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STRATEGIES FOR FILLING THE LONGITUDINAL PHASE SPACE IN THE INJECTION INTO A BOOSTER PROTON SYNCHROTRON AT VERY HIGH INTENSITY CURRENTS

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ABSTRACT

A theory of the longitudinal painting in high intensity circular proton synchrotron accelerators is developed. The study of the uniform filling of a limited region of the booster phase space (x,y) and a uniform line density on a finite interval of the x-axis is achieved by describing the particle beam behavior via the Vlasov equation with no space charge forces. Analytical painting laws are found for both the cases of small oscillations and pendulum-like motion.

1. INTRODUCTION

In the proton accelerators recently proposed by several national and international groups for future basic physics at intermediate energies, the current intensity appears as the new fundamental characteristic parameter.

To achieve the requested values for this current in the final beams, the procedure of injecting particles, without losses, from one accelerating structure (ring or linac) to the next, plays a very significant role. In particular it looks quite important to transfer the beam from the linac to the first circular machine which, for instance in the EHF project[1], is the booster synchrotron. To be more specific, in order to accelerate 100 μA one must reach a reasonable compromise between the number of particles accelerated at each machine cycle and the repetition rate: such a compromise is particularly delicate for the synchrotron that operates at lower energies (the booster in the EHF) because the space-charge problems become more serious.

The booster has its RF-buckets already formed at injection and the linac bunches, often called microbunches, arriving synchronously with the buckets, have to fill them in a controlled uniform way; this requires either the same frequency for the booster and linac RF (out of question) or, at least, a linac beam "modulated" with the booster RF. The latter solution is possible and must be adopted.

To fill the booster with the inserted amount of protons (~2.5×10^{13}) needs several injections turns (~200) of the linac beam and the only efficient multiturn injection scheme is based on the H^- charge exchange stripping process. The injection must be arranged so that the high brightness linac beam ($3\pi \times 10^{-6}$ rad m normalized transverse emittances and $0.6 \times 10^{-4}\pi$ eVs longitudinal bunch area for EHF) is used to "paint" the much larger emittances of the booster beam ($25\pi \times 10^{-6}$ rad m and 0.075 eVs respectively).

"Painting" is a term widely used, nowadays, to mean a rule for populating a two-dimensional phase space with many small spots in such a way that one finally ends up filling a much larger surface. The spots represent the small bunches of the linac and the surfaces of the phase planes considered here are externally limited by closed continuous curves, named separatrices, whose areas give the emittances. Painting in the transverse phase space has been already achieved in existing machines, but the use of simultaneous painting in the longitudinal phase plane is a new concept introduced in the very recent proposals for next generation facilities. The importance of an appropriate filling of the three phase planes is to reduce the transverse and longitudinal space charge forces and, consequently, the possibility of losses in very intense current beams.

An additional practical advantage of performing a longitudinal painting stays in using the synchrotron oscillations to make the injected beam miss the H^- stripping foil more frequently on successive machine revolutions: the scattering in the foil gets reduced as well as foil heating and activation, thus enhancing its lifetime.

In this paper we deal with essentially the "longitudinal painting" neglecting its coupling with the transverse motion, which will be considered in the near future. Our purpose consists of working out a general mathematical frame which i) enables us to deduce various alternative ways of doing the longitudinal painting, according to the constraints emerging from the special system under examination, and ii) to formulate, correspondingly, an adequate code.

A painting procedure constitutes a law of motion for the center of the linac microbunch (or microbunches if we inject more than one per turn) that can be realized in practice by using the flexibility of the linac; it has to allow us to fill the desired part of the bucket area in a uniform stationary way. The cartesian coordinates (x, y) commonly employed are $x=\phi$, the particle phase and $y = 2\pi\Delta E/\omega_s$ where ΔE is the energy difference between the particle and the ideal synchronous particle of frequency ω_s. As is well known, x and y are canonically conjugate.

To be more precise we here summarize the main features of the longitudinal dynamics:

a) the equation of motion can be written in a Hamiltonian form and Liouville's theorem applies;
b) if no space-charge potential is included into the equations, the synchrotron oscillations are pendulum-like (small oscillations are harmonic);
c) at injection, for a limited number of turns (~200 in EHF)

the synchronous phase ϕ_s can be chosen zero ($\phi_s=0$), below the transition energy, as one does for a storage ring, thus getting a nice simplification in the oscillation equations.

In this paper we emphasize mostly the analytical consequences of the point a) (Liouville's theorem) and develop detailed calculations in a simplified model that allows us to derive mathematical formulae in a compact form: we assume the linac microbunches as Gaussian functions in the (x,y) plane and treat them in the delta Dirac function limit. Furthermore, we are also interested in finding those painting laws which provide the closest phase space density, in the x variable, to the ideal constant-uniform one.

In Sec.2 the Vlasov equations relative to both the cases of harmonic and pendulum-like motion are solved starting from a given initial beam distribution. Sec.3 contains some preliminaries and a few considerations about the problem of painting. In Sec.4 a connection is established between a phase space radial density and its linear projection. In Sections 5 and 6, respectively, a model of uniform painting under the harmonic approximation and in the pendulum-like case is dealt with.

Finally, in the Appendixes A, B and C details of the calculations are reported.

2. THE VLASOV EQUATION

We suppose that a charge-particle beam in an accelerator is represented by a collection of non-interacting particles moving in a given external electromagnetic field. Then, the beam behaviour may be described by the Vlasov equation, which assumes smooth beam distributions which hold rigorously in the limit of an infinite number of micro-particles each carrying an infinitesimal charge[2].

If we consider a distribution of particles in the phase space (x,y), where x and y are canonically conjugate variables (say the coordinate and the momentum, respectively), the Vlasov equation reads

$$\rho_t + \dot{x}\,\rho_x + \dot{y}\,\rho_y = 0 \,, \tag{2.1}$$

where $\rho = \rho(x,y,t)$ is the beam distribution, subscripts denote partial derivatives,

$$\dot{x} = \frac{dx}{dt} = \frac{\partial H}{\partial y} \quad , \quad \dot{y} = \frac{dy}{dt} = -\frac{\partial H}{\partial x} \,, \tag{2.2}$$

and H is the Hamiltonian of the (conservative) system.

i) The harmonic case

Let us deal with a harmonic system with Hamiltonian

$$H = \frac{\Omega}{2}\,(\,x^2 + y^2\,). \tag{2.3}$$

(The mass of the particle is taken equal to unity). Then Eqs. (2.2) give

$$\dot{x} = \Omega y \,, \qquad \dot{y} = -\Omega x \,, \tag{2.4}$$

and the Vlasov equation (2.1) becomes

$$\rho_t + \Omega y \rho_x - \Omega x \rho_y = 0. \tag{2.5}$$

In order to find the general solution of Eq. (2.5), we shall exploit the method of characteristics (Ref. 3, vol. IV, p. 302), that associates with Eq. (2.5) the following system of ordinary differential equations

$$dt = \frac{dx}{\Omega y} = -\frac{dy}{\Omega x} \,. \tag{2.6}$$

Solving Eq. (2.6), we get

$$x^2 + y^2 = C^2 \,, \tag{2.7}$$

$$x = C \sin \Omega(t+\alpha) \,, \tag{2.8a}$$
$$y = C \cos \Omega(t+\alpha) \,, \tag{2.8b}$$

where C and α are constants of integration. We observe that (2.7) expresses the conservation of the total energy (see (2.3)).

