v^A(\tau) = \mathrm{tr}_{\mathcal{H}_A^v}\, e\left((L_0^R - \frac{c_R}{24})\,\tau\right), \qquad (5.27)$$

where η is the Dedekind η - function. The irreducible positive energy representations of the total chiral algebra $\mathcal{A}$ are then labeled by the pair (A,α) and act on the space

$$\mathcal{H}_{A\alpha} = \bigoplus_{v \in Z_k} \mathcal{H}_{e_v} \otimes \mathcal{H}_A^v\,, \qquad \frac{1}{k}(g\, e_v - 2\, pv) = \alpha \pmod{2N}\,. \qquad (5.28)$$

Their characters are given by

$$\chi_{A\alpha}(\tau,\zeta) = \mathrm{tr}_{\mathcal{H}_{A\alpha}}\, e\left((L_0 - \frac{c}{24})\,\tau + \zeta\, g\, J_0\right) =$$

$$= \sum_{v=0}^{k-1} c_v^A(\tau)\; \Theta_{k\alpha + 2vN,\, kN}(\tau,\zeta) \qquad (c = 1 + c_R)\,, \qquad (5.29)$$

where $\Theta_{m,M}$ is the classical theta function (*)

$$\Theta_{m,M}(\tau,\zeta) = \sum_{n \in Z} e\left(M\,(n + \frac{m}{2M})^2\,\tau\right)\, e\left(\zeta\,(n M + \frac{m}{2})\right). \qquad (5.30)$$

(*) Our $\Theta_{m,M}(\tau,\zeta)$ coincides with $\Theta_{m,M}(\tau,-\zeta,0)$ of [GQ] (apart from a typographical error in their Eq. (3.38); $\Theta_{m,M}(\tau,0)$ of (5.32) is denoted by $\Theta_{M,m}(\tau)$ in [PT 2, 3].

We assume (as usual, in a rational CQFT) that these characters span a finite dimensional unitary representation of the modular group S L (2, **Z**). (For a coset space model one can, in fact, derive this basic property.) One can then read from (5.29) the transformation law for the string functions. The behaviour of $c^A{}_v$ under the translation $\tau \rightarrow \tau+1$ follows directly from (5.27); we have:

$$c_v^A(\tau+1) = e\left(\Delta_A^v - \frac{c}{24}\right) c_v^A(\tau) , \qquad (5.31)$$

where $\Delta_A{}^v$ is the lowest eigenalue of $L_o{}^R$ in $\mathcal{H}_A{}^v$. Using also the obvious property of the theta series (5.30)

$$\Theta_{m,M}(\tau+1, \zeta) = e\left(\frac{m^2}{4M}\right) \Theta_{m,M}(\tau, \zeta) \qquad (5.32)$$

we find

$$\chi_{A\alpha}(\tau+1, \zeta) = \sum_{v=0}^{k-1} e\left(\Delta_A^v - \frac{c}{24} + \frac{(k\,\alpha + 2\,vN)^2}{4\,k\,N}\right) c_v^A(\tau)\, \Theta_{k\alpha+2vN,kN}(\tau, \zeta). \qquad (5.33)$$

Demanding that the right hand side of (5.33) is a multiple of $\chi_{A\alpha}(\tau, \zeta)$ we find a restriction on the conformal weights:

$$\Delta_A^v + \frac{v^2}{k} N = \Delta_A^0 \mod \mathbf{Z}. \qquad (5.34)$$

We observe that this relation is consistent with the monodromy property (5.12). Indeed, we have

$$\varphi(e^{2\pi i} z)\, \mathcal{H}_A^v = e\,(\Delta_A^{v+1} - \Delta_A^v)\, \mathcal{H}_A^v = e\left(-2\frac{pv}{k}\right) \mathcal{H}_A^v \qquad (5.35)$$

since $N = p \pmod k$ according to (5.5).

<u>Remark 5.2</u> Noting that $N = p \mod k$ we can rewrite (5.34) in the form

$$\Delta^{v}_{A} = \Delta^{o}_{A} - p\,\frac{v^2}{k} \pmod{Z}. \qquad (5.34')$$

We note that for $p = 1$ this relation is satisfied (for odd k) by the weights (5.26) due to the equivalence between the representations $[\ell]$ and $[k-\ell]$ of the algebra $\mathcal{A}_{PF}$ of standard parafermionic currents.

Under the second generator of the modular group, the inversion $\tau \to -1/\tau$, the Θ – function transforms as

$$\Theta_{m,M}(\tau,\zeta) \to e\left(-\frac{\zeta^2}{4\tau}\right)\,\Theta\left(-\frac{1}{\tau},\frac{\zeta}{\tau}\right) =$$

$$= (-2Mi\tau)^{-\frac{1}{2}} \sum_{m'=1-M}^{M} e\left(\frac{m\,m'}{4M}\right)\,\Theta_{m'M}(\tau,\zeta). \qquad (5.36)$$

We assume that the characters $\chi_{\alpha A}$ transform according to a finite dimensional unitary representation under this inversion:

$$e\left(\frac{\zeta^2}{4\tau}\right)\,\chi_{A\alpha}\left(-\frac{1}{\tau},\frac{\zeta}{\tau}\right) = \sum_{B\cdot}\ \sum_{\beta\in Z_{2N}} S^{B\beta}_{A\alpha}\ \chi_{B\beta}(\tau,\zeta). \qquad (5.37)$$

Inserting the expansion (5.29) in the left hand side and using (5.36) we find

$$e\left(\frac{\zeta^2}{4\tau}\right)\,\chi_{A\alpha}\left(-\frac{1}{\tau},\frac{\zeta}{\tau}\right) =$$

$$= (-2ikN\tau)^{-\frac{1}{2}} \sum_{M,v=0}^{k-1} c^{A}_{\mu}\left(-\frac{1}{\tau}\right) \sum_{\beta=1-N}^{N} e\left(\frac{(\alpha k + 2\mu N)(\beta k + 2vN)}{4kN}\right)\Theta_{\beta k+2vN,\,kN}(\tau,\zeta)\ .$$

Expanding also the right hand side of (5.37) in terms of Θ, recalling the assumption that $2N$ and k (or $2p$ and k) are coprimes so that

$$\delta^{\beta k+2\nu N}_{\sigma k+2\rho N} = \delta^{\beta}_{\gamma}\,\delta^{\nu}_{\rho} \qquad (\beta, \gamma \in Z_{2N}\,, \quad \nu, \rho \in Z_k)$$

and using the independence of $\Theta_{mM}(\tau, \zeta)$, we derive

$$c^{A}_{\mu}\left(-\frac{1}{\tau}\right) = \sqrt{\frac{-2iN\tau}{k}}\,\sum_{B,\nu} e\left(-\frac{N\mu\nu}{k}\right)\, e\left(-\frac{k\,\alpha\,\beta}{4N}\right) S^{B\beta}_{A\alpha}\, c^{B}_{\nu}(\tau)\,.$$

Since the left hand side does not depend on α and β we deduce that

$$S^{B\beta}_{A\alpha} = S_{AB}\; e\left(\frac{k\,\alpha\,\beta}{4N}\right) \qquad (5.38)$$

which leads to a factorized form of the law (5.37) and to the relation

$$\eta\left(-\frac{1}{\tau}\right) c^{A}_{\mu}\left(-\frac{1}{\tau}\right) = \sqrt{\frac{2N}{k}}\;\eta(\tau)\sum_{B,\,\nu} S_{AB}\; e\left(-\mu\nu\frac{N}{k}\right) c^{B}_{\nu}(\tau) \qquad (5.39)$$

for the characters (5.27) of the parafermionic theory. This is an important result, since the S - transformation of conformal characters carries information about the fusion rules and the quantum symmetry of the underlying CQFT ([V], [MS], [DV], [BYZ] and [AGS]).

