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FUSION MATRICES AND C < 1 (QUASI) LOCAL CONFORMAL THEORIES

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ABSTRACT

General properties of the fusion matrices and their explicit expression given by the $U_q(sl(2))$ quantum 6j-symbols are exploited to analyze some two dimensional c<1 conformal theories. The primary fields structure costants of the local theory, corresponding to the (A_{10},E_6) modular invariant, and of the $\mathbf{Z}_2$ quasilocal analogs of the (A,D) and (A_{10},E_6) series, are computed. The list of the $\Gamma_0(2)$ submodular invariant partition functions on the torus is extended.

1. Introduction

It has been noticed recently by many people that the crossing (fusion) matrices of the rational conformal theories can be identified with the quantum 6j - symbols [1,2] for special "rational phase" values of the deformation parameters. The minimal (unitary) conformal theories of [3] with central charge $c(m) = 1 - \frac{6}{m(m+1)}$ correspond to the quantum group $SL_q(2)$ with $q = \exp\left(\frac{2\pi i m}{m+1}\right)$ and $q = \exp\left(\frac{2\pi i\,(m+1)}{m}\right)$.

The kowledge of the crossing matrices allows the complete description of the nondiagonal minimal theories corresponding to the (A,E) modular invariants in [4]. The case (A,D) has already been done [5] exploiting some general properties of the crossing matrix, without using its explicit form. It was pointed out that with the same technique one can build not only the integer spin (A,D) series, but also local theories involving fermions, as well as a series describing for any c(m) a theory with $\mathbf{Z}_2$ statistics. The latter contains the fermionic and (A,D) algebras as subalgebras. In this paper we give explicitly the structure constants for these quasilocal theories and also analyze another nondiagonal minimal theory - that corresponding to the (A_{10},E_6) invariant, which is the only one among the (A,E) cases, with which one can associate again all the three possibilities. We compute the primary fields structure constants in all three cases and also extend the classification of $\Gamma_0(2)$ submodular invariants on the torus in [6]. The $\mathbf{Z}_2$ theories, existing for all values of c(m), generalize the $\mathbf{Z}_N$ parafermionic theory [7] for N=2. Similarly one can expect that the analysis of the local correlation functions described by the c>1 extended Virasoro algebra W_N representations [8] will give as a byproduct a large class of theories with $\mathbf{Z}_N$ statistics, generalizing the models in [7]. Finally we give some examples of local theories with different values $c \neq \bar{c}$ of the left and right central charges.

2. Locality requirements and structure constants.

We consider representations $\hat{F} = (F,\bar{F})$ of $Vir \oplus \overline{Vir}$ with $c=\bar{c}=1 - \frac{6}{m(m+1)}$, m integer, $m \geq 3$, and scaling dimensions $(\Delta_F,\Delta_{\bar{F}})$ given by the Kac determinant formula with $F=(rs)$, $\bar{F}=(r\bar{s})$, $1 \leq r \leq m-1$, $1 \leq s, \bar{s} \leq m$. The primary fields structure constants $\hat{D}^{\hat{F}}_{\hat{A}\hat{C}}$ are determined from the 4-point SL(2,C)-invariant function [9]

$$<\varphi^a_{\hat{A}}(x_1)\varphi^c_{\hat{C}}(x_2)\varphi^a_{\hat{A}}(x_3)\varphi^c_{\hat{C}}(x_4)>=\sum_{\hat{F}} J_F(A,C,A,C;Z)N_{\hat{F}}(\hat{A},\hat{C},\hat{A},\hat{C})J_{\bar{F}}(\bar{A},\bar{C},\bar{A},\bar{C};\bar{Z}) \quad (2.1a)$$

defined for noncoinciding arguments; $Z = (z_1,z_2,z_3,z_4)$, $z_i = x^1_i+ix^2_i, (x^1_i,x^2_i) \in R^2$. The indices a,c distinguish different fields labelled by the same scale dimensions. For simplicity of notation we skip them in the r.h.s. of (2.1).

In a proper basis the conformal block $J_F(A,C,A,C;Z)$ has a simple behaviour, e.g., at $z_{12} \to 0$ (and $z_{34} \to 0$)

$$J_F(A,C,A,C;Z) \approx S_F(A,C,A,C) \frac{z^{\Delta_F-\Delta_A-\Delta_C}}{z_{13}^{2\Delta_A} z_{24}^{2\Delta_C}} [1+O(z)], \qquad (2.1b)$$

$$z = \frac{z_{12}\, z_{34}}{z_{13}\, z_{24}}, \quad z_{ij} = z_i - z_j.$$

For (A,C,F) consistent with the chiral fusion rules one can choose a branch of the multivalued conformal block such that the constants $S_F(A,C,A,C)$ are real [9]. They provide the Dotsenko-Fateev (DF) diagonal theory structure constants $D^F_{AC} = D^F_{CA} = D^A_{FC}$; $D^1_{AA} = 1$; namely

$$(S_F(A,C,A,C))^2 = (D^F_{AC})^2. \qquad (2.2a)$$

Furthermore a direct inspection shows that (choosing the positive sign of the square root appearing in the explicit expression [9] for S_F)

$$S_F(A,C,A,C) > 0 \quad , \quad S_T(A,A,C,C) \geq 0, \qquad (2.2b)$$

where

$$(S_T(A,A,C,C))^2 = S_T(A,A,A,A)\; S_T(C,C,C,C) = D^T_{AA}\; D^T_{CC}. \qquad (2.2c)$$

Hence the structure constants $D^{\hat{F}}_{\hat{A}\hat{C}}$ extracted from (2.1) are expressed by the DF structure constants

$$(D^{\hat{F}}_{\hat{A}\hat{C}})^2 = N^{\hat{F}}_{\hat{A}\hat{C}}\, D^F_{AC}\, D^{\bar{F}}_{\bar{A}\bar{C}}, \quad N^{\hat{F}}_{\hat{A}\hat{C}} = e^{-i\pi s(\hat{F})}\; N_{\hat{F}}(\hat{A},\hat{C},\hat{A},\hat{C}), \qquad (2.3)$$

assuming that all D^F_{AC} can be chosen to be of the same sign (e.g., positive), which is justified at least for A = C.

The phase in (2.3), involving the spin $s(\hat{F})=\Delta_F-\Delta_{\bar{F}}$, accounts for a choice of the 2-point function normalization $d_{\hat{A}\hat{A}} = \exp(i\pi s(\hat{A}))$, consistent with the reflection positivity condition.