Now we require that at the time t = t', x(t) and y(t) take, respectively, the values x' and y', i.e.

$$x(t') = x' \,, \qquad y(t') = y' \,. \tag{2.9}$$

Then from (2.8) and (2.9) we obtain

$$C = \sqrt{x'^2 + y'^2} \quad , \qquad (2.10a)$$

$$\alpha = - t' + \frac{1}{\Omega} \tan^{-1} \frac{x'}{y'} \ . \qquad (2.10b)$$

Substitution from (2.10) into (2.8) yields

$$x = \sqrt{x'^2 + y'^2} \ \sin (\Omega \Delta t + \tan^{-1} \frac{x'}{y'}) \ , \qquad (2.11a)$$

$$y = \sqrt{x'^2 + y'^2} \ \cos (\Omega \Delta t + \tan^{-1} \frac{x'}{y'}) \ , \qquad (2.11b)$$

where $\Delta t = t - t'$.

Eqs. (2.11) represent a rotation in the phase space (x,y) at the constant angular speed of Ω.

The inverse transformation of (2.11) is given by

$$x' = \sqrt{x^2 + y^2} \ \sin (-\Omega \Delta t + \tan^{-1} \frac{x}{y}) \ , \qquad (2.12a)$$

$$y' = \sqrt{x^2 + y^2} \ \cos (-\Omega \Delta t + \tan^{-1} \frac{x}{y}) \ . \qquad (2.12b)$$

Sometimes it could be useful to write the relations (2.11) and (2.12) in a more explicit form. This can be easily done by means of the addition formulae for the sine and cosine, keeping in mind that

$$\sin (\tan^{-1} \frac{x}{y}) = \frac{x}{\sqrt{x^2 + y^2}} \ , \qquad (2.13a)$$

$$\cos (\tan^{-1} \frac{x}{y}) = \frac{y}{\sqrt{x^2 + y^2}} \ . \qquad (2.13b)$$

In doing so, from (2.11) and (2.12) we obtain, respectively,

$$x = x'\cos \Omega\Delta t + y'\sin\Omega\Delta t \ , \qquad (2.14a)$$

$$y = -x'\sin \Omega\Delta t + y'\cos \Omega\Delta t \ , \qquad (2.14b)$$

and

$$x' = x \cos \Omega\Delta t - y \sin \Omega\Delta t \ , \qquad (2.15a)$$

$$y' = x \sin \Omega\Delta t + y \cos \Omega\Delta t \ . \qquad (2.15b)$$

If $\rho_0(x(t'),y(t'),t')$ is a known beam distribution at the time t', the general solution of the Vlasov equation (2.5) arises formally from ρ_0 replacing x(t') and y(t') with the expressions at the right-hand sides of (2.12) (or, equivalently, of (2.15)), namely

$$\rho(x, y, t; t') =$$

$$= \rho_0(\sqrt{x^2+y^2}\,\sin(-\Omega\Delta t+\tan^{-1}\frac{x}{y}),\sqrt{x^2+y^2}\,\cos(-\Omega\Delta t+\tan^{-1}\frac{x}{y});t')=$$

$$= \rho_0(x\cos\Omega\Delta t - y\sin\Omega\Delta t,\ x\sin\Omega\Delta t + y\cos\Omega\Delta t;\ t') . \qquad (2.16)$$

ii) The pendulum-like case

The Hamiltonian of a pendulum-like system can be written as

$$H = \Omega\,(\frac{y^2}{2} + 1 - \cos x) . \qquad (2.17)$$

Hence, Eqs. (2.2) provide

$$\dot{x} = \Omega\, y, \quad \dot{y} = -\Omega \sin x , \qquad (2.18)$$

and the Vlasov equation (2.1) reads

$$\rho_t + \Omega\, y\, \rho_x - \Omega \sin x\, \rho_y = 0 . \qquad (2.19)$$

The general solution of Eq. (2.19) can be found resorting to the same procedure previously used in the harmonic case.

Eq. (2.19) allows the characteristics system

$$dt = \frac{dx}{\Omega y} = -\frac{dy}{\Omega \sin x} . \qquad (2.20)$$

The equation

$$\frac{dx}{y} = -\frac{dy}{\sin x} \qquad (2.21)$$

furnishes

$$\frac{y^2}{2} - \cos x = {}^2C \quad , \qquad (2.22)$$

which resembles the total energy of the pendulum (see (2.17)).

On the other hand, with the help of (2.21) the equation

$$dt = \frac{dx}{\Omega y} \qquad (2.23)$$

can be put into the form

$$\sqrt{2}\ \Omega\ dt = \frac{dx}{\sqrt{\cos x - \cos x_0}}\ , \tag{2.24}$$

where x_0 is an arbitrary fixed value of x such that

$$C^2 = -\cos x_0\ . \tag{2.25}$$

Then Eq. (2.24) can be integrated following the scheme reported in Appendix A. We have

$$x = 2\ \sin^{-1}\ [k\ sn\ (\Omega\ (t + \alpha),\ k)]\ , \tag{2.26}$$

where sn is an elliptic function of the Jacobi type (Ref.4,p.91), α is a constant of integration, and

$$k = \sin \frac{x_0}{2} \tag{2.27}$$

means the modulus of sn. Such a quantity is connected with the total energy H through the relation

$$k^2 = \frac{H}{2\Omega}\ , \tag{2.28}$$

as we can immediately see by comparing (2.22) with (2.17) and using (2.25) and (2.27).

The last equation which remains to be examined is (see (2.20))

$$\Omega\ dt = -\frac{dx}{\sin\ x} = -\frac{dy}{\sqrt{1-(C^2 - \frac{y}{2})^2}}\ , \tag{2.29}$$

where (2.22) has been used.

In what follows we check that (2.29) is compatible with the expression of y coming from the first of (2.18)s, that is

$$y = \frac{1}{\Omega}\ \dot{x} = \frac{1}{\Omega}\ \frac{d}{dt}\ \{2\ \sin^{-1}\ [k\ sn\ (\Omega\ (t+\alpha),\ k)]\} =$$

$$= 2\ \frac{k\ cn(\Omega(t+\alpha),k)\ \ dn(\Omega(t+\alpha),k)}{\sqrt{1-k^2 sn^2\ (\Omega(t\ +\ \alpha),\ k)}} =$$

$$= 2\ k\ cn\ (\Omega(t+\alpha),k)\ , \tag{2.30}$$

where cn and dn design elliptic Jacobi functions which are related to sn by

$$cn^2(.) + sn^2(.) = 1, \quad dn^2(.) = 1 - k^2\, sn^2(.)\ . \tag{2.31}$$

In fact, by differentiating (2.30) we have

$$dy = -2\, k\, sn\, (\Omega(t + \alpha), k)\, dn\, (\Omega\, (t + \alpha), k)\, \Omega\, dt\ . \tag{2.32}$$

Furthermore, since

$$C^2 = 2\, k^2 - 1 \tag{2.33}$$

(see (2.25) and (2.27)), inserting (2.30) into the denominator of the right-hand side of (2.29), we get

$$\sqrt{1-(C^2 - \frac{y^2}{2})^2} = 2\, k\, sn\, (\Omega\, (t+\alpha), k)\, dn\, (\Omega\, (t+\alpha), k)\ . \tag{2.34}$$

Thus, by virtue of (2.32) and (2.34) we obtain that (2.29) is identically satisfied.