6. The k = 3 series.

We shall now consider the QFT representations of the first term in the family of models with $\mathbf{Z}_k$ - currents of conformal weights (3.3), the $\mathbf{Z}_3$ parafermionic algebra generated by a pair of conjugate fields of spin 4/3 introduced and first studied by Zamolodchikov and Fateev [ZF3]. There exists a discrete series of unitary representations of this algebra ([ZF3], [R], [GS2]) corresponding to λ in (3.8) (3.9) equal to $n/4$ ($n \in Z$); it corresponds, in particular, to central charge of the reduced Virasoro algebra given by

$$c_R = c_3(n) = 2\left(1 - \frac{12}{(n+2)(n+6)}\right), \quad n = (0), 1, 2, \dots . \tag{6.1}$$

This series is realized by a coset space construction

$$\frac{su(3)_1 \times so(3)_n}{so(3)_{n+4}} \tag{6.2}$$

put forward by Goddard and Schwimmer [GS2] and studied in more detail in Appendices B and C.

We shall now express the string functions (5.27) in terms of the Kac-Peterson branching coefficients $b_{pq}{}^{r}(\tau)$ [KP], [KW] for the coset

$$\frac{so(3)_4 \times so(3)_n}{so(3)_{n+4}} . \tag{6.3}$$

(The lower index indicates, as usual, the level of the corresponding affine Kac-Moody algebra - cf. [GO]. $\hat{so}(3)_4$ appears as a conformal subalgebra of $\hat{su}(3)_1$ in the sense that they yield the same Virasoro central charge - see the discussion in Appendix B, in particular, Eq. (B. 12).)

The label A of the representations of the parafermionic algebra, introduced in Sec. 5.C. will be identified with the pair p, q while the parafermionic index ν will be related to r, where each letter stands for twice the isospin of the corresponding so(3) factor. We have the following branching rule for the product of two $\hat{so}(3)$ characters

$$\chi_{n,p}(\tau)\ \chi_{4\,r}(\tau) = \sum_{\substack{q=0 \\ q-p\ \mathrm{even}}}^{n+4} b^{r}_{pq}(\tau)\ \chi_{n+4,q}(\tau) \tag{6.4}$$

$$p = 0, 1, \ldots, n\,,\quad q = 0, 1, \ldots, n+4 \qquad (r = 0, 2, 4)$$

(the first index of each χ indicating the level).

The relation between the $\mathbf{Z}_3$ charge ν and the isospin r/2 of $\hat{so}(3)_4$ is indicated by the fact that the conformal weights of the (integer isospin) positive energy representations of $\hat{so}(3)_4$ are given by

$$\Delta_{4,r} = \frac{r(r+2)}{24} \tag{6.5a}$$

and hence

$$\Delta_{4,2} - \Delta_{4,0} = \frac{1}{3}\,,\quad \Delta_{4,4} - \Delta_{4,2} = \frac{2}{3}\,. \tag{6.5b}$$

Recalling the monodromy relations (5.12) which give

$$\varphi\,(e^{2\pi i} z)\ \mathcal{H}^{(\nu)}_{pq} = e\left(-\frac{\nu}{3}\right)\varphi(z)\ \mathcal{H}^{(\nu)}_{pq}\,, \tag{6.6}$$

we deduce that when acting on $\mathcal{H}^{(\nu)}$ φ raises the conformal weight by $\Delta_\varphi^{(\nu)}$, where (for $\nu = -1 \sim \nu = 2 \pmod 3$)

$$\Delta^{(0)}_\varphi = \frac{1}{3}\ (\mathrm{mod}\ \mathbf{Z}),\quad \Delta^{(1)}_\varphi = 0\ (\mathrm{mod}\ \mathbf{Z}),\quad \Delta^{(-1)}_\varphi = \frac{2}{3}\ (\mathrm{mod}\ \mathbf{Z}). \tag{6.7}$$

This allows us to relate the representations with $\nu=0$ with $r=0$ and 4 and $\nu=\pm 1$ with $r=2$. In fact, one can express the string functions $c_\nu{}^{pq}$ (5.27) in terms of the branching coefficients as follows:

$$c_0^{pq} = b_{pq}^0 + b_{pq}^4 , \qquad c_\pm^{pq} = b_{pq}^2 . \tag{6.8}$$

Using the coset space expression (B.15) we can reproduce the result of Ravanini [R] for the minimal conformal weights Δ_{pq} for $q=p+2m$ ($m=0, \pm 1, \ldots$):

$$\Delta_{p\, p+2m} = \frac{[\, p\,(n+6) - (p+2m)\,(n+2)\,]^2 - 16}{16\,(n+2)\,(n+6)} + \frac{\kappa}{12} \tag{6.9a}$$

where

$$\kappa = 0 \quad \text{for} \quad m = 0 \pmod 2, \qquad \kappa = 1 \quad \text{for} \quad m = 1 \pmod 2. \tag{6.9b}$$

Proceeding to the complete theory (that includes the $U(1)_J$ vertex operator) and denoting $\chi_{A\alpha}(\tau, 0)$ (5.29) by $\chi_\alpha{}^{pq}(\tau)$ we find (using (5.33))

$$\chi_\alpha^{pq}(\tau+1) = e\left(\Delta_{pq} + \frac{2}{3}\kappa + \frac{3\,\alpha^2}{4\,(3\,s-4)} - \frac{c}{24}\right) \chi_\alpha^{pq} \tag{6.10}$$

(where κ is given by (6.9b) for $q = p + 2\,m$).

Eq. (6.4) allows to derive

$$\chi_\alpha^{pq}\left(-\frac{1}{\tau}\right) = \left[\frac{1}{(n+2)\,(n+6)\,s\,(3s-4)}\right]^{\frac{1}{2}} \sum_{\substack{0 \le p' \le n \\ 0 \le q' \le n+4 \\ p'-q' \text{ even}}} \sum_{\beta=5-3s}^{3s-4} e^{3i\pi \frac{\alpha\beta}{3s-4}} \cdot$$

$$\cdot \sin\left(\frac{\pi\,(p+1)\,(p'+1)}{n+2}\right) \cdot \sin\left(\frac{\pi\,(q+1)\,(q'+1)}{n+6}\right) \chi_\beta^{p'q'}(\tau) \tag{6.11}$$

which agrees with the general structure (5.39) of the modular transformation $S : \tau \to -1/\tau$.

The modular invariant partition functions:

$$Z = \sum_{\substack{\alpha, \bar{\alpha} \\ p, q \\ \bar{p}, \bar{q}}} N_{pq\alpha, \bar{p}\bar{q}\bar{\alpha}} \; \chi_{\alpha}^{pq}(\tau) \; \bar{\chi}_{\bar{\alpha}}^{\bar{p}\bar{q}}(-\bar{\tau}) \tag{6.12}$$

with integer factorizable $N_{pq\alpha, \bar{p}\bar{q}\bar{\alpha}}$:

$$N_{pq\alpha, \bar{p}\bar{q}\bar{\alpha}} = M_{p\bar{p}}(n) \;\; M_{q\bar{q}}(n+4) \;\; m_{\alpha\bar{\alpha}} \tag{6.13}$$

can be classified using the classification of the $\hat{so}(3)$ partition functions [CIZ], [G] (of levels n and $n+4$):

$$Z_{\hat{so}(3)_n} = \sum_{p, \bar{p} = 0}^{n} M_{p\bar{p}}(n) \;\; \chi_{n,p}(\tau) \;\; \bar{\chi}_{n\bar{p}}(-\bar{\tau}) \tag{6.14}$$

and the classification of the modular invariants made out of Θ - functions [GQ]:

$$Z_{3s-4} = \frac{1}{|\eta|^2} \sum_{\alpha, \bar{\alpha} = 5-3s}^{3s-4} m_{\alpha\bar{\alpha}} \;\; \Theta_{3s-4, \alpha} \;\; \bar{\Theta}_{3s-4, \bar{\alpha}} \, . \tag{6.15}$$

Concluding, we observe that the family of QFT representations of the chiral algebra becomes reacher when the difference $n_2 = 2s - 1/2\, s_2$ increases. For $n_2 = 0$ we have for each spin s just a finite number of modular invariant 2-dimensional theories (depending on the factorization of s into primes) all with $c = 1$ - see [BMT] [PT1]. For $n_2 = 1$, the cases including the standard $\mathbf{Z}_k$ - parafermions, we have a 2-parameter set of such finite families; they are labeled by the spin s and the level k (corresponding to total central charge $c = 3k/k+2$) - see [PT2]. For the case $n_2 = 2$ treated here we needed another quantum number n (on top of s and k) to enumerate the finite families of models. We conjecture that this tendency will persist for higher values of n_2.