The 4-point conformal blocks $\{J_F(A,C,B,D;Z)\}$ span a representation space of the braid group B_4 on the Riemann sphere with generators $\mathbf{B}^{12}$, $\mathbf{B}^{23}$, $\mathbf{B}^{34}$ (see [9-13]); e.g.,

$$J_F(A,C,B,D;z_1,z_2,z_3,z_4) = \sum_T \mathbf{B}^{12}(\varepsilon_{12})_{FT}\, J_T(C,A,B,D;z_2,z_1,z_3,z_4), \qquad (2.4)$$

etc.; ε_{12}=sign (Im z_{12}). We have for the product of the left and right braid transformations $\mathbf{B} \otimes \bar{\mathbf{B}}$ the explicit expressions

$$\mathbf{B}^{12}_{FJ}(\varepsilon_{12})\, \bar{\mathbf{B}}^{12}_{\bar{F}\bar{J}}(\varepsilon_{12}) = e^{-i\pi\varepsilon_{12}[s(\hat{A})+s(\hat{C})-s(\hat{F})]}\, \delta_{FJ}\, \delta_{\bar{F}\bar{J}}, \qquad (2.5a)$$

$$\mathbf{B}^{34}_{FJ}(\varepsilon_{34})\, \bar{\mathbf{B}}^{34}_{\bar{F}\bar{J}}(\varepsilon_{34}) = e^{-i\pi\varepsilon_{34}[s(\hat{B})+s(\hat{D})-s(\hat{F})]}\, \delta_{FJ}\delta_{\bar{F}\bar{J}},$$

$$\mathbf{B}^{23}_{FT}(\varepsilon_{23})\, \bar{\mathbf{B}}^{23}_{\bar{F}\bar{T}}(\varepsilon_{23}) = e^{i\pi\varepsilon_{23}[s(\hat{A})+s(\hat{D})-s(\hat{F})-s(\hat{T})]} \begin{Bmatrix} C & A \\ B & D \end{Bmatrix}_{FT} \begin{Bmatrix} \bar{C} & \bar{A} \\ \bar{B} & \bar{D} \end{Bmatrix}_{\bar{F}\bar{T}} \qquad (2.5b)$$

$$\mathbf{B}^{13} = \mathbf{B}^{12}(\varepsilon_{23})\, \mathbf{B}^{23}(\varepsilon_{13})\, \mathbf{B}^{12}(\varepsilon_{12}) = \mathbf{B}^{23}(\varepsilon_{12})\, \mathbf{B}^{12}(\varepsilon_{13})\, \mathbf{B}^{23}(\varepsilon_{23}), \qquad (2.5c)$$

$$\mathbf{B}^{13}(\varepsilon) \otimes \bar{\mathbf{B}}^{13}(\varepsilon) = e^{i\pi\varepsilon[s(\hat{D})s-(\hat{A})-s(\hat{B})s-(\hat{C})]} \begin{Bmatrix} A & C \\ B & D \end{Bmatrix} \otimes \begin{Bmatrix} \bar{A} & \bar{C} \\ \bar{B} & \bar{D} \end{Bmatrix}, \qquad (2.5d)$$

for $\varepsilon_{12} = \varepsilon_{23} = \varepsilon_{13} = \varepsilon$,
where up to normalization $\begin{Bmatrix} A & C \\ B & D \end{Bmatrix}$ are the crossing matrices introduced in [9]. They satisfy the relations

$$\begin{Bmatrix} A & C \\ B & D \end{Bmatrix} \begin{Bmatrix} A & C \\ B & D \end{Bmatrix}^t = \mathbf{1} = \begin{Bmatrix} A & C \\ B & D \end{Bmatrix} \begin{Bmatrix} B & C \\ A & D \end{Bmatrix}. \qquad (2.6)$$

The first of these equalities holds for a proper normalization of the conformal blocks. In particular, the normalization in (2.1), (2.2) is consistent with it. For an explicit expression for $\begin{Bmatrix} A & C \\ B & D \end{Bmatrix}$ see the next section. The transformations (2.5) can be derived using the explicit expressions for the DF multiple contour integrals.

One has for the correlation function (2.1a)

$$(\mathbf{B}(\varepsilon))^2 \otimes (\bar{\mathbf{B}}(\varepsilon))^2 = \mathbf{1} \otimes \mathbf{1} \text{ iff}$$

$$s(\hat{A}) + s(\hat{C}) - s(\hat{F}) = 0 = 2s(\hat{A}) = 2s(\hat{C}) = s(\hat{T}) \mod 1, \qquad (2.7)$$

in which case the braid group elements do not depend on ε_{ij}.

Accordingly the correlation functions are singled-valued with respect to all coordinate differences. The locality requires further

$$\begin{Bmatrix} A & C \\ A & C \end{Bmatrix}_{TF} N_{F\bar{F}}(\hat{A},\hat{C},\hat{A},\hat{C}) \begin{Bmatrix} \bar{A} & \bar{C} \\ \bar{A} & \bar{C} \end{Bmatrix}_{\bar{F}\bar{T}} = N_{T\bar{T}}(\hat{A},\hat{C},\hat{A},\hat{C}) \ , \tag{2.8a}$$

$$N_{\hat{F}}(\hat{C},\hat{A},\hat{A},\hat{C}) = (-1)^{\ s(\hat{A})+s(\hat{C})-s(\hat{F})}\ p(\hat{A},\hat{C})\ N_{\hat{F}}(\hat{A},\hat{C},\hat{A},\hat{C}) \ , \tag{2.8b}$$

$$N_{\hat{T}}(\hat{A},\hat{A},\hat{C},\hat{C}) = (-1)^{\ s(\hat{T})}\ N_{\hat{T}}(\hat{A},\hat{A},\hat{C},\hat{C}) =$$

$$= p(\hat{A},\hat{C}) \begin{Bmatrix} A & A \\ C & C \end{Bmatrix}_{TF} (-1)^{\ s(\hat{A})+s(\hat{C})-s(\hat{F})}\ N_{F\bar{F}}(\hat{A},\hat{C},\hat{A},\hat{C}) \begin{Bmatrix} \bar{C} & \bar{A} \\ \bar{A} & \bar{C} \end{Bmatrix}_{\bar{F}\bar{T}} \ , \tag{2.8c}$$

where p(A,C) is a ± sign present in the fermion case. Accordingly

$$\hat{D}^{\hat{F}}_{\hat{C}\hat{A}} = p(\hat{A},\hat{C})(-1)^{\ s(\hat{A})+s(\hat{C})-s(\hat{F})}\ \hat{D}^{\hat{F}}_{\hat{A}\hat{C}} \ , \tag{2.9a}$$

$$\hat{D}^{\hat{T}}_{\hat{A}\hat{A}}\hat{D}^{\hat{T}}_{\hat{C}\hat{C}} = N_{\hat{T}}(\hat{A},\hat{A},\hat{C},\hat{C}) \ \sqrt{D^{T}_{AA}\,D^{\bar{T}}_{\bar{A}\bar{A}}}\ \sqrt{D^{T}_{CC}\,D^{\bar{T}}_{\bar{C}\bar{C}}}, \tag{2.9b}$$

if $N_{\hat{T}}(\hat{A},\hat{A},\hat{C},\hat{C}) \neq 0$.

We fix the overall constant in (2.1) by

$$N_1(\hat{A},\hat{A},\hat{C},\hat{C}) \ = \ e^{i\pi[s(\hat{A})+s(\hat{C})]} = \hat{D}^{1}_{\hat{A}\hat{A}}\ \hat{D}^{1}_{\hat{C}\hat{C}} \tag{2.10}$$

in agreement with the choice for $d_{\hat{A}\hat{A}} = \hat{D}^{1}_{\hat{A}\hat{A}}$ in (2.3).