At this point we demand that

$$x(t') = x', \quad y(t') = y'\ . \tag{2.35}$$

Then Eqs. (2.26) and (2.30) imply

$$x' = 2\, \sin^{-1}\, [k\, sn\, (\Omega(\, t' + \alpha), k)], \tag{2.36a}$$

$$y' = 2\, k\, cn\, (\Omega\, (t' + \alpha), k)\ , \tag{2.36b}$$

from which we derive

$$\alpha = -\, t' + \frac{1}{\Omega}\, tn^{-1} \left[2\, \frac{\sin \frac{x'}{2}}{y'} \right], k)\ , \tag{2.37}$$

$$k^2 = \frac{y'^2}{4} + \sin^2 \frac{x'}{2}\ , \tag{2.38}$$

where tn^{-1} indicates the inverse of the elliptic function

$$tn\, (.) = sn\, (.)/cn\, (.)\ . \tag{2.39}$$

Introducing (2.37) and (2.38) in Eqs. (2.26) and (2.30), we provide

$$\sin \frac{x}{2} = k' sn \left[\Omega \Delta t + tn^{-1}\left[2\, \frac{\sin \frac{x'}{2}}{y'}\, , k'\right], k'\right], \tag{2.40a}$$

$$y = 2\, k' cn \left[\Omega \Delta t + tn^{-1}\left[2\, \frac{\sin \frac{x'}{2}}{y'}, k'\right], k'\right], \tag{2.40b}$$

and

$$k' = k\,(x', y') = \sqrt{\frac{y'^2}{4} + \sin^2\frac{x'}{2}} \ . \tag{2.41}$$

Eqs. (2.40) admit the inverse transformations

$$\sin\frac{x'}{2} = k \ \mathrm{sn}\left[-\Omega\Delta t + \mathrm{tn}^{-1}\left(2\,\frac{\sin\frac{x}{2}}{y}, k\right), k\right], \tag{2.42a}$$

$$y' = 2\,k \ \mathrm{cn}\left[-\Omega\Delta t + \mathrm{tn}^{-1}\left(2\,\frac{\sin\frac{x}{2}}{y}, k\right), k\right], \tag{2.42b}$$

where

$$k = k\,(x\ ,\ y\ ) = \sqrt{\frac{y^2}{4} + \sin^2\frac{x}{2}} \ . \tag{2.43}$$

Eqs. (2.40) and (2.42) imply $k = k'$. One expects this result because of the energy conservation.

On the analogy of the harmonic case, it might be convenient to express (2.40) and (2.42) more explicitly. This can be achieved resorting to the addition formulae for sn and cn, i.e. (Ref. 5, p. 574)

$$\mathrm{sn}\,(u \pm v) = \frac{\mathrm{sn}\,u\ \mathrm{cn}\,u\ \mathrm{dn}\,v \pm \mathrm{sn}\,v\ \mathrm{cn}\,u\ \mathrm{dn}\,u}{1 - k^2\ \mathrm{sn}^2u\ \mathrm{sn}^2v}, \tag{2.44a}$$

$$\mathrm{cn}\,(u \pm v) = \frac{\mathrm{cn}\,u\ \mathrm{cn}\,v \mp \mathrm{sn}\,u\ \mathrm{dn}\,u\ \mathrm{sn}\,v\ \mathrm{dn}\,v}{1 - k^2\ \mathrm{sn}^2u\ \mathrm{sn}^2v}. \tag{2.44b}$$

In fact, by setting

$$\tan\varphi = 2\,\frac{\sin\frac{x'}{2}}{y'}\ , \tag{2.45}$$

we obtain (see Appendix A)

$$\mathrm{tn}^{-1}(\tan\varphi) = \mathrm{cn}^{-1}(\cos\varphi)\ . \tag{2.46}$$

Therefore,

$$\mathrm{cn}\,[\mathrm{tn}^{-1}(\tan\varphi)] = \mathrm{cn}\,[\mathrm{cn}^{-1}(\cos\varphi)] = \cos\varphi =$$

$$= \frac{1}{\sqrt{1 + \tan^2\varphi}} = \frac{y}{2k}\ , \tag{2.47}$$

$$\mathrm{sn}\,[\mathrm{tn}^{-1}(\tan\varphi)] = \mathrm{sn}\,[\mathrm{sn}^{-1}(\sin\varphi)] = \sin\varphi =$$

$$= \frac{\tan\varphi}{\sqrt{1+\tan^2\varphi}} = \frac{1}{k}\sin\frac{x}{2}, \qquad (2.48)$$

$$\mathrm{dn}\,[\mathrm{tn}^{-1}(\tan\varphi)] = \mathrm{dn}\,[\mathrm{dn}^{-1}(\Delta\,\varphi)] = \Delta\,\varphi =$$

$$= \sqrt{1-k^2\sin^2\varphi} = \sqrt{1-k^2\frac{\tan^2\varphi}{1+\tan^2\varphi}} = \cos\frac{x}{2}, \qquad (2.49)$$

where (2.45) and (2.43) have been employed.

If we identify u with $\Omega\Delta t$ and v with $\mathrm{tn}^{-1}(\tan\varphi)$, and put (2.47), (2.48) and (2.49) in (2.44a) and (2.44b), Eqs. (2.40) can be written as

$$\sin\frac{x}{2} = \frac{y\cos\frac{x'}{2}\mathrm{sn}(\Omega\Delta t,k')+2\sin\frac{x'}{2}\mathrm{cn}(\Omega\Delta t,k')\mathrm{dn}(\Omega\Delta t,k')}{2\,[1-\sin^2\frac{x'}{2}\,\mathrm{sn}^2(\Omega\Delta t,k')]} \qquad (2.50a)$$

$$x = \frac{y'\mathrm{cn}(\Omega\Delta t,k') - \sin x'\,\mathrm{sn}(\Omega\Delta t,k')\,\mathrm{dn}(\Omega\Delta t,k')}{1-\sin^2\frac{x'}{2}\,\mathrm{sn}^2(\Omega\Delta t,k')} \qquad (2.50b)$$

where k' is given by (2.41).

Eqs. (2.42) can be elaborated in a similar fashion. We get

$$\sin\frac{x'}{2} = \frac{-y\cos\frac{x}{2}\mathrm{sn}(\Omega\Delta t,k)+2\sin\frac{x}{2}\mathrm{cn}(\Omega\Delta t,k)\mathrm{dn}(\Omega\Delta t,k)}{2\,[1-\sin^2\frac{x}{2}\,\mathrm{sn}^2(\Omega\Delta t,k)]} \qquad (2.51a)$$

$$y' = \frac{y\,\mathrm{cn}(\Omega\Delta t,k) + \sin x\,\mathrm{sn}(\Omega\Delta t,k)\,\mathrm{dn}(\Omega\Delta t,k)}{1-\sin^2\frac{x}{2}\,\mathrm{sn}^2(\Omega\Delta t,k)} \qquad (2.51b)$$

where k is expressed by (2.43).