Acknowledgments

This work was started during the visit of P. Furlan at the Institute for Nuclear Research and Nuclear Energy in Sofia in October 1988 and was continued during the visit of R.R. Paunov and I.T. Todorov at the International School for Advanced Studies in Trieste in December 1988, and during a second visit of R.R. Paunov, sponsored by the Sezione di Trieste of I.N.F.N. in 1989. Part of this work was done during the visit of I.T. Todorov at l'Institut des Hautes Etudes Scientifiques in Bures-sur-Yvette, France in 1989. The authors would like to thank all these institutions for hospitality and financial support.

Discussions with Galen Sotkov are gratefully acknowledged.

Appendix A

Contribution of zero charge quasiprimary fields to the small distance expansion of the 4-point function

We shall start by evaluating the contribution of the general term in the OPE

$$z_{34}^{2\Delta}\, \varphi^*(z_3)\, \varphi(z_4) = 1 + \sum_{\ell\geq 2} a_\ell \frac{\Gamma(2\ell)}{[\Gamma(\ell)]^2} \int_{z_4}^{z_3} \frac{(z_3-\zeta)^{\ell-1}\,(\zeta-z_4)^{\ell-1}}{z_{34}^{\ell-1}}\, O_\ell\,(\zeta)\, d\zeta \qquad \text{(A.1)}$$

to the 4-point function $(1-\eta)^{-2\Delta} F_4(\eta)$ (2.25). Here O_ℓ is a quasiprimary conformal field of spin (=dimension) ℓ. If N_ℓ is the normalization constant of the 2-point function of O_ℓ,

$$< O_\ell\,(z_1)\, O_\ell\,(z_2) > = N_\ell\, z_{12}^{-2\ell}\ , \qquad \text{(A.2)}$$

then the 3-point function $< \varphi^* \varphi\, O_\ell >$ consistent with (A.1) will be

$$z_{12}^{2\Delta} < \varphi^*(z_1)\, \varphi(z_2)\, O_\ell\,(z_3) > = N_\ell\, a_\ell \left(\frac{z_{12}}{z_{13}\, z_{23}}\right)^{\ell} . \qquad \text{(A.3)}$$

Inserting (A.1) and (A.3) into the left hand side of (2.25) we find

$$(1-\eta)^{-2\Delta} F_4(\eta) = 1 + \sum_{\ell\geq 2} N_\ell\, a_\ell\, f_\ell\,(\eta) \qquad \text{(A.4)}$$

with

$$f_\ell\,(\eta) = \frac{\Gamma(2\ell)}{\Gamma^2(\ell)} \frac{z_{12}^{\ell}}{z_{34}^{\ell-1}} \int_{z_4}^{z_3} \frac{(z_3-\zeta)^{\ell-1}\,(\zeta-z_4)^{\ell-1}}{(z_1-\zeta)^{\ell}\,(z_2-\zeta)^{\ell}}\, d\zeta\ .$$

Using the fact that f_ℓ is Möbius invariant and setting $z_1 \to \infty$, $z_2=1$, $z_3=\eta$, $z_4=0$, we express f_ℓ in terms of a hypergeometric function:

$$f_\ell(\eta) = \frac{\Gamma(2\ell)}{\Gamma^2(\ell)}\, \eta^\ell \int_0^1 \frac{t^{\ell-1}(1-t)^{\ell-1}}{(1-t\eta)^\ell}\, dt =$$

$$= \frac{\Gamma(2\ell)}{\Gamma^2(\ell)} \sum_{v=0}^{\infty} \frac{\Gamma^2(\ell+v)}{v!\,\Gamma(2\ell+v)}\, \eta^{\ell+v} = \eta^\ell\, F(\ell,\ell\,;2\ell\,;\eta) =$$

$$= \eta^\ell + \frac{\ell}{2}\,\eta^{\ell+1} + \frac{\ell(\ell+1)^2}{4(2\ell+1)}\,\eta^{\ell+2} + \frac{\ell(\ell+1)(\ell+2)^2}{24(2\ell+1)}\,\eta^{\ell+3} + \ldots \quad \text{(A.5)}$$

The functions f_ℓ satisfy the relation

$$f_\ell(\eta) = (-1)^\ell\, f_\ell\left(\frac{-\eta}{1-\eta}\right), \qquad \text{(A.6)}$$

which reflects the symmetry of the original integral under the exchange $z_3 \leftrightarrow z_4$. For even ℓ this allows to rearrange the sum in the right hand side of (A.5) as a power series in the invariant variable $\eta/(1-\eta) - \eta = \eta^2/(1-\eta)$:

$$f_\ell(\eta) = \frac{\eta^\ell}{(1-\eta)^{\ell/2}} \left\{1 - \frac{\ell^2}{8(2\ell+1)}\,\frac{\eta^2}{(1-\eta)} + \ldots\right\} \text{ for } \ell = 2n\,; \quad \text{(A.7a)}$$

for odd ℓ it can be written as $[1/2(\eta + \eta/(1-\eta))]^\ell$ times a power series in $\eta^2/(1-\eta)$:

$$f_\ell(\eta) = \eta^\ell \left(\frac{1-\frac{1}{2}\eta}{1-\eta}\right)^\ell \left\{1 - \frac{\ell(3\ell+1)}{8(2\ell+1)}\,\frac{\eta^2}{1-\eta} + \ldots\right\} \text{ for } \ell = 2n+1\,. \quad \text{(A.7b)}$$

The second equation (2.25) is then obtained by inserting (A.7) into (A.4) with $N_2 = c/2$, $a_2 N_2 = \Delta$, $N_3 = 1$, $a_3 = A$.

Going to higher order terms in the expansion (A.1) we encounter, typically, more than one quasiprimary (i.e. Möbius covariant) field of the same dimension. Such a situation first appears for $\ell = 4$. Indeed, the OPE of T with itself involves the composite conformal field [Z]

$$\Lambda(z) = :T^2(z): \tag{A.8a}$$

where the normal product is defined (through the OPE) to be orthogonal to T :

$$:T(z_1)\,T(z_2): = T(z_1)\,T(z_2) - \frac{c}{2z_{12}^4} - \frac{12}{z_{12}^5}\int_{z_2}^{z_1} (z_1-\zeta)(\zeta-z_2)\; T(\zeta)\, d\zeta\,. \tag{A.8b}$$

The normalization of Λ is computed from the 4-point function

$$\langle T(z_1)\,T(z_2)\,T(z_3)\,T(z_4)\rangle\; z_{12}^4\, z_{34}^4 =$$

$$c\,\left\{ \frac{c}{4}\left[1 + \frac{\eta^4}{(1-\eta)^2}\left(2 + \frac{4\eta^2}{1-\eta} + \frac{\eta^4}{(1-\eta)^2}\right)\right] + \frac{2\eta^2}{1-\eta}\left(1 + \frac{\eta^2}{1-\eta}\right)\right\} \tag{A.9a}$$

and from the relation (derived from (A.5) and (A.7))

$$12 \langle T(z_1)T(z_2)\, \frac{z_{12}^4}{z_{34}}\int_{z_4}^{z_3} (z_3-\zeta)(\zeta-z_4)T(\zeta)d\zeta\rangle = 12c\, \frac{z_{12}^2}{z_{34}}\int_{z_4}^{z_3} \frac{(z_3-\zeta)(\zeta-z_4)}{(z_1-\zeta)^2 (z_2-\zeta)^2}\, d\zeta$$

$$= 2c\,\frac{\eta^2}{1-\eta}\left(1 - \frac{1}{10}\,\frac{\eta^2}{1-\eta} + \frac{1}{70}\,\frac{\eta^4}{(1-\eta)^2}\right) + O\left(\frac{\eta^8}{(1-\eta)^4}\right). \tag{A.9b}$$