The locality of the 3-point function implies for its normalization constant the relation

$$d_{\hat{A}\hat{C}\hat{F}} = p(\hat{A},\hat{C})\ (-1)^{\ s(\hat{A})+s(\hat{C})-s(\hat{F})}\ d_{\hat{C}\hat{A}\hat{F}} \quad ,\text{etc.}, \tag{2.11}$$

and hence a set of symmetry relations for the structure constants, $\hat{D}^{\hat{F}}_{\hat{A}\hat{C}} = \dfrac{d_{\hat{A}\hat{C}\hat{F}}}{d_{\hat{F}\hat{F}}}$. One of these relations is recovered in (2.9a). Due to the symmetry of the DF constants D^{F}_{AC} it follows that

$$N^{\hat{F}}_{\hat{A}\hat{C}} = N^{\hat{F}}_{\hat{C}\hat{A}} = (-1)^{2s(\hat{C})} N^{\hat{A}}_{\hat{C}\hat{F}} = N^{\hat{F}}_{\hat{A}\hat{C}} . \tag{2.12}$$

The last equality follows from (2.8a,c) and the symmetry (2.6) $\left(\begin{Bmatrix} A & C \\ B & D \end{Bmatrix}^t = \begin{Bmatrix} B & C \\ A & D \end{Bmatrix}\right)$ of the crossing matrix; $\hat{A} \equiv (\bar{A}, A)$.

Given $\hat{A}$, $\hat{C}$, there are in general more than one solutions for $N^{\hat{F}}_{\hat{A}\hat{C}}$ of eq. (2.8a). They correspond to different field interpretations of the 4-point function, which is accounted for in the notation in (2.1a).

3. Relations for the fusion matrices.

The fusion matrices $\begin{Bmatrix} A & C \\ B & D \end{Bmatrix}$ have the factorized form (see [9] for notation)

$$\begin{Bmatrix} A & C \\ B & D \end{Bmatrix}_{JT} = \underset{\sim}{\alpha}^{(s')}_{j't'} (a',b',c';\rho') \ \underset{\sim}{\alpha}^{(s)}_{jt}(a,b,c;\rho) \tag{3.1}$$

where $\rho = \frac{m}{m+1}$, $\rho' = \frac{1}{\rho}$, $a=(1-s_A)\rho+r_A-1$, $a'=-\rho'a$, $A=(r_A s_A)$, etc., and $J=J(a,c,j,j')$, $T=T(b,c,t,t')$, $D=D(a,b,c;s,s')$, are computed according to the formulae in [9]. The factorization (3.1) reflects the underlying $\widehat{su}(2)\mathrm{x}\widehat{su}(2)$ structure of the minimal models. The matrices $\alpha^{(s)}(a,b,c;\rho)$ satisfy a relation ((A.13) in [5a]) which in terms of $\begin{Bmatrix} A & C \\ B & D \end{Bmatrix}$ reads

$$\sum_K \begin{Bmatrix} C & A \\ B & D \end{Bmatrix}_{JK} e^{-i\pi\varepsilon\Delta_K(A,B)} \begin{Bmatrix} A & B \\ C & D \end{Bmatrix}_{KT} =$$

$$= e^{-i\pi\varepsilon(\Delta_A+\Delta_B+\Delta_C+\Delta_D)} e^{i\pi\varepsilon[\Delta_J(A,C)+\Delta_T(C,B)]} \begin{Bmatrix} A & C \\ B & D \end{Bmatrix}_{JT}, \ \varepsilon = \pm 1, \tag{3.2}$$

and is equivalent to the Yang-Baxter equation (2.5c); $\Delta_K(A,B)$ indicates that K appears in the product of A and B. (Such a relation appears in a more general framework in [13]). Comparing (A. 13) in [5a] and the q-analog of the Racah identity (6.17) in [2] (with $C_j \rightarrow C_j/2$ everywhere) we obtain the exact relation between the crossing matrix $\underset{\sim}{\alpha}^{(s)}(a,b,c;\rho)$ and the quantum 6j-symbols for $q = \exp(2\pi i \rho)$

$$\underset{\sim}{\alpha}^{(s)}_{jk}(a,b,c,\rho) = (-1)^{(j-1)(1+r_2+r_3)+(k-1)(1+r_1+r_2)+(s-1)(r_1+r_2+r_3)}$$

$$\cdot \begin{Bmatrix} j_1(a) & j_2(c) & j_5(a,c,j) \\ j_3(b) & j_4(d) & j_6(b,c,k) \end{Bmatrix}_q \tag{3.3a}$$

with

$$\begin{Bmatrix} j_1 & j_2 & j_5 \\ j_3 & j_4 & j_6 \end{Bmatrix}_q = \frac{\sqrt{s[(2j_5+1)\rho]s[(2j_6+1)\rho]}}{s[(1)\rho]} (-1)^{j_1+j_2-j_3-j_4-2j_5} \begin{Bmatrix} j_1 & j_2 & j_5 \\ j_3 & j_4 & j_6 \end{Bmatrix}_q^{RW} \qquad (3.3b)$$

denoting $s(x)=\sin(\pi x)$. Here $2j_j+1 = s_j, j = 1,2,3,4$, and

$$j_5(a,c,j) = j_1+j_2-j+1,\ j_6(b,c,k) = j_3+j_2-k+1 \qquad (3.4a)$$

if a, c, b are given by $(1-s_i)\rho + r_i-1$, $i = 1, 2, 3$ respectively, and $d = (1+s_4)\rho - r_4-1$; $s = j_1+j_2+j_3-j_4+1$, or

$$j_5(a,c,j) = j_2-j_1+j-1,\ j_6(b,c,k) = j_2-j_3+k-1, \qquad (3.4b)$$

if a, b, d are given by $(1-s_i)\rho + r_i-1$, $i = 1, 3, 4$ respectively, and $c = (1+s_2)\rho - r_2-1$; $s = j_1+j_3+j_4-j_2+1$, etc..