We observe that (2.50) and (2.51) can be obtained correspondingly each from the other interchanging, respectively, x, y, $\Omega\Delta t$, k with x',y', $-\Omega\Delta t$, k' and viceversa, keeping in mind that sn is an odd function and cn and dn are even functions.

We are now ready to write down the general solution of the Vlasov equation (2.19). Following the same procedure applied in the case of harmonic motion, we shall start from a given initial beam distribution $\rho_0(x(t'),y(t'),t')$ where x(t') and y(t') are replaced by the right-hand terms of (2.42) (or (2.51)). In such a way we are led to the formula

$$\rho(x,y,t;t')=\rho_0\{2\ \sin^{-1}[k\ \mathrm{sn}\ (-\Omega\Delta t\ +\ \mathrm{tn}^{-1}\left[2\frac{\sin\frac{x}{2}}{y},k\right],\ k)],$$

$$2\ k\ \mathrm{cn}\ (-\Omega\Delta t\ +\ \mathrm{tn}^{-1}\left[2\frac{\sin\frac{x}{2}}{y},k\right],\ k);\ t'\}\ =$$

$$=\rho_0\{2\ \sin^{-1}\frac{-y\cos\frac{x}{2}\mathrm{sn}(\Omega\Delta t,k)+2\sin\frac{x}{2}\mathrm{cn}(\Omega\Delta t,k)\mathrm{dn}(\Omega\Delta t,k)}{2\ [1-\sin^2\frac{x}{2}\ \mathrm{sn}^2(\Omega\Delta t,k)]},$$

$$\frac{y\mathrm{cn}(\Omega\Delta t,k)\ +\ \sin x\ \mathrm{sn}(\Omega\Delta t\ ,k)\ \mathrm{dn}(\Omega\Delta t,k)}{1-\sin^2\frac{x}{2}\ \mathrm{sn}^2\ (\Omega\Delta t,k)};\ t'\},\qquad (2.52)$$

with k defined by (2.43).

Before closing this Section, we notice that for small values of x, x' and k, the pendulum-like solution (2.52) reproduces just the harmonic solution (2.16), as one expects. This can be easily deduced looking at Eqs. (2.51) in which

$$\sin\frac{x'}{2}\sim x'/2\ ,\ \cos\frac{x}{2}\sim 1,\ \mathrm{sn}\ (\Omega\Delta t,k)\sim\sin\Omega\Delta t,$$

$$\mathrm{cn}\ (\Omega\Delta t,k)\sim\cos\Omega\Delta t,\ \mathrm{dn}\ (\Omega\Delta t,k)\sim 1,\qquad (2.53)$$

where the modulus k is considered so small that we may take into account only the lowest order quantities in the approximation formulae of sn, cn and dn in terms of circular functions (Ref. 5, p. 573).

3. STATIONARY CONDITION FOR THE LINE PHASE SPACE DENSITY

a) The problem of painting

The problem of painting in the longitudinal phase space (x,y) can be formulated, mathematically, in the following way.

Let $(x_0(t'),\ y_0(t'))$ be the coordinates of the center of a microbunch, injected at the time t' and represented by the distribution $g_0(x - x_0(t'),\ y - y_0(t'))$.

After the introduction of the trajectory $(x_0(t),\ y_0(t))$ of the center of the microbunch, that allows the wanted painting, we must have for the microbunch distribution at the time $t > t'$:

$$\rho(x,y,t) = \int_{-\infty}^{t} i(t')\, g_0(x' - x_0(t'), y' - y_0(t'))dt', \tag{3.1}$$

where i(t) (a weight varying with time) denotes the so-called injection function (whose shape is determined by the linac device and whose dimensions are those of number of particles/time), and

$$x' = x'(x,y,t;t'), \quad y' = y'(x,y,t;t'), \tag{3.2}$$

are the retarded coordinates at the time t'.

The actual coordinates (x,y), formally expressed by

$$x = x\,(x',y',t;t'), \quad y = y\,(x',y',t;t'), \tag{3.3}$$

constitute the inverse transformation of (3.2).

The goal one has to achieve in solving a painting problem, is to obtain the functions of painting $x_0(t)$, $y_0(t)$ in such a way that the line phase space density

$$p(x,t) = \int_{-\infty}^{+\infty} \rho(x,y,t)\, dy, \tag{3.4}$$

with $\rho(x,y,t)$ given by (3.1), is stationary (time independent), i.e. $p(x,t) = p(x)$.

b) The stationary condition and the Vlasov equation

Let us write the Vlasov equation (2.1) as

$$\rho_t + X(y)\,\rho_x + Y(x)\,\rho_y = 0, \tag{3.5}$$

where X and Y are defined by

$X(y) = \dot{x} = \Omega\, y, Y(x) = \dot{y} = -\Omega\, x$ (the harmonic approximation) or $X(y) = \dot{x} = \Omega\, y$, $Y(x) = \dot{y} = -\Omega \sin x$ (the pendulum-like case).

Keeping in mind that x, y are canonically conjugate variables, where y plays the role of momentum, one expects that the condition

$$\lim_{x,y \to \pm\infty} \rho(x,y,t;t') = 0 \tag{3.6}$$

is accomplished.

Integrating (3.5) with respect to y and taking into account

(3.6), we deduce

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}\rho(x,y,t;t')\,dy + \Omega\frac{\partial}{\partial x}\int_{-\infty}^{+\infty} y\rho(x,y,t;t')\,dy = 0, \qquad (3.7)$$

from which we find

$$\int_{-\infty}^{t} dt'\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}\rho(x,y,t;t')\,dy+\Omega\int_{-\infty}^{t} dt'\frac{\partial}{\partial x}\int_{-\infty}^{+\infty} y\rho(x,y,t;t')\,dy=0, \qquad (3.8)$$

for $t'\in(-\infty,t)$, where the microbunch distribution ρ is supposed such that the interchange of the derivative $\partial/\partial x$ and the operation of integration is meaningful.

If we require that p(x,t) is stationary, we have

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial t}\int_{-\infty}^{t} dt'\int_{-\infty}^{+\infty}\rho(x,y,t;t')\,dy = \int_{-\infty}^{+\infty}\rho(x,y,t;t')\,\Big|_{t'=t}dy +$$

$$+\int_{-\infty}^{t} dt'\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}\rho(x,y,t;t')\,dy = 0\,. \qquad (3.9)$$

Combining together Eqs. (3.8) and (3.9), we obtain

$$\int_{-\infty}^{+\infty}\rho(x,y,t;t')\,\Big|_{t'=t}dy - \Omega\frac{\partial}{\partial x}\int_{-\infty}^{t} dt'\int_{-\infty}^{+\infty} y\rho(x,y,t;t')dy = 0, \qquad (3.10)$$

that is another way to express the stationary condition (3.9).