We obtain

$$< T(z_1)T(z_2) : T(z_3)T(z_4) : > = \frac{c}{z_{13}^2 z_{14}^2 z_{23}^2 z_{24}^2} \left(\frac{5c+22}{10} + (c - \frac{1}{35}) \frac{\eta^2}{1-\eta} + O\left(\frac{\eta^4}{(1-\eta)^2}\right) \right) \tag{A.10a}$$

or, in the limit $z_4 \to z_3$,

$$z_{13}^4 z_{23}^4 < T(z_1)T(z_2)\,\Lambda(z_3) > = \frac{c}{10}(5c+22) = z_{12}^8 < \Lambda(z_1)\,\Lambda(z_2) > . \tag{A.10b}$$

Similarly, the 3-point function of Λ with the pair $\varphi{*}\varphi$ is evaluated from the 4-point function (computed, e.g., in [FST])

$$z_{12}^{2\Delta} z_{34}^4 < \varphi{*}(z_1)\varphi(z_2)T(z_3)T(z_4) > = \frac{c}{2} + 2\Delta \frac{\eta^2}{1-\eta} + \Delta^2 \frac{\eta^2}{(1-\eta)^4} \tag{A.11}$$

by first subtracting the constant term and the integral

$$\frac{12 z_{12}^{2\Delta}}{z_{34}} \int_{z_4}^{z_3} (z_3-\zeta)(\zeta-z_4) < \varphi{*}(z_1)\,\varphi(z_2)\,T(\zeta) > d\zeta =$$

$$= \frac{12\Delta}{z_{34}} z_{12}^2 \int_{z_4}^{z_3} \frac{(z_3-\zeta)(\zeta-z_4)\,d\zeta}{(z_1-\zeta)^2 (z_2-\zeta)^2} = 2\Delta \frac{\eta^2}{1-\eta}\left(1 - \frac{\eta^2}{10(1-\eta)}\right) + O\left(\frac{\eta^6}{(1-\eta)^3}\right). \tag{A.12}$$

We find

$$< \varphi{*}(z_1)\,\varphi(z_2)\,\Lambda(z_3) > = \frac{\Delta(5\Delta+1)\, z_{12}^{4-2\Delta}}{5\, z_{13}^4\, z_{23}^4} . \tag{A.13}$$

Putting the above remarks together we obtain the following concretization of the expansion (A.1)

$$z_{34}^{2\Delta}\, \varphi^*(z_3)\, \varphi(z_4) = 1 + \frac{12\Delta}{c} \int_{z_4}^{z_3} \frac{(z_3-\zeta)(\zeta-z_4)}{z_{34}}\, T(\zeta)\, d\zeta\, +$$

$$+ \frac{30A}{z_{34}^2} \int_{z_4}^{z_3} (z_3-\zeta)^2 (\zeta-z_4)^2\, W(\zeta)\, d\zeta\, +$$

$$+ \frac{140}{z_{34}^3} \int_{z_4}^{z_3} (z_3-\zeta)^3 (\zeta-z_4)^3 \left[\frac{5\Delta(5\Delta+1)}{c(5c+22)} \Lambda(\zeta) + B\, W_4(\zeta) \right] d\zeta$$

$$+ z_{34}^5\, R_5(z_3,z_4)$$

$$\left(z_{12}^8 < W_4(z_1)\, W_4(z_2) > = 1,\; < \Lambda(z_1)\, W_4(z_2) > = 0 \right), \qquad \text{(A.14)}$$

W_4 being another primary field (of dimension 4). Inserting this OPE into the 4-point function of $\varphi^{(*)}$ we obtain the following extension of (2.25):

$$(1-\eta)^{-2\Delta}\, F_4(\eta) = 1 + \frac{\Delta^2}{5c} \frac{\eta^2}{1-\eta} \left(10 - \frac{\eta^2}{1-\eta}\right) +$$

$$+ A^2 \eta^3 \left(\frac{1 - \frac{1}{2}\eta}{1-\eta} \right)^3 + \left[\frac{2\Delta^2}{5c} \frac{(5\Delta+1)^2}{5c+22} + B^2 \right] \frac{\eta^4}{(1-\eta)^2} + O(\eta^5). \qquad \text{(A.15)}$$

Appendix B

Coset space construction of the Z_k parafermion algebras for $p = 2,\ m_V = 0.$

The objective of this Appendix is to give an expanded version of the Goddard - Schwimmer [GS2] coset space construction.

For $k \geq 4$ it is based on the coset

$$\frac{su\,(k)_1 \times so\,(k)_n}{so\,(k)_{n+2}}\ ; \tag{B.1a}$$

for $k = 3$ it corresponds to

$$\frac{su\,(3)_1 \times so\,(3)_n}{so\,(3)_{n+4}}\ . \tag{B.1b}$$

The corresponding central charges of Vir are obtained from (3.9) for $\lambda = n/2$ and $\lambda = n/4$, respectively:

$$c_R = c\,(k, n) = k - 1 + \frac{n\,k\,(k-1)}{2\,(n+k-2)} - \frac{(n+2)\,k\,(k-1)}{2\,(n+k)} =$$

$$= (k-1)\ \left[1 - \frac{k\,(k-2)}{(n+k)\,(n+k-2)}\right] \tag{B.2a}$$

$$c_R = c_3\,(n) = 2 + \frac{3n}{n+2} - \frac{3\,(n+4)}{n+6} = 2\ \left[1 - \frac{12}{(n+2)\,(n+6)}\right]. \tag{B.2b}$$

Remark B.1 If we substitute n in (B.2b) by $2n$ then it will appear as a special case of (B.2a) (corresponding to $k = 3$) provided that we also allow half-integer values of n in that case - see [GS2].

We start by recalling the vertex operator construction of $\hat{su}(k)_1$ ([FK], [GO]). The Cartan subalgebra of the $\hat{su}(k)_1$ - current algebra is spanned by $k-1$ currents $H^i(z)$ satisfying

$$H^i(z_1)\, H^j(z_2) = \delta_{ij}\, z_{12}^{-2} + :H^i(z_1)\, H^j(z_2):, \quad i, j = 1,\dots, k-1. \qquad \text{(B.3)}$$

We introduce k vectors $\rho_1,\dots,\rho_k$ in the $su(k)$ weight space with the properties of the weight vectors of the defining (k - dimensional), representation of $su(k)$:

$$\rho_1 + \dots + \rho_k = 0 \qquad \text{(B.4a)}$$

$$\rho_i\, \rho_j = \delta_{ij} - \frac{1}{k} \qquad i, j = 1, \dots, k, \qquad \text{(B.4b)}$$

so that the (positive) roots of $su(k)$ are given by

$$\alpha_{ij} = \rho_i - \rho_j \ (> 0 \text{ for } i < j) \qquad \text{(B.5a)}$$

the simple roots being

$$\alpha_i \equiv \alpha_{i\, i+1} = \rho_i - \rho_{i+1} \quad , \qquad i = 1, \dots, k-1. \qquad \text{(B.5b)}$$

The fundamental weights $\lambda_1, \dots, \lambda_{k-1}$ characterized by the property

$$(\lambda_i, \alpha_j) \equiv \lambda_i \cdot \alpha_j = \delta_{ij} \qquad \text{(B.6a)}$$

are expressed as sums of the ρ's:

$$\lambda_1 = \rho_1,\ \ \lambda_2 = \rho_1+\rho_2,\ \ \lambda_3 = \rho_1+\rho_2+\rho_3,\ldots,\ \lambda_{k-1} = -\rho_k\,. \qquad \text{(B.6b)}$$

The charged current corresponding to the root α_{ij} is given by the vertex operator $U(z, \rho_i - \rho_j)$ where $U(z, \alpha)$ is characterized by the small distance OPEs

$$H^i(z_1)\ U(z_2, \alpha) \approx \frac{\alpha^i}{z_{12}}\ U(z_2, \alpha)\,, \qquad \text{(B.7)}$$

$$U(z_1, \alpha)\ U(z_2, \beta) \approx \varepsilon(\alpha, \beta)\ z_{12}^{\alpha\cdot\beta}\ U(z_2, \alpha+\beta) \quad (\alpha+\beta\neq 0) \qquad \text{(B.8a)}$$

$$z_{12}^2\ U(z_1, \alpha)\ U(z_2, -\alpha) \approx 1 + z_{12}\ \alpha\cdot H(z). \qquad \text{(B.8b)}$$

The phase factors $\varepsilon(\alpha, \beta)$ have an important part in what follows and are studied in detail in Appendix C. Here we shall find a restriction on these factors coming from the requirement that the orthogonal subalgebra $\hat{so}(k)$ of $\hat{su}(k)$ has a canonical form.