Using the explicit expression for the Racah-Wigner 6j-symbols $\begin{Bmatrix} jjj \\ jjj \end{Bmatrix}_q^{RW}$ in [2] one gets for the original matrix α of [9]

$$\alpha_{jk}^{(s)}(a,b,c;\rho) = \sum_p \frac{\prod_0^{s-p-1} s(1+a+(j-1+i)\rho) \prod_0^{p-k-1} s(1+d+(s-j+i)\rho)}{\prod_0^{s-k-1} s(a+d+(s-k-1+i)\rho)} \qquad (3.5)$$

$$\cdot \frac{\prod_0^{j+k-p-2} s(1+b+(s-j+i)\rho) \prod_0^{p-j-1} s(1+c+(j-1+i)\rho)}{\prod_0^{k-2} s(b+c+(k-2+i)\rho)}$$

$$\cdot \prod_1^{p-j} \frac{s((k-p+j-1+i)\rho)}{s(i\rho)} \prod_1^{s-p} \frac{s((p-k+i)\rho)}{s(i\rho)} ,$$

$$\underset{\sim}{\alpha}_{jk}^{(s)}(a,b,c;\rho) = \sqrt{\frac{X_j^{(s)}(a,b,c;\rho)}{X_k^{(s)}(b,a,c;\rho)}}\ \alpha_{jk}^{(s)}(a,b,c;\rho).$$

The sum in (3.5) runs between max(j,k) and min(s,j+k-1). For j=s or k=s this formula recovers the expressions found in [9]. The constants $X_j^{(s)}(a,b,c;\rho)$, given explicitly in [9], modify the first equality for α in (2.6). Due to the symmetry [5a]:

$$\alpha_{jk}^{(s)}(a,b,c;\rho) = \alpha_{s+1-j\ s+1-k}^{(s)}(b,a,d;\rho) = \alpha_{s+1-j\ k}^{(s)}(d,c,b;\rho) = \alpha_{j\ s+1-k}^{(s)}(c,d,a;\rho) \quad (3.6)$$

which holds for $\underset{\sim}{\alpha}$ as well, it is enough to use, say (3.4a), to recover (3.5).

The relations (3.6) are examples of the symmetries satisfied by the 6j-symbols, all of which are listed in [2]. When m+1-even (or, for q=exp(i2πρ'), m-even), there are more relations:

$$\begin{Bmatrix} j_1 & j_2 & j_5 \\ j_3 & j_4 & j_6 \end{Bmatrix}_q = (-1)^{(j_2+j_4-j_5-j_6)} \begin{Bmatrix} j_1 & \bar{j}_2 & \bar{j}_5 \\ j_3 & \bar{j}_4 & j_6 \end{Bmatrix}_q = \quad (3.7a)$$

$$= (-1)^{(j_2+j_3-j_6)} \begin{Bmatrix} \bar{j}_1 & j_2 & \bar{j}_5 \\ j_3 & \bar{j}_4 & j_6 \end{Bmatrix}_q = (-1)^{(j_1+j_2-j_5)} \begin{Bmatrix} \bar{j}_1 & \bar{j}_2 & j_5 \\ \bar{j}_3 & \bar{j}_4 & j_6 \end{Bmatrix}_q ,$$

$$\begin{Bmatrix} j_1 & j_2 & j_5 \\ j_3 & j_4 & j_6 \end{Bmatrix}_q = (-1)^{(j_1+j_2+j_3+j_4)} \begin{Bmatrix} \bar{j}_1 & \bar{j}_2 & j_5 \\ \bar{j}_3 & \bar{j}_4 & j_6 \end{Bmatrix}_q , \quad (3.7b)$$

$\bar{j}_i = -j_i - 1 + \frac{m+1}{2}$.

The first set of these relations has been derived in [5a]. In terms of $\underset{\sim}{\alpha}$ it says that $\underset{\sim}{\alpha}^{(s)}(a,b,c;\rho)$ is invariant up to a sign whenever some of the parameters a,b,c change by an (odd) integer. The exact form of these sign factors (the same for α and $\underset{\sim}{\alpha}$) found in [5a] are now manifest, given the explicit expression (3.5). The second relation (3.7b) is a consequence of (3.7a) and of the standard crossing symmetry relations of the 6j- symbols. In terms of $\underset{\sim}{\alpha}$ it reads

$$\underset{\sim}{\alpha}_{\underline{jk}}^{(m+1-s)}(\underline{a},\underline{b},\underline{c};\rho) = (-1)^{(s_1+s_2+s_3+s_4)/2} \underset{\sim}{\alpha}_{jk}^{(s)}(a,b,c;\rho) , \quad (3.7c)$$

where $\underline{a} = a - (m+1-2s_1)\rho$, etc., $\underline{j} = j + m+1-s_1-s_2$; $\underline{k} = k+m+1-s_3-s_2$. Finally we rewrite (3.7) for the full crossing matrix in the cases which will be used below:

$$\begin{Bmatrix} A & C \\ A & C \end{Bmatrix}_{FT} = (-1)^{2\,l(C)-l(F)-l(T)} \begin{Bmatrix} A & \sigma(C) \\ A & \sigma(C) \end{Bmatrix}_{\sigma(F)\sigma(T)}$$

$$= (-1)^{2\,l(A)+2l(C)} \begin{Bmatrix} \sigma(A) & \sigma(C) \\ \sigma(A) & \sigma(C) \end{Bmatrix}_{FT} , \quad (3.8)$$

$$= (-1)^{(s_T-1)/2}\begin{Bmatrix} \sigma(C) & A \\ A & \sigma(C) \end{Bmatrix}_{\sigma(F)T} = (-1)^{(s_F-1)/2}\begin{Bmatrix} A & A \\ \sigma(C) & \sigma(C) \end{Bmatrix}_{F\sigma(T)},$$

$\sigma(A) = (r_A\ m+1-s_A)$, $A = (r_A, s_A)$; $l(A) = \Delta_A - \Delta_{\sigma(A)}$.

It should be stressed that neither the quantum Clebsch-Gordan coefficients, nor the 6j-symbols are necessarily zero for spins inconsistent with the rational theory fusion rules. Nevertheless it can be shown that all identities for the 6j-symbols reduce for $q^N=1$ to identities consistent with the fusion rules. To do that one needs new relations for the q-6j-symbols (for pairs $(j, \underset{\sim}{j})$, $j + \underset{\sim}{j} + 1 = k(m+1)$, extending the list in (3.7) [14].

4. Local and quasilocal (A,D) type series.

Let $\hat{A}=(A,A)$, $\hat{C}=(C,\sigma(C))$. Then $l(C)=s(\hat{C})$ in (3.8) and the locality conditions (2.7,8) have a solution for $\hat{F}=(F, \sigma(F))$, $\hat{T}=(T,T)$, $(s_T-1=2s_A=0 \bmod 2)$

$$N_{\hat{F}}(\hat{A},\hat{C},\hat{A},\hat{C}) = e^{i\pi s(\hat{F})}\delta_{\bar{F}\sigma(F)},\ N_{\hat{T}}(\hat{A},\hat{A},\hat{C},\hat{C}) = e^{i\pi s(\hat{C})}(-1)^{(s_T-1)/2}\delta_{T\bar{T}} \qquad (4.1a)$$

$$s_C + s_F = 0 = s_A - 1 \mod 2 . \qquad (4.1b)$$

Similarly for $\hat{A}=(A, \sigma(A))$, $\hat{C}=(C, \sigma(C))$, $s(\hat{F})=0=s(\hat{T})$ one finds

$$N_{\hat{F}}(\hat{A},\hat{C},\hat{A},\hat{C}) = p(\hat{A},\hat{C})\delta_{\bar{F}F},\ N_{\hat{T}}(\hat{A},\hat{A},\hat{C},\hat{C}) = e^{i\pi(s(\hat{C})+s(\hat{A}))}\delta_{\bar{T}T} \qquad (4.2a)$$