Integrating (3.10) with respect to x and taking account of (3.6) we get the following necessary condition for the occurrence of (3.9):

$$\int_{-\infty}^{+\infty} dx\int_{-\infty}^{+\infty}\rho(x,y,t;t')\,\Big|_{t'=t}dy = 0. \qquad (3.11)$$

Under the hypothesis that the microbunch distribution ρ is a non-negative (or non-positive) function in the phase space (x,y), the vanishing of the double integral (3.11) implies

$$\rho(x,y,t;t')\,\Big|_{t'=t} = 0. \qquad (3.12)$$

This result means that there exists a value of the time, say t_0, such that

$$\rho(x,y,t;t') = 0 \quad \text{for } t' > t_0 \ . \tag{3.12'}$$

In other words, with the aid of (2.16) or (2.52), from the relation (3.11) we deduce

$$\rho(x,y,t;t')\,|_{t'=t} = \rho_0(x,y,t) = 0, \tag{3.13}$$

which must hold identically, i.e. for every t lying in the interval in which one is interested.

Eq. (3.13) tells us that for getting the stationary condition $\partial p(x,t)/\partial t = 0$, it is necessary that the initial microbunch distribution ρ_0 vanishes for every time t' greater than a certain fixed value t_0. Thus ρ_0 must take the form

$$\rho_0(x,y;t') = \tilde{\rho}_0(x,y;t')\ \theta(t_0 - t'), \tag{3.14}$$

where θ is the Heaviside unit function. This means that the condition (3.13) turns out to be verified for $t' > t_0$.

From (3.14) we infer that in order to achieve a stationary situation after a given time t_0, the injection process has to be stopped at the time t_0 itself. However, we remark that this is only a necessary condition.

c) An exact result concerning a simple filling strategy

Let us consider a simple filling strategy, where a great number of microbunches are injected continuously and at injection times belonging to the interval $(0,2\pi/\Omega)$, where Ω is the synchrotron frequency, at the same locus (x_0,y_0) in the longitudinal phase space (x,y).

Adopting this filling mechanism, dubbed "no-painting" scheme[6], the painting action is left to the synchrotron motion of the microbunches.

Furthermore, let us suppose, by way of example, that the distribution of any microbunch, injected at the time t', has the shape

$$\rho(x,y;t=t') = \rho[\alpha(x-x_0)^2 + \beta(y-y_0)^2], \tag{3.15}$$

where α, β are positive constants.

Under the approximation of particle harmonic motion, the time evolution of the microbunch distribution $\rho(x,y,t)$, for $t > t'$, is ruled by the Vlasov equation (2.5). Thus (see (2.16))

$$\rho(x,y;t>t') = \rho\,\{\alpha\,[(x\cos\Omega\Delta t - y\sin\Omega\Delta t) - x_0]^2 + \\ + \beta\,[(x\sin\Omega\Delta t + y\cos\Omega\Delta t) - y_0]^2\}, \qquad (3.16)$$

with $\Delta t = t-t'$.

Then, the line phase space density $p(x,t)$ takes the form

$$p(x,t) = \\ = \int_{-\infty}^{+\infty} dy \int_0^{2\pi/\Omega} \begin{array}{l}\rho\,\{\alpha[(x\cos\Omega\Delta t - y\sin\Omega\Delta t)-x_0]^2 + \\ +\beta[(x\sin\Omega\Delta t + y\cos\Omega\Delta t)-y_0]^2\}\,dt',\end{array} \qquad (3.17)$$

for $t > \frac{2\pi}{\Omega}$.

At this point we recall that, as is well known (see, for example, Ref. 3, vol. II, p. 401), if a function $f(\xi)$, defined for any real value of ξ, is periodic with period T, one has

$$\int_{\xi_0}^{\xi_0+T} f(\xi)d\xi = \int_0^T f(\xi)d\xi\ . \qquad (3.18)$$

Performing the change of variable $\tau = -\Omega\Delta t$ and applying the property (3.18), from (3.7) we get

$$p(x,t) = \frac{1}{\Omega}\int_{-\infty}^{+\infty} dy \int_{-\Omega t}^{-\Omega t+2\pi} \begin{array}{l}\rho\ \{\alpha\,[x\cos\tau + y\sin\tau - x_0]^2 + \\ +\beta\,[-x\sin\tau + y\cos\tau - y_0]^2\}d\tau =\end{array}$$

$$= \frac{1}{\Omega}\int_{-\infty}^{+\infty} dy \int_0^{2\pi} \begin{array}{l}\rho\ \{\alpha\,[x\cos\tau + y\sin\tau - x_0]^2 + \\ +\beta\,[-x\sin\tau + y\cos\tau - y_0]^2\}d\tau =\end{array}$$

$$= p(x). \qquad (3.19)$$

This result shows that, within a simple and physically reasonable model of filling strategy, the line phase space density turns out to be exactly stationary.

d) A weak stationary condition

To tackle the problem of painting, one needs to invoke an appropriate definition of "stationary condition" for the x-axis projection of the phase space microbunch density.

A definition of stationary condition weaker than that considered in a) may be proposed as follows.

Let us assume that a given ensemble of microbunches, injected in an accelerator machine, is endowed with a characteristic length parameter, say a, generally depending on the injection process. Then, we introduce the line density

$$F(x,t;a) = \frac{1}{a}\int_{x-\frac{a}{2}}^{x+\frac{a}{2}} p(x',t)dx', \tag{3.20}$$

where p(x',t) is given by (3.4) and a is a positive quantity.

We remark that (3.20) recovers the line phase space density (3.4) when a tends to zero, i.e.

$$\lim_{a\to 0} F(x,t;a) = p(x,t). \tag{3.21}$$

The stationary condition for F (x,t;a) is reached whenever

$$\partial F(x,t;a) / \partial t = 0. \tag{3.22}$$

In other words, (3.22) (if satisfied) means that there exists a value of $a>0$ such that, for every x, the number of microbunches injected in the range $(x-a/2,\ x+a/2)$ is constant in time. Of course, in the case of a physical situation where the stronger stationary condition $\partial p(x,t)/\partial t=0$ holds, then (3.22) should be fulfilled for any value of x and a.

Furthermore, we observe that

$$N = \lim_{a\to +\infty}\int_{x-\frac{a}{2}}^{x+\frac{a}{2}} p(x',t')dx' \tag{3.23}$$

which should be independent both from the variables t and x, provides the total number of injected microbunches.

Eq. (3.22) can be viewed in the light of the Vlasov equation (3.10). In doing so, by the integration of (3.7) with respect to the variable x' from x-a/2 to x+a/2, we obtain

$$\frac{\partial}{\partial t}\int_{x-\frac{a}{2}}^{x+\frac{a}{2}} p(x',t)dx' + \Omega\int_{-\infty}^{+\infty} y\,[\rho(x+\tfrac{a}{2},y;t) - \rho(x-\tfrac{a}{2},y;t)]\,dy = 0, \tag{3.24}$$

from which we arrive at the (necessary) condition

$$\int_{-\infty}^{+\infty} y\,[\rho(x+\tfrac{a}{2},y;t) - \rho(x-\tfrac{a}{2},y;t)]\,dy = 0, \tag{3.25}$$

if (3.22) is supposed to be valid.

As a simple example of application of (3.25), let us take the injection of a sole microbunch, described by the Gaussian distribution

$$\rho(x,y;t=t') = C\exp\{-\alpha[(x-x_0(t'))^2 + y^2]\}, \tag{3.26}$$

where C and α are positive constants, and $(x_0(t'),0)$ denotes the center of the microbunch injected at the time t'.