We first recall that the standard hermitian generators of $so(3)$

$$s_1 = s_{23} = \begin{pmatrix} 0&0&0 \\ 0&0&-i \\ 0&i&0 \end{pmatrix},\ \ s_2 = s_{31} = \begin{pmatrix} 0&0&i \\ 0&0&0 \\ -i&0&0 \end{pmatrix},\ \ s_3 = s_{12} = \begin{pmatrix} 0&-i&0 \\ i&0&0 \\ 0&0&0 \end{pmatrix}$$

satisfying $[s_1, s_2] = is_3$ etc. are expressed in terms of the canonical $su(3)$ Cartan-Weyl generators

$$E_{12} = E(\alpha_{12}) = \begin{pmatrix} 0&1&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix} \quad \text{etc.}$$

by $s_{j\ell} = i\,(E_{\ell j} - E_{j\ell})$. Using this relation as a guide we define the basic currents $V_{j\ell}(z)$ of $\hat{so}(k)$ current algebra in terms of the $\hat{su}(k)$ vertex operators by

$$V_{j\ell}\ (z) = i\left[\,U(z, \alpha_{\ell j}) \;-\; U(z, \alpha_{j\ell})\right] . \tag{B.9}$$

The standard OPE for the V's

$$z_{12}^{2}\ V_{ij}(z_1)\ V_{\ell m}(z_2) \approx 2\,(\delta_{i\ell}\ \delta_{jm} - \delta_{im}\,\delta_{j\ell}) +$$

$$i\,z_{12}\left[\,\delta_{i\ell}\ V_{jm}(z_2) + \delta_{jm}\,V_{i\ell}\ (z_2) - \delta_{j\ell}\ V_{im}(z_2) - \delta_{im}\,V_{j\ell}\ (z_2)\right] \tag{B.10}$$

are satisfied provided that

$$\varepsilon\,(\alpha_{i\ell}\,, \alpha_{\ell m}) = -1 = -\,\varepsilon\,(\alpha_{\ell i}, \alpha_{m\ell})\quad \text{for}\quad i \neq \ell \neq m \neq i\,. \tag{B.11}$$

(In writing down the first term in the right hand side of (B.10) we have used the unitarity condition $\varepsilon\,(\alpha, -\alpha) = 1$ – see Eq. (C.5).)
Eq. (B.10) corresponds to a level 2 representation of $\hat{so}(k)$ for $k \geq 4$ and to a level 4 representation of $\hat{so}(3)$. (To see the latter point we note that the Cartan-Weyl basis for $\hat{su}(2)$ - for which the level is given by the normalization of the 2-point function - is related to the orthonormal $\hat{so}(3)$ basis by

$$H\,(z) = \sqrt{2}\ V_{12}\,(z)\,,\quad E_{\pm}\,(z) = V_{23}\,(z) \pm i\,V_{31}\,(z)\,,$$

so that we have to multiply the numerator of the 2-point function of V_{ij} by 2 to get the level. For $k \geq 4$ a current $E\alpha_1$ corresponding to a long root α_1 is related to V_{ij} by an equation of the type

$$E_{\alpha_1} = \frac{1}{2}\,(V_{31} + V_{14} + i\,V_{32} + i\,V_{24})$$

and has hence the same normalization as the V's.) We note that $\hat{so}(3)_4 \subset \hat{su}(3)_1$ and $\hat{so}(k)_2 \subset \hat{su}(k)_1$ for $k \geq 4$ are conformal imbeddings in the sense of [BETZ] and [AGO], - i.e. they carry equal Virasoro central charges $c\,(\mathcal{G}) = k d_G/(k+\tilde{h})$ where d_G is the dimension of $\mathcal{G}$, $\tilde{h}$ is the dual Coxeter number (see [GO] or [FST]):

$$c\,(\hat{so}(3)_4) = \frac{4\cdot 3}{4+2} = 2 = c\,(\hat{su}(3)_1); \qquad \text{(B.12a)}$$

$$c\,(\hat{so}(k)_2) = \frac{k\,(k-1)}{k} = k - 1 = c\,(\hat{su}(k)_1), \quad \text{for } k \geq 4\,. \qquad \text{(B.12b)}$$

In all cases c is equal to the rank $k - 1$ of $su(k)$ so that the stress energy tensor is given by the abelian Sugawara formula

$$T\,(z, \hat{su}(k)_1) = \frac{1}{2} : H^2(z) : \equiv \frac{1}{2} \sum_{i=1}^{k-1} : H^i(z)\,H^i(z) : \quad . \qquad \text{(B.13)}$$

Let the currents $J_{ij}(z)$ span the factor $\hat{so}(k)_n$ of the coset (B.1). Then the <u>gauge currents</u> (which give rise to the denominator in (B.1)) are given by the sum

$$Y_{ij}(z) = J_{ij}(z) + V_{ij}(z)\,. \qquad \text{(B.14)}$$

They generate the diagonal $\hat{so}(k)$ subalgebra whose level is the sum of the level (n) of the second factor in the numerator of (B.1) and of the level of the $\hat{so}(k)$ that is conformally imbedded in $\hat{su}(k)_1$. The (reduced) stress energy tensor of the coset space theory is given by a linear combination of Sugawara tensors:

$$(T_R=)\ T(z) = T(z, \hat{su}(k)_1) + T(z, \hat{so}(k)_n) - T(z, \hat{so}(k) \cdot Y_{ij}). \qquad \text{(B.15)}$$

(By construction T commutes with Y.)

The basic parafermionic currents $\varphi^{(*)}(z)$ are constructed in terms of the vertex operators U and the currents J_{ij} by the condition that they are primary conformal fields of weight $\Delta = 2 - 2/k$ which commute with Y_{ij}:

$$\varphi(z) = \sqrt{\frac{n}{k\,[n+2\,(k-1)]}}\ \Big\{ \sum_{i=1}^{k} U(z, 2\rho_i) +$$

$$+ \frac{2}{n} \sum_{1 \le i < j \le k} J_{ij}\ U(z, \rho_i + \rho_j) \Big\} \qquad \text{(B.16a)}$$

$$\varphi^*(z) = \sqrt{\frac{n}{k\,[n+2\,(k-1)]}}\ \Big\{ \sum_{i=1}^{k} U(z, -2\rho i) +$$

$$+ \frac{2}{n} \sum_{1 \le i < j \le k} J_{ij}(z)\ U(z, -\rho_i - \rho_j) \Big\} . \qquad \text{(B.16b)}$$