$$s_C + s_A = 0 = s_F - 1 \mod 2 . \qquad (4.2b)$$

If $\sigma(A)=A$ (or $\sigma(C)=C$) the second formula (4.2) is valid only for $s(\hat{C})$ (respectively $\sigma(\hat{A})$) integer. Similarly (4.1) holds for $\sigma(C) = C$ only if $s(\hat{F})$ is integer. Hence, in the fermionic case we have $s(\hat{A}) = s(\hat{C}) = \frac{1}{2}$ mod 1 in (4.2) and $s(\hat{C}) = s(\hat{F}) = \frac{1}{2}$ mod 1 in (4.1). We recover the local fermionic fusion algebras of [5], to be denoted

$$\mathbb{A}^{f} = \mathbb{A}_{o}^{odd} \oplus \mathbb{A}_{1}^{f},\quad \mathbb{A}_i \times \mathbb{A}_j = \mathbb{A}_{i+j},\ i, j = 0,1 \mod 2$$

$$\mathbb{A}_{o}^{odd} = \{\hat{A} = (A,A) : A = (r_A\ s_A),\ s_A \text{ - odd}\} \qquad (4.3)$$

$$\mathbb{A}_{1}^{f} = \{\hat{C} = (C, \sigma(C)) : C = (r_C\ s_C),\ s_C = \frac{m-1}{2} \mod 2\}$$

with structure constants given by (2.3) and (4.1,2). In particular, identifying $\mathcal{A}_0^{odd}$ with the corresponding subalgebra of the DF (A,A) series, we have $D^{\hat{T}}_{\hat{A}\hat{A}} = D^{T}_{AA}$ and hence (4.1a) determines not only $(D^{\hat{T}}_{\hat{C}\hat{C}})^2$ but $D^{\hat{T}}_{\hat{C}\hat{C}}$ itself.

In the bosonic case the fusion algebra $\mathcal{A}^b$ has the same $\mathbf{Z}_2$-graded structure [5] with the same subalgebra $\mathcal{A}_0^{odd}$ as in (4.3) and with $\mathcal{A}_1^b$ given by

$$\mathcal{A}^{\,b}_{\,1} = \{\hat{C} = (C,\sigma(C)),\ C = (r_C\, s_C),\ s_C = \frac{m+1}{2} \bmod 2\} \tag{4.4}$$

This subspace includes the scalars $\varphi^{(1)}_{\hat{C}}, \hat{C}=(C,\sigma(C)=C)$ which for m+1=6 mod 4 (the (A,D_{even}) series in the notation of [4]) have the same dimensions Δ_C as the DF fields $\varphi^{(o)}_{\hat{C}}$ in $\mathcal{A}_0^{odd}$. Indeed (4.1a) and (4.1b) for $\sigma(A)=A$ provide two different solutions. The fusion algebra $\mathcal{A}^b$ for m+1=6 mod 4 can be recast in another, equivalent form, using linear combinations of $\varphi^{(o)}_{\hat{A}}$ and $\varphi^{(1)}_{\hat{A}}$, different for m+1=6 mod 8 and m+1=10 mod 8 [5b] (see also [15]).

Both $\mathcal{A}^b$ and the local fermionic algebra $\mathcal{A}^f$ can be realized as local subalgebras of a bigger algebra, in which the fields obey in general a $\mathbf{Z}_2$ type statistics. The corresponding mixed correlation functions (2.1) have the property that

$$(\mathbf{B}(\varepsilon))^2 \otimes (\bar{\mathbf{B}}(\varepsilon))^2 = \pm\ \mathbf{1} \otimes \mathbf{1} \tag{4.5}$$

i.e., they can be either single-valued (with respect to z_{13}, z_{24}) or double-valued (with respect to z_{12}, z_{14}, z_{23},z_{34}). The conditions in (2.7) are replaced by

$$2s(\hat{A})+2s(\hat{C})-2s(\hat{F}) = 1 \bmod 2,\quad 2s(\hat{A}) = 2s(\hat{C}) = s(\hat{T}) = 0 \bmod 1 \tag{4.6}$$

and $p(\hat{A},\hat{C}) = 1$. The equations replacing (2.8) are obtained by multiplying the r.h.s. of (2.8a), (2.8b) and the second equality of (2.8c) by (-1), ($\mp\varepsilon_{12}$) and ($\pm\varepsilon_{23}$) respectively (cf. (2.5c)). (This means, in particular, that the correlation function (2.1a) is antisymmetric in x_1, x_3, if $\varepsilon_{12} = \varepsilon_{23}$).

One obtains the same solutions for the constants $\pm N_{\hat{F}}(\hat{A},\hat{C},\hat{A},\hat{C})$, $N_{\hat{T}}(\hat{A},\hat{A},\hat{C},\hat{C})$ (and $\mp N_{\hat{F}}(\hat{A},\hat{C},\hat{A},\hat{C})$, $N_{\hat{T}}(\hat{A},\hat{A},\hat{C},\hat{C})$, with $p(\hat{A},\hat{C}) = 1$) as in (4.1a) ((4.2a), respectively) but with the conditions (4.1b), (4.2b) replaced by (cf. (4.6))

$$s(\hat{A}) = 0,\ 2s(\hat{C})+2s(\hat{F}) = 1 \bmod 2, \text{ i.e., } \quad s_C+s_F+1 = 0 = s_A \bmod 2, \tag{4.1b'}$$

and

$$s(\hat{F}) = 0,\ 2s(\hat{A})+2s(\hat{C}) = 1 \quad \text{mod } 2,\ \text{ i.e., }\ s_A+s_C+1 = 0 = s_F \quad \text{mod } 2, \qquad (4.2b')$$

respectively.

If in (4.2a), (4.2b'), $s(\hat{A})=0$, i.e, $\sigma(A)=A$, which now happens for m+1=4 mod 4, the formulae (4.1a) (for $2s(\hat{C})=1$ mod 2) and (4.2a) provide two different solutions of the modified analogs of (2.8), reflecting the doubling of the scalar fields labelled by $\hat{A} = (A,\sigma(A) = A)$. Unlike the previous case, the two scalar fields, to be denoted also $\varphi^{(o)}_{\hat{A}}$ and $\varphi^{(1)}_{\hat{A}}$, are not mutually local. In particular (4.1a), (4.1b') for $\hat{C}=(C,\sigma(C)=C)$ describe the product $\varphi^{(o)}_{\hat{A}}\,\varphi^{(1)}_{\hat{C}}$ creating fermions and, in the other channel, the products $\varphi^{(o)}_{\hat{A}}\,\varphi^{(o)}_{\hat{A}}$, $\varphi^{(1)}_{\hat{C}}\,\varphi^{(1)}_{\hat{C}}$, producing scalars with structure constants, differing in general by a sign. The scalars $\varphi^{(o)}_{\hat{A}}$ and $\varphi^{(1)}_{\hat{A}}$ generalize for arbitrary c(m) the Ising model order, disorder fields σ, μ.