Assuming, for definiteness, the particle harmonic motion approximation, the evolution of (3.26) is governed by the Vlasov equation (2.19); therefore

$$\rho(x,y;t>t') = C\exp\{-\alpha\,[(x\cos\Omega\Delta t - y\sin\Omega\Delta t - x_0(t'))^2 + $$
$$+(x\sin\Omega\Delta t + y\cos\Omega\Delta t)^2]\}. \tag{3.27}$$

Inserting (3.27) into (3.25), we are led to the constraint

$$a\,(x + x_0\cos\Omega\Delta t) = 0, \tag{3.28}$$

from which we argue that no value of the parameter $a>0$ can be determined such that (3.28) is satisfied for any x, i.e. the stationary condition (3.25) cannot be achieved via the injection of one microunch only. This result, although obviously expected, can be reproduced as well starting from the requirement of the stronger stationary condition.

4. DISTRIBUTION OF PARTICLES WITH A RADIALLY SYMMETRIC PHASE SPACE DENSITY

For later use, it is of interest to establish a connection between a phase space radial density and its projection (line density).

For this purpose, let us consider in the phase space (x,y) a radially symmetric particle distribution of the type

$$\rho(x,y) = \rho_0(x^2+ y^2)\ \theta\ (R^2- x^2- y^2)\ , \qquad (4.1)$$

where R is a given constant and θ means Heaviside's unit function (that is equal to unity for $x^2+ y^2\leq R^2$, and zero elsewhere).

By definition, the line density p(x) reads

$$p(x) = \int_{-\infty}^{+\infty} \rho(x,y)\ dy, \qquad (4.2)$$

which becomes upon substitution of (4.1):

$$p(x) = \int_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} \rho_0(x^2+y^2)\ dy = 2\int_0^{\sqrt{R^2 - x^2}} \rho_0(x^2+y^2)\ dy, \qquad (4.3)$$

with $|x| \leq R$.

In the polar coordinates frame $x = r\cos\beta$, $y = r\sin\beta$, Eq.(4.3) can be written as

$$p(x) = 2\int_x^R \frac{r\rho_0(r)}{\sqrt{r^2 - x^2}}\ dr, \qquad (4.4)$$

while the number of particles is

$$N = 2\pi\int_0^R r\ \rho_0(r)\ dr. \qquad (4.5)$$

Therefore, the radial density n(r) is expressed by

$$n(r) = 2\pi r\ \rho_0(r). \qquad (4.6)$$

Taking account of (4.6). (4.4) yields

$$p(x) = \frac{1}{\pi}\int_x^R \frac{n(r)}{\sqrt{r^2 - x^2}}\ dr. \qquad (4.7)$$

Furthermore, by letting

$$z = R^2 - r^2 \quad , \quad w = R^2 - x^2 \quad , \tag{4.8}$$

(4.4) takes the form of the Abel integral equation

$$g(w) = \int_0^w \frac{\chi(z)}{\sqrt{w - z}} \, dz \tag{4.9}$$

where

$$g(w) = p(x), \quad \chi(z) = \rho_0(r) \ . \tag{4.10}$$

As is well known. Eq. (4.9) affords the solution[7]

$$\chi(z) = \frac{1}{\pi} \frac{d}{dz} \int_0^z \frac{g(w)}{\sqrt{z - w}} \, dw =$$

$$= \frac{1}{\pi} g(0) \frac{1}{\sqrt{z}} + \frac{1}{\pi} \int_0^z \frac{g'(w)}{\sqrt{z - w}} \, dw \ , \tag{4.11}$$

where $g'(w) = dg/dw$.

Below we deal with some simple special applications of (4.11).

i) Let us choose $g'(w) = 0$, i.e. $g(0) = \text{const}$. Then (4.11) gives

$$\chi(z) = \text{const} \frac{1}{\pi} \frac{1}{\sqrt{z}} \tag{4.12}$$

and, consequently (see (4.6) and (4.8)):

$$n(r) = \text{const} \frac{2r}{\sqrt{R^2 - r^2}} \ . \tag{4.13}$$

If $\text{const} = 1/(2R)$, (4.13) coincides exactly with the radial density quoted in Ref. 6, corresponding to the constant projection $p(x) = 1/(2R)$.

ii) Let us start from $g(w) = \text{const} \times w \equiv \text{const} \times (R^2 - x^2)$. Then (4.11) gives

$$\chi(z) = \frac{\text{const}}{\pi} \int_0^z \frac{1}{\sqrt{z - w}} \, dw = \text{const} \times \frac{2}{\pi} \sqrt{z} \, , \qquad (4.14)$$

which, by setting const $= \dfrac{3}{4R^3}$, yields the elliptic radial density

$$n(r) = \frac{3r}{R^3} \sqrt{R^2 - r^2} \, . \qquad (4.15)$$

5. DISTRIBUTION OF PARTICLES IN A PHASE SPACE POLAR COORDINATE SYSTEM

Here we shall easily prove that in a phase space polar coordinate system, a distribution function $\rho(x,y,t;t')$ leads to a time independent phase space density.

In doing so, assuming for simplicity the small oscillations approximation, we notice that if

$$\rho\,(x,y,t = t';t') = \rho_0(x,y,t') = \rho_0(r \cos \beta, \; r \sin \beta;t') \equiv$$
$$\equiv \tilde{\rho}_0(r,\beta;t') \qquad (5.1)$$

denotes the initial distribution, then the evolution of the distribution for $t > t'$ is formally given by (see Sec. 2)

$$\tilde{\rho}\,(r,\beta,t > t';t') = \rho_0[r \cos(\beta - \Omega \Delta t), r \sin(\beta - \Omega \Delta t);t']. \qquad (5.2)$$

We can determine the full particle distribution by integrating (5.2) over all the injection times $t' \in (0,t'_M)$, with t'_M the maximum injection time. We get

$$\tilde{\rho}\,(r,\beta;t) = \int_0^{t'_M} \tilde{\rho}\,(r,\beta,t > t';t') \; dt' \; . \qquad (5.3)$$

Then, we can define the phase space density $P(r,t)$ as

$$P(r,t) = \int_0^{2\pi} \tilde{\rho}\,(r,\beta;t) \; d\beta = \int_0^{2\pi} d\beta \int_0^{t'_M} \tilde{\rho}\,(r,\beta,t > t';t') \; dt' =$$

$$= \int_0^{t'_M} dt' \int_0^{2\pi} \tilde{\rho}_0[r \cos(\beta - \Omega \Delta t), r \sin(\beta - \Omega \Delta t);t'] \; d\beta, \qquad (5.4)$$

which can be written as

$$P(r,t) = \int_0^{t'_M} dt' \int_{-\Omega\Delta t}^{-\Omega\Delta t+2\pi} \tilde{\rho}_0(r \cos\beta', r \sin\beta';t')d\beta' =$$

$$= \int_0^{t'_M} dt' \int_0^{2\pi} \tilde{\rho}_0(r \cos\beta', r \sin\beta';t')d\beta' \equiv P(r), \qquad (5.5)$$

where the change of variable $\beta' = \beta - \Omega\Delta t$ and (3.8) have been used.