(In verifying the normalization factor we note that each of the $k + 1/2\,k\,(k-1)$ terms in the right hand side of (B.16) is orthogonal to the others, the 2-point functions of the vertex operators are normalized to 1 and the 2-point functions of the currents J are normalized to n.) To verify commutation with $Y_{\ell m}$ we use the OPEs

$$Y_{\ell m}(z_1) \sum_{1\le i<j\le k} J_{ij}(z_2)\ U(z_2, \rho_i+\rho_j) \approx n\, z_{12}^{-2}\ U(z_2, \rho_i+\rho_j) +$$

$$+\frac{i}{z_{12}} \Big\{ \Big[\sum_{i=1}^{m-1} - \sum_{i=m+1}^{k} - \Big(\sum_{i=1}^{\ell-1} - \sum_{i=\ell+1}^{k} \Big) \varepsilon(\rho_m-\rho_\ell, \rho_\ell+\rho_i) \Big] J_{i\ell}\ (z_2)\, U(z_2, \rho_i+\rho_m)$$

$$+ \Big[\Big(\sum_{i=1}^{m-1} - \sum_{i=m+1}^{k} \Big) \varepsilon(\rho_\ell-\rho_m, \rho_m+\rho_i) - \sum_{i=1}^{\ell-1} + \sum_{i=\ell+1}^{k} \Big] J_{im}\,(z_2)\, U(z_2, \rho_i+\rho_\ell) \Big\}$$

$$\text{for } \ell < m \qquad \text{(B.17a)}$$

$$Y_{\ell m}(z_1) \sum_{i=1}^{k} U(z_2, 2\rho_i) \approx 2\, i\, z_{12}^{-2}\ \varepsilon(\rho_\ell-\rho_m, 2\rho_m)\ U(z_2, \rho_\ell+\rho_m)$$

$$(\ell < m) \qquad \text{(B.17b)}$$

and give the following values of the phase factors ε:

$$\varepsilon(\rho_\ell-\rho_m, \rho_m+\rho_i) = \varepsilon(\rho_m-\rho_\ell, \rho_i+\rho_\ell) = 1 \qquad \text{for } 1 \le i \le \ell-1,$$

$$\varepsilon(\rho_\ell-\rho_m, \rho_m+\rho_i) = \varepsilon(\rho_m-\rho_\ell, \rho_i+\rho_\ell) = -1 \qquad \text{for } \ell+1 \le i \le m-1$$

$$\varepsilon(\rho_\ell-\rho_m, \rho_m+\rho_i) = \varepsilon(\rho_m-\rho_\ell, \rho_i+\rho_\ell) = 1 \qquad \text{for } m+1 \le i \le k$$

$$\varepsilon(\rho_\ell-\rho_m, 2\rho_m) = -\varepsilon(\rho_m-\rho_\ell, 2\rho_\ell) = -i, \qquad \text{(B.18)}$$

$$\varepsilon(\rho_\ell-\rho_m, \rho_\ell+\rho_m) = -\varepsilon(\rho_m-\rho_\ell, \rho_\ell+\rho_m).$$

In order to prove that the OPE of $\varphi^*(z_1)\ \varphi(z_2)$ coincides with (1.23) we impose the following additional restrictions on the ε's:

$$\varepsilon(2\rho_\ell, -\rho_\ell - \rho_m) = \varepsilon(\rho_\ell + \rho_m, -2\rho_m) = i\ ,$$

$$\varepsilon(2\rho_m, -\rho_\ell - \rho_m) = \varepsilon(\rho_\ell + \rho_m, -2\rho_\ell) = -i\ ; \tag{B.19a}$$

$$\varepsilon(\rho_i+\rho_\ell\ , -\rho_i-\rho_j) = \varepsilon(\rho_i+\rho_j, -\rho_i-\rho_\ell) = +1 \qquad \text{for}\ \ i<j<\ell\ ,$$

$$\varepsilon(\rho_i+\rho_j\ , -\rho_i-\rho_\ell) = \varepsilon(\rho_i+\rho_\ell\ , -\rho_i-\rho_j) = 1 \qquad \text{for}\ \ j<\ell<i\ ,$$

$$\varepsilon(\rho_i+\rho_j\ , -\rho_i-\rho_\ell) = \varepsilon(\rho_i+\rho_\ell\ , -\rho_i-\rho_j) = -1 \qquad \text{for}\ \ j<i<\ell\ . \tag{B.19b}$$

(We note that Eqs. (B.19) are consistent with the conjugation property (C.6) of $\varepsilon\ (\alpha, \beta)$.) The absence of a spin 1 contribution to the OPE $\varphi^*\varphi$ is a consequence of (B.4a). In computing the spin-2 contribution we have used the identities

$$\sum_{i=1}^{k} \rho_i \otimes \rho_i = \mathbb{1} \tag{B.20a}$$

$$\sum_{1\le i<j\le k} (\rho_i+\rho_j) \otimes (\rho_i+\rho_j) = (k-2)\ \mathbb{1}\ , \tag{B.20b}$$

where $\mathbb{1}$ stands for the $(k-1) \times (k-1)$ unit matrix. The 2-cocycle $\varepsilon\ (\alpha, \beta)$ satisfying all the above conditions is constructed explicitely in Appendix C for the special case $k = 3$ considered in Sec. 6. (Some results concerning these cocycles for arbitrary $k \geq 3$ are also presented there.)

The stress tensor (B.15) for the Z_k parafermionic theory can be written in the form

$$4\,\frac{k-1}{k\,c\,(k,n)}\,T(z) = \frac{2}{k}\,\frac{k+n-2}{n+2k-2} : H^2(z) : + \frac{2}{n\,k\,(n+2k-2)} \sum_{i \neq j} \Big\{ : J_{ij}^2(z) : -$$

$$- (n+k-2)\, J_{ij}\, V_{ij} \Big\} \,. \tag{B.21}$$

We observe that for $n \to \infty$ the limit theory also exists and appears as an $\hat{su}(k)_1$ - current algebra theory in complete analogy with the limit of the [BPZJ minimal conformal models.

Appendix C

Two - cocycles associated with the vertex operator construction for level - 1 representations of $\hat{su}(k)$.

The vertex operator $U(z, \alpha)$ characterized by the OPEs (B.7) (B.8) can be written as a (normal ordered) exponential of an integral of a current (belonging to the Cartan subalgebra) times a constant operator of the form

$$\hat{c}_\alpha = \Gamma_\alpha \, c_\alpha \, . \qquad \text{(C.1)}$$

Here $\alpha \to \Gamma_\alpha$ is a unitary representation of the abelian group of weights, the weight lattice $\mathcal{W}$; Γ_α shifts the weight by α:

$$H_0^i \, \Gamma_\alpha = \Gamma_\alpha (H_0^i + \alpha^i) \, .$$

c_α is a "Klein factor" that commutes with the currents (and hence with all observables). The operators $\hat{c}_\alpha$ give rise to a projective representation of $\mathcal{W}$ with the phase factors $\varepsilon(\alpha, \beta)$ (B.8a) appearing as multipliers:

$$\hat{c}_\alpha \, \hat{c}_\beta = \varepsilon(\alpha, \beta) \, \hat{c}_{\alpha+\beta} \, . \qquad \text{(C.2)}$$

The associativity of the operator multiplication

$$(\hat{c}_\alpha \hat{c}_\beta) \hat{c}_\gamma = \hat{c}_\alpha (\hat{c}_\beta \hat{c}_\gamma)$$

implies the <u>cocycle condition</u>

$$\varepsilon(\alpha, \beta) \, \varepsilon(\alpha+\beta, \gamma) = \varepsilon(\alpha, \beta+\gamma) \, \varepsilon(\beta, \gamma). \qquad \text{(C.3)}$$

Demanding further the "unitary relations"

$$\hat{c}_\alpha^* = \hat{c}_{-\alpha} \,, \quad \hat{c}_\alpha \hat{c}_{-\alpha} = \hat{c}_0 = \mathbb{1} \qquad \text{(C.4)}$$

we deduce

$$\varepsilon\,(\alpha, -\alpha) = 1 = \varepsilon\,(0, \alpha) \qquad \text{(C.5)}$$

$$\overline{\varepsilon\,(\alpha, \beta)} = \varepsilon\,(-\beta, -\alpha)\,, \quad |\,\varepsilon\,(\alpha, \beta)\,| = 1\,. \qquad \text{(C.6)}$$

A redefinition of the operators $\hat{c}_\alpha$,

$$\hat{c}_\alpha \to u\,(\alpha)\,\hat{c}_\alpha\,, \qquad u\,(\alpha) \in C\,,\ |\,u\,(\alpha)\,| = 1 = \overline{u\,(\alpha)}\,u\,(-\alpha) \qquad \text{(C.7a)}$$

leads to a gauge transformation of the phase factors ε, - i.e., to their multiplication by a trivial cocycle:

$$\varepsilon\,(\alpha, \beta) \to \frac{u\,(\alpha)\,u\,(\beta)}{u\,(\alpha+\beta)}\ \varepsilon\,(\alpha, \beta)\,. \qquad \text{(C.7b)}$$