The field content of the quasilocal algebra $\mathcal{A}^{(\mathbf{Z}_2)}$ for any c(m) is described by the left-right combinations provided by the corresponding (A,A), (A,D) invariants and the invariant in [6]. The fusion rules in (4.3) are completed by

$$\begin{aligned} \mathcal{A}_o^{\text{even}} \times \mathcal{A}_1^{b(f)} &= \mathcal{A}_1^{f(b)} \\ \mathcal{A}_1^{f} \times \mathcal{A}_1^{b} &= \mathcal{A}_o^{\text{even}} \\ \mathcal{A}_o^{\text{even}} &= \{\hat{A} = (A,A),\ A = (r_A,s_A),\ s_A\text{-even}\} \end{aligned} \qquad (4.7)$$

and the fields in $\mathcal{A}_o^{\text{even}}$ can be identified with the DF scalar fields. Hence the $\mathbf{Z}_2$-graded structure of the full operator algebra $\mathcal{A}^{(\mathbf{Z}_2)}$ is preserved with $\mathcal{A}_o = \mathcal{A}_o^{\text{odd}} \oplus \mathcal{A}_o^{\text{even}}$ and $\mathcal{A}_1 = \mathcal{A}_1^{b} \oplus \mathcal{A}_1^{f}$. The subalgebra $\mathcal{A}_o$ coincides with the DF scalar algebra, corresponding to the (A,A) modular invariant.

There are more submodular invariants on the torus, connected with models with $\mathbf{Z}_2$ twisted boundary conditions, besides the one found in [6] (see Appendix A). In particular there is a new submodular invariant for any m+1=4 mod 4. However, this invariant is a linear combination of the invariant in [6] and the (A,A), (A,D) modular invariants. Accordingly, the corresponding field algebra does not differ from $\mathcal{A}^{(\mathbf{Z}_2)}$ for m+1=4 mod 4.

All (A,D) type correlation functions discussed here (and in [5]) are consistent with the chiral fusion rules iff the same holds for the corresponding pairs of diagonal DF correlations. To show this one has to use again the relations (3.8) for the fusion matrices[1].

5. Local and quasilocal (A,E_6) type algebras.

Before considering the (A,E_6) examples let us first extend the result in (4.1,2) using the symmetry relations (3.8). The equations (c.f. (2.8), (2.10))

$$\begin{Bmatrix} A & C \\ A & C \end{Bmatrix}_{TF} e^{i\pi s(\hat{F})} N^{\hat{F}}_{\hat{A}\hat{C}} \begin{Bmatrix} \bar{A} & \bar{C} \\ \bar{A} & \bar{C} \end{Bmatrix}_{\bar{F}\bar{T}} = e^{i\pi s(\hat{T})} N^{\hat{T}}_{\hat{A}\hat{C}} \tag{5.1}$$

$$1 = p(\hat{A},\hat{C}) \begin{Bmatrix} A & C \\ A & C \end{Bmatrix}_{1F} N^{\hat{F}}_{\hat{A}\hat{C}} \begin{Bmatrix} \bar{C} & \bar{A} \\ \bar{A} & \bar{C} \end{Bmatrix}_{\bar{F}1}$$

are invariant under $\hat{C} \to (\sigma(C),\bar{C})$, $\hat{F} \to (\sigma(F),\bar{F})$, $\hat{T} \to (\sigma(T),\bar{T})$ (or $\hat{A} \to (\sigma(A),\bar{A})$, $\hat{C} \to (\sigma(C),\bar{C})$ with

$$N^{(\sigma(F),\bar{F})}_{A(\sigma(C),\bar{C})} = p(\hat{A},\hat{C})\, p(\hat{A},(\sigma(C),\bar{C})) N^{\hat{F}}_{\hat{A}\hat{C}} \,, \tag{5.2a}$$

iff $s_C + s_F = 0 \mod 2$, or

$$N^{\hat{F}}_{(\sigma(A),\bar{A})(\sigma(C),\bar{C})} = p(\hat{A},\, \hat{C})\, p((\sigma(A),\bar{A}),(\sigma(C),\bar{C})) N^{\hat{F}}_{\hat{A}\hat{C}} \tag{5.2b}$$

iff ,$s_A + s_C = 0 \mod 2$. Furthermore, one has

$$N^{\hat{F}}_{\hat{A}\hat{C}} = N^{\sigma(\hat{F})}_{\hat{A}\sigma(\hat{C})} = N^{\hat{F}}_{\sigma(\hat{A})\sigma(\hat{C})} \tag{5.3}$$

$\sigma(\hat{F}) = (\sigma(F), \sigma(\bar{F}))$, for $s_A - s_{\bar{A}} = 0 \mod 2$, which is the case for all A-D-E spin combinations, as well as, for the other examples under consideration.

The relations (5.2) generalize (4.1,2) and reduce to them for $s(\hat{A})=0$ or $s(\hat{F})=0$ respectively. Whenever the transformation in (5.2) leaves invariant the local (A,E) type algebra (as is always the case for (A,E_7), (A,E_8)) it can be used along with (2.12), (5.3) to find all structure constants, given a minimal set of independent ones.

Let us consider the local field algebra corresponding to the (A,E_6) modular invariant. Using the explicit expression for the crossing matrix in section 3, we look for a solution of eqs. (2.8) consistent with the spin combinations of the (A,E_6) invariant. We find the following results.

The fusion algebra is consistent with the Ising type fusion rules of the extended chiral algebra [17]. All constants $N^{\hat{F}}_{\hat{A}\hat{C}}$ are non negative (and hence $\hat{D}^{\hat{F}}_{\hat{A}\hat{C}}$ are real) and $N^{(1,1)}_{\hat{A}\hat{A}}=1$;

(1,1) = ((r 1), (r 1)). The independent nonzero constants are given by (2.3) with

$$N^{(5,5)}_{(5,11)(1,7)} = 1,\ N^{(1,7)}_{(5,5)(5,5)} = \sqrt{2} = N^{(7,1)}_{(5,11)(5,11)}$$

$$N^{(7,7)}_{(5,5)(5,5)} = 2,\ N^{(1,7)}_{(4,4)(4,4)} = \frac{1}{\sqrt{2}},\ N^{(5,11)}_{(4,4)(4,8)} = \sqrt{\frac{3}{2}}, \tag{5.4a}$$

$$N^{(5,5)}_{(4,4)(4,4)} = \frac{3}{2},\ N^{(7,7)}_{(4,4)(4,4)} = \frac{1}{2},\ N^{(4,8)}_{(4,4)(5,5)} = \frac{\sqrt{3}}{2}.$$

Here, for example, (1,7) stands for $\hat{A} = (A,\bar{A})$ with $A = (r_A\ 1)$, $\bar{A} = (r_A\ 7)$, r_A - arbitrary; any triple r_A, r_C, r_F is consistent with the chiral fusion rules. All the rest nonzero constants are recovered applying the symmetry relations (5.3), (2.12) and (5.2) for combinations consistent with the content of the (A,E_6) series. We stress again that the sign factors in front of the odd spin contributions in the correlation functions are included in the expansion coefficients $N_{\hat{F}}(\hat{A},\hat{C},\hat{A},\hat{C}) = (-1)^{s(\hat{F})} N^{\hat{F}}_{\hat{A}\hat{C}}$ (cf. (2.1a), (2.3)). Note that, unlike the (A,D) case, here and for all exceptional (A,E) series as well, the factor $(-1)^{s(\hat{F})}$ depends in general not only on s_F, $s_{\bar{F}}$ but also on r_F of $F = (r_F s_F)$, $\bar{F} = (r_F s_{\bar{F}})$.