A similar result can be obtained as well in the case of particle pendulum-like motion (see Ref. 6).

6. MODEL OF UNIFORM PAINTING UNDER THE HARMONIC APPROXIMATION

Case I: injection of microbunches along the x-axis

Let us suppose that a great number of microbunches is injected in a booster of a proton synchrotron machine along the x-axis of the longitudinal phase space (x,y). Assuming a continuous injection mechanism, we consider an initial microbunch distribution of the delta function type, namely

$$\rho_0(x,y;t') = I\ \delta(x-x_0(t'))\delta(y), \qquad (6.1)$$

for injection times t' such that $0 \leq t' \leq t'_M$, where $(x_0(t'),0)$ are the coordinates of the injection point at the time t', and I denotes a constant injection function. In what follows, we shall normalize I in order to have one particle only injected in the interval $(0,t'_M)$, so that $I = \frac{1}{t'_M}$.

The form (6.1), which tells us that the microbunch distribution is particle-like at every time t' lying in the interval $(0,t'_M)$, can be regarded as the limit of a Gaussian distribution (see Appendix B).

In the case in which the particle motion is harmonic, the time evolution of the distribution (6.1) is given by (see Sec. 2)

$$\rho(x,y,t;t') = I\ \delta[x \cos\Omega\Delta t - y \sin\Omega\Delta t - x_0(t')] \times$$

$$\times\ \delta(x \sin \Omega\Delta t + y \cos \Omega\Delta t), \tag{6.2}$$

where $\Delta t = t - t'$, and (x,y) denotes the position of the injection point $(x_0(t'),0)$ at the time t.

The center of the microbunch distribution (6.2) occurs at the point (x,y) such that

$$x \cos \Omega\Delta t - y \sin \Omega\Delta t - x_0(t') = 0, \tag{6.3a}$$

$$x \sin \Omega\Delta t + y \cos \Omega\Delta t = 0\ , \tag{6.3b}$$

which imply

$$x = x_0(t') \cos \Omega\Delta t \quad , \quad y = -x_0(t') \sin \Omega\Delta t\ . \tag{6.4}$$

The total microbunch distribution is obtained by integrating (6.2) over all the injection times $t' \in (0,t'_M)$, i.e.

$$\rho(x,y,t) = \int_0^{t'_M} \rho(x,y,t;t')\ dt' = I \int_0^{t'_M} \delta\ [x \cos \Omega\Delta t - y \sin \Omega\Delta t$$

$$-x_0(t')]\delta(x \sin \Omega\Delta t + y \cos \Omega\Delta t)\ dt'. \tag{6.5}$$

Let us perform the calculation of the integral (6.5) using the second delta function. To this end, we recall that if $f(\xi)$ is a single-valued function, the following property

$$\delta\ [f(\xi)] = \sum_j \frac{1}{|\ f'(\xi)\ |_{\xi=\xi_j}}\ \delta\ (\xi-\xi_j) \tag{6.6}$$

holds, where $f' = df/d\xi$ and ξ_j are the (isolated and simple) zeros of $f(\xi)$.

The zeros of the argument of $\delta(x \sin \Omega\Delta t + y \cos \Omega\Delta t)$ take place at

$$t'_j = t + \frac{1}{\Omega}\ (\theta - j\pi), \tag{6.7}$$

where $\theta = \tan^{-1}(y/x)$ and j is a (positive or negative) integer.

Then, with the help of (6.6) we can write

$$\delta(x \sin \Omega\Delta t + y \cos \Omega\Delta t) =$$

$$= \sum_j \frac{1}{\Omega\ |\ x \cos \Omega\Delta t - y \sin \Omega\Delta t\ |_{t'=t'_j}}\ \delta(t'-t_j) =$$

$$= \sum_j \frac{1}{\Omega r}\ \delta(t'-t'_j), \tag{6.8}$$

where

$$r = x \cos \theta + y \sin \theta \tag{6.9}$$

and (6.7) has been taken into account.

Inserting (6.8) in (6.5), we get

$$\rho(x,y,t) \equiv \rho(r,\theta,t) = \frac{I}{\Omega} \Sigma_j \frac{1}{r} \int_0^{t'_M} \delta(r-x_0(t'))\delta(t'-t'_j)\ dt' =$$

$$= \frac{1}{\Omega} \Sigma_j \frac{1}{r} \delta(r-x_0(t'_j)), \qquad (6.10)$$

where the summation index j runs over all those values for which $0 \leq t' \leq t'_M$.

Now we shall evaluate the number of particles, $\Delta N(r)$, contained in a circular ring of radii r and $r+\Delta r$, where $\Delta r << r$. We have

$$\Delta N(r) = \int_0^{2\pi} d\theta \int_r^{r+\Delta r} r'\rho(r',\theta,t)\ dr' \simeq r\Delta r \int_0^{2\pi} \rho(r,\theta,t)\ d\theta\ . \qquad (6.11)$$

To calculate (6.11). we need to evaluate

$$\int_0^{2\pi} \rho(r,\theta,t)\ d\theta = \frac{I}{\Omega} \Sigma_j \frac{1}{r} \int_0^{2\pi} \delta(r-x_0(t'_j))\ d\theta. \qquad (6.12)$$

At this point we make the hypothesis that $x_0(t')$ is a monotonic function for $t' \in (0,t'_M)$. This means that the microbunch carrying out the painting passes once only through each point (x,y). Consequently, in (6.10) only one term contributes. If we suppose also that $x_0(t')$ is a continuous and monotonic function and $x_0(0)=x_1$, $x_0(t'_M) = x_2$, the microbunch distribution $\rho(r.\theta,t)$ turns out to be different from zero for all values of r belonging to the interval of extremes x_1 and x_2. Therefore, in (6.12) only one contribution survives..

To compute (6.12) we exploit formula (6.6), which entails the determination of the zeros of the argument of the delta function $\delta(r-x_0(t'))$ in terms of the variable θ. We find (see (6.7))

$$r = x_0(t'_j) = x_0[t + \frac{1}{\Omega} (\theta_0 - j\pi)]. \qquad (6.13)$$

Calling m the value of the index j corresponding to such a contribution, from (6.13) we have

$$\theta_0 = -\Omega\ [t - x_0^{-1}(r) - m \frac{\pi}{\Omega}\ ]. \qquad (6.14)$$

Then

$$\delta(r-x_0(t')) = \frac{\Omega}{|\dot{x}_0[x_0^{-1}(r)]|}\delta(\theta-\theta_0), \qquad (6.15)$$

where $\dot{x}_0 = \frac{dx_0}{dt'}$.