An invariant under such gauge transformations is the symmetry factor S defined by

$$\hat{c}_\alpha \hat{c}_\beta = S\,(\alpha, \beta)\,\hat{c}_\beta \hat{c}_\alpha \quad \text{or} \quad \varepsilon\,(\alpha, \beta) = S\,(\alpha, \beta)\,\varepsilon\,(\beta, \alpha). \qquad \text{(C.8)}$$

Commuting $\hat{c}_\gamma$ with both sides of (C.2) we deduce the additivity property

$$S\,(\alpha + \beta, \gamma) = S\,(\alpha, \gamma)\ S\,(\beta, \gamma)\,. \qquad \text{(C.9)}$$

On the other hand, (C.6) and (C.8) imply

$$S(\alpha, \beta)\ S(\beta, \alpha) = 1 = S(\alpha, \pm\alpha) \qquad \text{(C.10a)}$$

$$\overline{S(\alpha, \beta)} = S(\beta, \alpha) = S(-\beta, -\alpha)\,. \qquad \text{(C.10b)}$$

(It follows from these relations that $S(\alpha, \beta)$ also satisfies the cocycle condition (C.3).) The significance of the S-factors is exhibited by the theorem of Goddard et al. [G] [GNOS] asserting that for each symmetry factor S satisfying (C.9) (C.10) there exists a unique up to a gauge transformation cocycle ε obeying (C.2) (C.5) (C.6). We shall, therefore, first determine the symmetry factors in the case of the su(k) weight lattice.

Let $\mathcal{R}$ be the root lattice of su(k),

$$\mathcal{R} = \{\ \alpha = m_1\alpha_1 + \ldots + m_{k-1}\alpha_{k-1}\ ;\ m_i \in \mathbf{Z},\ i = 1,\ldots, k-1\ \} \qquad \text{(C.11)}$$

where $\alpha_i = \rho_i - \rho_{i+1}$ are the simple roots of su(k) - see Eq. (B.5). The weight lattice $\mathcal{W}$ of su(k) is an Euclidean lattice spanned by linear combinations with integer coefficients of the vectors ρ_j satisfying (B.4). It can be defined as the dual of $\mathcal{R}$ in the sense that for each $\lambda \in \mathcal{W}(\supset \mathcal{R})$ we have $(\lambda, \alpha) \equiv \lambda\cdot\alpha \in \mathbf{Z}$ for all $\alpha \in \mathcal{R}$.

We postulate that the symmetry factor satisfies

$$S(\alpha, \beta) = (-1)^{\alpha\cdot\beta} = S(\beta, \alpha) \qquad \text{for } \alpha, \beta \in \mathcal{R}, \qquad \text{(C.12)}$$

a property that guarantees the local commutativity between currents.

Remark C.1. In general it is not enough to assume that just one of the vectors α or β in (C.12) belongs to the root system $\mathcal{R}$. If that were the case we would have using (B.4b) (C.9) (C.10a) and (C.12)

$$S(\rho_i, \pm\rho_j) = S(\rho_i - \rho_j, \pm\rho_j)\ S(\rho_j, \pm\rho_j) = -1 \qquad \text{for } i \neq j\,; \qquad \text{(C.13)}$$

on the other hand, according to (B.4a),

$$(-1=)\ S(\rho_1, -\rho_k) = S(\rho_1, \rho_1 + \ldots + \rho_{k-1}) = S(\rho_1, \rho_2) \ldots S(\rho_1, \rho_{k-1}) = (-1)^{k-2}$$

which is a contradiction for even k. As a matter of fact we can always choose the Klein factors in such a way that the current commutes with all vertex operators and hence Eq. (C.12) remains true when only $\alpha \in \mathcal{R}$. However we have already restricted the Klein factors by demanding (C.4) (and hence (C.5)) and the above argument shows that, in general, one cannot simultaneously require (C.12) for $\alpha \in \mathcal{R}$, $\beta \in \mathcal{W}$. We shall demonstrate that this is, however, possible in the important special case in which $k = 3$.

The general solution of (C.9) (C.10) can be written in the form

$$S(\alpha, \beta) = e\left(\frac{1}{2} A(\alpha, \beta)\right) \equiv e^{i\pi A(\alpha, \beta)} \qquad \text{(C.14a)}$$

where $A(\alpha, \beta)$ is a bilinear skewsymmetric form:

$$A(\alpha, \beta) = -A(\beta, \alpha) = -A(-\alpha, \beta), \qquad A(\alpha+\beta, \gamma) = A(\alpha, \gamma) + A(\beta, \gamma). \qquad \text{(C.14b)}$$

To satisfy (C.12) we must add the requirement

$$A(\alpha, \beta) = \alpha \cdot \beta \mod 2 \qquad \text{for } \alpha, \beta \in \mathcal{R} \qquad \text{(C.15)}$$

(This is consistent with skew-symmetry since $\alpha \cdot \beta = -\alpha \cdot \beta \mod 2$ for $\alpha \in \mathcal{R}$.)

In the special case $k = 3$ there is a unique up to a sign form A satisfying all the above conditions (including (C.13)):

$$A(\alpha,\beta) = \alpha \wedge \beta = m_1 n_2 - m_2 n_1 \quad \text{for } \alpha = m_1\rho_1 + m_2\rho_2,\ \beta = n_1\rho_1 + n_2\rho_2. \qquad \text{(C.16)}$$

In order to verify that this skew form obeys (C.15) we note that

$$\rho_1 \wedge \rho_2 = 1 = \rho_2 \wedge \rho_3 \; (= -\rho_2 \wedge \rho_1) = \rho_3 \wedge \rho_1 \qquad \text{(C.17)}$$

and hence

$$(\rho_1 - \rho_2) \wedge \rho_1 = 1 = (\rho_1 - \rho_2)\cdot\rho_1 , \; (\rho_1 - \rho_2) \wedge (\rho_2 - \rho_3) = 3 = \alpha_{12}\cdot\alpha_{23} + 2 . \qquad \text{(C.18)}$$

Given $S(\alpha, \beta)$ in the form (C.14) we can write an $\varepsilon_0(\alpha, \beta)$ which has the same additivity and conjugation properties as S:

$$\varepsilon_0(\alpha, \beta) = e\left(\frac{1}{4} A(\alpha, \beta)\right) . \qquad \text{(C.19)}$$

Thus, for $k = 3$ the general cocycle ε (satisfying (C.5)) is written in the form

$$\varepsilon(\alpha, \beta) = \frac{u(\alpha)\, u(\beta)}{u(\alpha + \beta)}\, \varepsilon_o(\alpha, \beta) , \quad \varepsilon_o(\alpha, \beta) = i^{\alpha\wedge\beta} \qquad \text{(C.20a)}$$

$$\overline{u(\alpha)} = u(-\alpha) , \quad u(\alpha)\, \overline{u(\alpha)} = 1 . \qquad \text{(C.20b)}$$

Condition (B.11) (the canonical commutation relations for so(3)) imposes the following restrictions on the phase factors $u(\alpha)$:

$$\frac{u(\alpha_{12})\, u(\alpha_{23})}{u(\alpha_{13})} = -i = \frac{u(\alpha_{23})\, u(\alpha_{31})}{u(\alpha_{21})} = \frac{u(\alpha_{31})\, u(\alpha_{12})}{u(\alpha_{32})} . \qquad \text{(C.21)}$$

(Due to (C.20b) the second and the third equation (C.21) are a consequence of the first.)

Condition (B.18) (which ensures the commutativity of Y_{ij} and $\varphi^{(*)}$) gives the relations

$$\frac{u(\rho_\ell - \rho_j)\, u(\rho_j + \rho_m)}{u(\rho_\ell + \rho_m)} = i = -\frac{u(\rho_j - \rho_\ell)\, u(\rho_\ell + \rho_m)}{-u(\rho_j + \rho_m)} = \frac{u(\rho_\ell - \rho_j)\, u(2\rho_j)}{u(\rho_j + \rho_\ell)} =$$

$$= -\frac{u(\rho_j - \rho_\ell)\, u(2\rho_\ell)}{u(\rho_j + \rho_\ell)} \qquad \text{for } j < \ell, \quad j \neq m \neq \ell . \qquad (C.22)$$

Eqs. (B.19) are satisfied if

$$\frac{u(\rho_j + \rho_m)\, u(-\rho_\ell - \rho_m)}{u(\rho_j - \rho_\ell)} = i = -\frac{u(\rho_\ell + \rho_m)\, u(-\rho_j - \rho_m)}{u(\rho_\ell - \rho_j)} \qquad \text{for } m \neq j < \ell \neq m$$

$$(C.23)$$

and

$$u(\rho_j + \rho_\ell) = u(2\rho_m) \qquad \text{for all } j, \ell, m . \qquad (C.24)$$

These conditions will be fulfilled if u is taken to be $-i$ for positive roots and i for negative roots:

$$u(\alpha_{12}) = u(\alpha_{13}) = u(\alpha_{23}) = -i = \overline{u(\alpha_{21})} = \ldots \qquad (C.25)$$

(and satisfies (C.24)).