The solution of eqs. (2.8a,c) gives also information on the relative sign of some of the structure constants. Assuming, e.g., $\hat{D}^{\hat{F}}_{(4,4)(4,4)} > 0$, we get

$$\hat{D}^{\hat{F}}_{(5,5)(5,5)} > 0,\ \hat{D}^{\hat{F}}_{(4,8)(4,8)} < 0, \tag{5.4b}$$

$$\hat{D}^{(7,1)}_{(5,11)(5,11)} = \hat{D}^{(7,1)}_{(7,1)(7,1)} = \hat{D}^{(7,1)}_{(5,5)(5,5)} = \hat{D}^{(7,1)}_{(7,7)(7,7)},$$

for all $\hat{F}$ consistent with the fusion rules implied by (5.4a).

Note that the constants $N^{\hat{F}}_{\hat{A}\hat{C}}$ in (5.4a) have factorized form $N^{\hat{F}}_{\hat{A}\hat{C}} = N^{F}_{AC}\, N^{\bar{F}}_{\bar{A}\bar{C}}$ with $N^{1}_{AA} = 1$ and three independent constants

$$N^{7}_{55} = \sqrt{2},\ N^{8}_{45} = \frac{1}{\sqrt{2}},\ N^{4}_{45} = \sqrt{\frac{3}{2}}. \tag{5.4c}$$

As in the (A,D) case we can associate naturally with the (A,E_6) type local bosonic algebra also a local algebra including fermions. While in the previous section the structure constants of the local fermionic algebra were recovered by the structure constants of the (A,A) series, using the symmetries of the crossing matrix, here, using (5.2), we get these constants from the structure constants of the bosonic local (A,E_6) type algebra.

The local fermionic operator algebra $\mathcal{A}^f = \mathcal{A}_0 \oplus \mathcal{A}_1^f$ consists of

$$\mathcal{A}_0 = \{(1,1), (5,5), (7,7), (11,11), (5,11), (11,5), (7,1), (1,7)\}$$
$$\mathcal{A}_1^f = \{(1,5), (7,5), (1,11), (7,11), (5,1), (5,7), (11,1), (11,7)\} \qquad (5.5)$$

where the subalgebra $\mathcal{A}_0$ is common for $\mathcal{A}^f$ and the (A,E_6) local bosonic algebra described by the structure constants in (5.4). The independent structure constants of $\mathcal{A}^f$ are given by

$$N^{(1,5)}_{(5,5)(5,1)} = 1,\ N^{(5,7)}_{(1,7)(5,7)} = \sqrt{2} = N^{(5,1)}_{(7,1)(5,1)},\ N^{(5,7)}_{(7,7)(5,7)} = 2,$$
$$N^{(1,1)}_{\hat{A}\hat{A}} = p(\hat{A},\hat{A}),\ \text{for all}\ \hat{A}\in \mathcal{A}^f. \qquad (5.6)$$

Finally replacing the conditions in (5.2) by $s_C+s_F = 1 \mod 2$ (or $s_A+s_C = 1 \mod 2$, respectively) we can convert the equations (5.1) to the modified equations, characterizing the $\mathbf{Z}_2$ quasilocal theory. Accordingly its structure constants are expressed through the constants of the (A,E_6) type local algebra. In this big quasilocal algebra the fields (A,A), (σ(A), σ(A)), (A,σ(A)), (σ(A),A) with $A = (r_A 4)$ appear twice. Each of these two sets, to be denoted $\mathcal{A}_1^+$, $\mathcal{A}_1^-$ consists of mutually local fields. The mixed correlation functions, as well as, the functions with $\varphi_A\in\mathcal{A}_1^{\pm}$, $\varphi_C\in \mathcal{A}_1^f$ are double-valued with respect to the corresponding coordinate differences as in the previous section. The operator product rules give

$$\mathcal{A}_1^{-(+)} \times \mathcal{A}_1^f = \mathcal{A}_1^{+(-)},\ \mathcal{A}_1^- \times \mathcal{A}_1^+ = \mathcal{A}_1^f,$$
$$\mathcal{A}_0 \times \mathcal{A}_1^{\pm(f)} = \mathcal{A}_1^{\pm(f)},\ \mathcal{A}_1^{\pm(f)} \times \mathcal{A}_1^{\pm(f)} = \mathcal{A}_0. \qquad (5.7)$$

Either subalgebra $\mathcal{A}_0 \oplus \mathcal{A}_1^+$ or $\mathcal{A}_0 \oplus \mathcal{A}_1^-$ can be identified with the local algebra corresponding to the (A,E_6) modular invariant.

The structure constants are computed, as described above, with $N^{\hat{F}}_{\hat{A}\hat{C}}$ running (up to a sign) along the set of values in (5.4). One has, e.g.,

$$N^{(5,1)}_{(4,8)^{\mp}(4,8)^{\pm}} = \pm N^{(5,11)}_{(4,8)\,(4,4)} = \pm\sqrt{\tfrac{3}{2}},\quad \text{etc..}$$

The field content of the $\mathbf{Z}_2$ quasilocal algebra is exhausted by the spin combinations of the (A,E_6) modular invariant and the new submodular invariant found for $m+1=12$ in Appendix A. In Appendix B we discuss some examples of local theories with different left and right central charges, $c\neq\bar{c}$.

When this work was completed we received a recent preprint [18] where the WZW local E_6-algebra structure constants have been computed, exploiting in addition some properties of the extended WZW algebra. Our results for the modifications $N_{\hat{A}\hat{C}}^{\hat{F}}$ (5.4a) of the squares of the structure constants agree. An explicit expression for α (looking somewhat more complicated than (3.5)) ha been independently derived in [18], solving recurent relations. Nondiagonal WZW models were also considered in [19], [20], [21]. The results of [21], concerning the WZW D series, are in agreement with [5]. The paper [19], addressing all nondiagonal cases, is based on a wrong, in general, assumption about the symmetries of the fusion matrices. The correct statement, relevant for the (A,D) series [5a], is provided by (3.8)[2)].

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Appendix A. Submodular invariant partition functions.

One can consider models with Z_2 twisted boundary conditions. The partition function with antiperiodic boundary conditions will not be invariant under the full modular group Γ but only under the subgroup

$$\Gamma_o(2) = \{A \in \Gamma : A = \begin{pmatrix} \pm 1 & 0 \\ * & \pm 1 \end{pmatrix} \pmod 2\}$$

generated by T^2: $\tau \to \tau + 2$ and TST: $\tau \to \tau / (\tau + 1)$ (see [6] for details).