REFERENCES

[AGO] R.C. Arcuri, J.F. Gomes, D.I. Olive, Conformal subalgebras and symmetric spaces, Nucl. Phys. B285 [FS 19] (1987) 327-339.

[AGS] L. Alvarez-Gaumè, C. Gomez, G. Sierra, Quantum group interpretation of some conformal field theories, Phys. Lett. B220 (1989) 142 - 152.

[BETZ] F.A. Bais, F. Englert, A. Taormina, P. Zizzi, Torus compactification for non-simply laced groups, Nucl. Phys. B279 (1987) 529-547.

[BMPT] D. Buchholz, G. Mack, R.R. Paunov, I.T. Todorov, An algebraic approach to the classification of local conformal field theories, IX International Congress on Mathematical Phisics, Swansea, Wales, July 1988, ed. by B. Simon, A. Trumau, I. Davis (Adam Hilger, Bristol, 1989) pp. 299-305.

[BMT] D. Buchholz, G. Mack, I.T. Todorov, The current algebra on the circle as a germ of local field theories, Nucl. Phys. B (Proc. Suppl.) 5B (1988) 20-56.

[BNY] J. Bagger, D. Nemeschansky and Sh. Yankielowicz, Virasoro algebras with central charge $c > 1$, Phys. Rev. Lett. 60 (1988) 389-392.

[BPZ] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, Nucl. Phys. B241 (1984) 333-380.

[BYZ] R. Brustein, S. Yankielowicz, J.-B. Zuber, Factorization and selection rules of operator product algebras in conformal field theories, Nucl. Phys. B313 (1989) 321 - 347.

[CI] Conformal Invariance and Applications to Statistical Mechanics, Eds. C. Itzykson, H. Saleur, J.-B. Zuber (World Scientific, Singapore 1988) 980 p.

[CIZ] A. Cappelli, C. Itzykson, J.-B. Zuber, The A-D-E classification of minimal and $A_1^{(1)}$ conformal invariant theories, Commun. Math. Phys. 113 (1987) 1-26.

[DHR] S. Doplicher, R. Haag, J.E. Roberts, Fields, observables and gauge transformations, I, II, Commun. Math. Phys. 13 (1969) 1-23, 15 (1969) 173-200.

[DR] S. Doplicher, J.E. Roberts, C*- algebra and duality of compact groups: why there is a compact group of internal symmetry in particle physics, Proc. Int. Conf. Math. Phys. Marseille, 1986, in: Mathematical Physics, Ed. by M. Mebkhout, R. Sénéor (World Scientific, Singapore, 1987).

[DV] R. Dijkgraaf, E. Verlinde, Modular invariance and the fusion algebra, Nucl. Phys. B (Proc. Suppl.) 5B (1988) 87-97.

[FLM] I.B. Frenkel, J. Lepowsky, A. Meurman, Vertex Operator Algebra and the Monster (Academic Press, N.Y. 1988).

[FQS] D. Friedan, Z. Qiu, S. Shenker, Conformal invariance, unitarity and critical exponents in two dimensions, Phys. Rev. Lett. 52 (1984) 1575-1578, Details of the non-unitarity proof for highest weight representations of the Virasoro algebra, Commun. Math. Phys. 107 (1986) 535-542.

[FS] D. Friedan, S. Shenker, The analyitic geometry of two-dimensional conformal field theory, Nucl. Phys. B281 (1987) 509-545.

[FST] P. Furlan, G.M. Sotkov, I.T. Todorov, Two-dimensional conformal quantum field theory, Riv. Nuovo Cim., 12 (1989) 1-202.

[G] P. Goddard, The vertex operator construction for non-simply laced Kac-Moody algebras II. Topological and Geometrical Methods in Field Theory, Eds. J. Hietarinta, J. Westerholm (World Scientific, Singapore, 1986) pp. 37-57.

[GNOS] P. Goddard, W. Nahm, D. Olive, A. Schwimmer, Vertex operators for non-simply laced algebras, Commun. Math. Phys. 107 (1986) 179-212.

[GO] P. Goddard, D. Olive, Kac-Moody and Virasoro algebras in relation to quantum physics, Int. J. Mod. Phys. 1 (1986) 303-414.

[GQ] D. Gepner, Z. Qiu, Modular invariant partition functions for parafermionic field theories, Nucl. Phys. B285 [FS 19] (1987) 423-453.

[GS1] P. Goddard, A. Schwimmer, Factoring out free fermions and superconformal algebras, Phys. Lett. B214 (1988) 209-214.

[GS2] P. Goddard, A. Schwimmer, Unitary construction of extended conformal algebras, Phys. Lett. B206 (1988) 62-70.

[HT] L.K. Hadjiivanov, I.T. Todorov, Conformal theories associated with an additive charge (Luigi Radicati Festschrift, to be published).

[KMQ] D. Kastor, E. Martinec, Z. Qiu, Current algebra and conformal discrete series, Phys. Lett. B200 (1988) 434-440.

[KP] V.G. Kac, D.H. Peterson, Infinite dimensional Lie algebras, theta functions and modular forms, Adv. Math. 53 (1984) 125-264.

[KW] V.G. Kac, M. Wakimoto, Modular and conformal invariance constraints in representation theory of affine algebras, Adv. Math. 70 (1988) 156-236.

[PT1] R.R. Paunov, I.T. Todorov, Modular invariant QFT models of u(1) conformal current algebra, Phys. Lett. B196 (1987) 519-526.

[PT2] R.R. Paunov, I.T. Todorov, Local quantum field theories involving the U(1) current algebra on the circle, Lett. Math. Phys. 17 (1989) 215 - 231.

[PT3] R.R. Paunov, I.T. Todorov, Local extensions of the U(1) current algebra and their positive energy representations, Proc. CIRM Conference on Infinite Dimensional Lie Algebras and Lie Groups (Luminy, July 1988), ed. by V. Kac.

[PT4] R.R. Paunov, I.T. Todorov, Rational conformal field theories and Zamolodchikov-Fateev parafermions, Ann. d. Phys. (to be published).

[R] F. Ravanini, An infinite class of new conformal field theories with extended algebras, Mod. Phys. Lett. A3 (1988) 397-412.

[V] E. Verlinde, Fusion rules and modular transformations in 2 D conformal field theory, Nucl. Phys. B300 [FS 22] (1988) 360 - 376.

[Z] A.B. Zamolodchikov, Infinite additional symmetries in two-dimensional conformal quantum field theory, Theor. Math. Phys. 65 (1986) 1205-1213.

[ZF1] A.B. Zamolodchikov, V.A. Fateev, Nonlocal (parafermion) currents in two-dimensional conformal quantum field theory and self-dual critical points in $\mathbf{Z}_N$-symmetric statistical systems, Sov. Phys. JETP 62 (1985) 215-225.

[ZF2] A.B. Zamolodchikov, V.A. Fateev, Disorder fields in two-dimensional conformal quantum-field theory and N=2 extended supersymmetry, Sov. Phys. JETP 63 (1986) 913-919.

[ZF3] A.B. Zamolodchikov, V.A. Fateev, Representations of the algebra of "parafermion currents" of spin 4/3 in two-dimensional conformal field theory. Minimal models and the tricritical Potts $\mathbf{Z}_3$ model, Teor. Math. Phys. 71 (1987) 451-462.