The modular invariant partition function for the minimal models

$$Z_{conf} = \frac{1}{2} \sum_{r,\bar{r}=1}^{m-1} \sum_{s,\bar{s}=1}^{m} \mathcal{N}^{(m)}_{r\bar{r}} \mathcal{N}^{(m+1)}_{s\bar{s}} \chi^{conf}_{rs} \chi^{conf*}_{\bar{r}\bar{s}}$$

is written [22] in terms of a pair of affine invariants

$$Z^{aff}_{k} = \sum_{l,\bar{l}=1}^{n-1} \mathcal{N}^{(n)}_{l\bar{l}} \chi^{aff}_{l;k} \chi^{aff*}_{\bar{l};k} ,$$

of level k=n-2, with n=m and n=m+1. The affine characters are proportional to $(\Theta_{l,n}-\Theta_{-l,n})$ where $\Theta_{l,n}$ are order 2n classical Θ-functions. On the Θ-functions acts an unitary representation of Γ with matrix elements of T and S given (up to a phase) by $T_{rs} = \exp(\frac{\pi i r^2}{2n}) \delta_{rs}$ and $S_{rs} = \frac{\exp(\pi i rs/n)}{\sqrt{2n}}$.

Let $M_{r\bar{r}}$ be a modular invariant, i.e. $M|_T = T^+MT = M$ and $M|_S = S^+MS = M$. Define $\underset{\sim}{M}_{r\bar{r}} = M_{r,n-\bar{r}}$. We immediately check that $\underset{\sim}{M}|_{T\, r\bar{r}} = \exp[\pi i(n/2-\bar{r})]\, \underset{\sim}{M}_{r\bar{r}}$ from where we get $\underset{\sim}{M}|_{TST} = \underset{\sim}{M}$ and $\underset{\sim}{M}|_{T^2} = (-1)^n \underset{\sim}{M}$. Thus if M is a Γ invariant and n is even then the so defined $\underset{\sim}{M}$ is a Γ_o (2) invariant. The Γ_o (2) submodular invariants one gets from the celebrated A-D-E classification [4] of the affine invariants, by the above anzatz, are

$$Z(\underset{\sim}{A}_{n-1}) = \sum_{j=1}^{n-1} \chi_j \chi^*_{n-j} , \quad Z(\underset{\sim}{D}_{(n/2)+1}) = \sum_{j=1 \text{ odd}}^{n-1} \chi_j \chi^*_{n-j} + \sum_{j=2 \text{ even}}^{n-2} |\chi_j|^2 , \tag{A.1}$$

$$Z(\underset{\sim}{E}_6) = (\chi_1 + \chi_7)(\chi^*_5 + \chi^*_{11}) + \text{c.c.} + |\chi_4 + \chi_8|^2$$

where n is even and in the case of $Z(\underset{\sim}{D})$ we have n=0 (mod 4). Note that when n=2 (mod 4) $Z(\underset{\sim}{D}) = Z(D)$ and $Z(\underset{\sim}{E}_7) = Z(E_7)$, $Z(\underset{\sim}{E}_8) = Z(E_8)$. We remark that $Z(\underset{\sim}{D}_{(n/2)+1}) = Z(A_{n-1}) + Z(\underset{\sim}{A}_{n-1}) - Z(D_{(n/2)+1})$ but inspite of the minus sign in the above expression, when written in terms of characters only nonnegative coefficients appears.

Expressing the conformal invariants by a pair of affine invariants one gets: $Z(\tilde{A}_{m-1},A_m)$ if m is even and $Z(A_{m-1},\tilde{A}_m)$ if m is odd (these were found in [6]); $Z(\tilde{D}_{(m/2)+1},A_m)$ for m=0 (mod 4); $Z(A_{m-1},D_{(m+3)/2})$ for m+1=0 (mod 4); $Z(A_{10},\tilde{E}_6)$ and $Z(\tilde{E}_6,A_{12})$.

Appendix B. Examples of local theories with $c \neq \bar{c}$.

The identity

$$\begin{Bmatrix} A & A \\ A & A \end{Bmatrix}_{1J} + \sqrt{2}\begin{Bmatrix} A & A \\ A & A \end{Bmatrix}_{\sigma(A)J} = \delta_{1J} + \sqrt{2}\,\delta_{\sigma(A)J} \tag{B.1}$$

satisfied by the fusion matrix $\{\begin{smallmatrix} A & A \\ A & A \end{smallmatrix}\}$ for A=(15), $J=(1s_J)$, and a similar identity for A=(17) (with $\sigma(A)$ replaced by A in (B.1)), can be used to build examples of local fields transforming as primary fields of $Vir \oplus \overline{Vir}$ with different values of the central charges, $c \neq \bar{c}$. (The simplest example is provided by $(\Delta_{1m},\Delta_{1m'})$ with $c=c(m)$, $\bar{c}=c(m')$ and m=m' mod 4.)

Let $c=c(11)$, $\bar{c} = c(m)$. Consider the 4-point function of $(\Delta_{17},\Delta_{1m})$ with m+1=6 mod 4, which satyisfy the locality conditions with two terms producing (Δ_1,Δ_1) and (Δ_{17},Δ_1) with structure constant $(\sqrt{2})^{1/2}\, D^{7}_{77}$. Similarly one can consider $(\Delta_{15},\Delta_{1m})$ with m+1=4 mod 4. On the contrary, the combinations $(\Delta_{17},\Delta_{1m})$, m+1=4 mod 4 and $(\Delta_{15},\Delta_{1m})$, m+1=6 mod 4, provide examples of local fermions.

Another example exists for $c=c(11)$, $\bar{c}=c(12)$. The combinations $(\Delta_{15},\Delta_{51})$, (Δ_{17},Δ_1), $(\Delta_{17},\Delta_{71})$, (Δ_1,Δ_{71}), (Δ_1,Δ_1) provide a local fusion algebra. The combination of E_6 and E_8 theories is expected to provide further examples.

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Footnotes

1) The Coulomb gas representation for the 4-point functions of [9] does not reproduce automatically neither the upper, nor the lower bounds of the minimal theory fusion rules, since there are nonzero structure constants D^{F}_{AC} for some unadmissible triples (A,C,F). A mechanism of compensation of unwanted contributions, demonstrated on examples in [16], is expected. It implies in addition a relation for the crossing matrix elements. The analogous problem holds for the $U_q(sl(2))$ representations if $q^N=1$ [14].

2) The squares of the structure constants of the local D_{odd} algebra are incidentally recovered due to compensation of the erroneous formulae (4.8), (4.9) of [19]. The results are still inconsistent, since, e.g., the constants $N_{\uparrow}(\hat{A},\hat{A},\hat{C},\hat{C}) / N_{1}(\hat{A},\hat{A},\hat{C},\hat{C})$ implied by (4.8) of [19] are incorrect for $s_A = s_C+1$ mod 2. (In (4.7,8) it is ignored that sign $(\hat{D}^{\hat{F}}_{\hat{A}\hat{A}} \hat{D}^{\hat{F}}_{\hat{C}\hat{C}} / \hat{D}^{1}_{\hat{A}\hat{A}} \hat{D}^{1}_{\hat{C}\hat{C}})$ is also restricted by locality).