Indeed, remembering that $\theta = \tan^{-1}(y/x) = \Omega\Delta t$, the application of (6.6) requires

$$\frac{d}{d\theta}(r-x_0(t')) = \frac{dr}{d\theta} - \dot{x}_0(t')\frac{dt'}{d\theta} = -\dot{x}_0(t')\frac{dt'}{d\theta} = \frac{1}{\Omega}\dot{x}_0(t'). \qquad (6.16)$$

Hence, by virtue of (6.13),

$$\frac{d}{d\theta}(r-x_0(t'))\,|_{\theta=\theta_0} = \frac{1}{\Omega}\dot{x}_0(t')\,|_{t'=t'_m} = \frac{1}{\Omega}\dot{x}_0[x_0^{-1}(r)]. \qquad (6.17)$$

Finally, by substituting (6.15) into (6.12), (6.11) becomes

$$\Delta N(r) \simeq \frac{I\Delta r}{|\dot{x}_0[x_0^{-1}(r)]|}. \qquad (6.18)$$

Hence, the average density of particles inside the circular ring is given by

$$P(r) = \frac{\Delta N(r)}{2\pi r\Delta r} = \frac{I}{2\pi r\,|\dot{x}_0[x_0^{-1}(r)]|}. \qquad (6.19)$$

We point out that this average density is obtained starting from a distribution function which is not rotationally invariant. In fact, in the case of continuous injection, the particle distribution in the phase space (x,y) is different from zero along a spiral whose parametric equations, in the parameter t', are given by (6.4), and whose pitch depends on the painting law, i.e. on the shape assigned to the function $x_0(t')$. In the case of a discrete injection mechanism, the particles will be distributed on points belonging to the same spiral.

We notice that the approximation (6.19) approaches the rotationally symmetric particle distribution as the spiral closes onto itself (reduced pitch).

After these considerations, in the following we shall derive the painting function $x_0(t')$ which allows two densities of physical interest, namely i) a uniform density inside a circle centered at the origin of the phase space, and ii) a uniform line density on a finite interval on the x-axis. In doing so, we shall replace the exact particle distribution with the average density (6.19).

i) We observe that in order to have a uniform density, the number of particles contained in a circular ring of radii r and $r+\Delta r$ has to be proportional to the area of the ring itself. This condition reads (see (6.19))

$$\Delta N(r) = \int_0^{2\pi} d\theta \int_r^{r+\Delta r} r' P(r')\, dr' =$$

$$= I \int_r^{r+\Delta r} \frac{1}{|\, x_0'[x_0^{-1}(r')] \,|}\, dr' = I \,|\, x_0^{-1}(r+\Delta r) - x_0^{-1}(r) \,| =$$

$$= \gamma\, [A(r+\Delta r) - A(r)], \tag{6.20}$$

where A(r) denotes the area of the circle of the radius r, γ is a constant of proportionality, and the change of variable $t' = x_0^{-1}(r)$ has been used.

From (6.20) we infer that

$$x_0^{-1}(r) = (\gamma/I)\, A(r) = (1/d)\, r^2 , \tag{6.21}$$

apart from an additive constant, where $d = I/(\pi\gamma)$.

Remembering that $x_0^{-1}(r)$ is the injection time of the beam on the x-axis at distance r from the origin of the phase space system, we get

$$x_0(t') = \sqrt{d\, t'} . \tag{6.22}$$

Hence, to obtain a uniform filling of a circle of radius R centered at the origin of the (x,y) frame, the injection has to be carried out by means of the painting law (6.22) from $t' = 0$ to $t' = (R^2/d)$.

A discrete version of (6.22) may be formulated by introducing the injection time

$$t' = n\, \tau , \tag{6.23}$$

where n is the injection turn number and τ is the (constant) interval between two injections. With the aid of (6.23), (6.22) provides

$$n = 1/(d\tau)\, x_{0,n}^2 \,, \tag{6.24}$$

where $x_{0,n}$ is the position of the center of the microbunch injected at the n-th turn.

ii) Keeping in mind Sec. 4, we may approach a uniform line density p(x) = 1/(2R) by choosing

$$\rho_0(r) = \frac{1}{2\pi R \sqrt{R^2 - r^2}} \,, \tag{6.25}$$

(see (4.13) and (4.6)).

Comparing (6.25) with (6.19), we find

$$\frac{I}{r\,|\,\dot{x}_0[x_0^{-1}(r)]\,|} = \frac{1}{R\sqrt{R^2 - r^2}} \,. \tag{6.26}$$

Since t' = $x_0^{-1}(r)$, (6.26) can be written as

$$x_0 \frac{dx_0}{dt'} = I\, R \sqrt{R^2 - x_0^2} \,, \tag{6.26'}$$

which yields

$$x_0(t') = R \sqrt{1 - I^2 (t'_M - t')^2} \,. \tag{6.27}$$

Therefore, to find uniform line density in the interval (-R, R) on the x-axis, one must exploit the painting law (6.27) from t'= 0 to t'= $t'_M = \frac{1}{IR}\sqrt{R^2 - x_1^2}$.

Following the procedure set up for the case i), one can also determine a painting law corresponding to (6.26)' when a discrete injection mechanism is assumed.

Case II: injection of microbunches along a curve in the phase space

Here we generalize the content of Case I, in the sense that, adopting again a continuous injection scheme, we consider the injection of a great number of microbunches along a curve in the longitudinal phase space (x,y).

We start from the initial microbunch distribution

$$\rho_0(x,y,t') = I\ \delta(x - x_0(t'))\ \delta(y - y_0(t')), \qquad (6.27)'$$

where $t' \in (0,t'_M)$ is the injection time, and $(x_0(t'),\ y_0(t'))$ are the coordinates of the injection point at the time t'.

Now (6.2) takes the form

$$\rho(x,y,t;t') = I\ \delta(x \cos \Omega\Delta t - y \sin \Omega\Delta t - x_0(t')) \times$$
$$\times\ \delta(x \sin \Omega\Delta t + y \cos \Omega\Delta t - y_0(t')). \qquad (6.28)$$

To carry out the integral

$$\int_0^{t'_M} \rho(x,y,t;t')\ dt',$$

where ρ is given by (6.28), let us perform a rotation of the (x,y)-frame which transforms the coordinates $(x_0(t'),\ y_0(t'))$ of the injection point at the time t', into the coordinates $(r_0(t'),0)$ along the x'-axis, where $r_0(t') = \sqrt{x_0^2\ (t') + y_0^2(t')}$.

The rotation angle is

$$\theta_0(t') = \tan^{-1}(y_0(t') / x_0(t')), \qquad (6.30)$$

while in the new frame the coordinates of the point (x,y) read

$$x' = x \cos \theta_0(t') + y \sin \theta_0(t'), \qquad (6.30a)$$
$$y' = -x \sin \theta_0(t') + y \cos \theta_0(t'), \qquad (6.30b)$$

with

$$\cos \theta_0(t') = \frac{x_0(t')}{r_0(t')}\ ,\ \sin \theta_0(t') = \frac{y_0(t')}{r_0(t')}\ , \qquad (6.31)$$

$$x'_0(t') = r_0(t'), \qquad y'_0(t') = 0\ . \qquad (6.32)$$

From (6.30) we deduce the inverse transformation

$$x = x' \cos \theta_0(t') - y' \sin \theta_0(t')\ , \qquad (6.33a)$$

$$y = x' \sin \theta_0(t') + y' \cos \theta_0(t')\ . \qquad (6.33b)$$

Inserting (6.33) and (6.31) in (6.28), we find

$$\rho(x',y',t;t')=I\ \delta[x'\cos (\Omega\Delta t+\theta_0) - y'\sin (\Omega\Delta t+\theta_0)-r_0\cos \theta_0] \times$$

$$\times\ \delta\ [x'\sin(\Omega\Delta t+\theta_0)+y'\cos(\Omega\Delta t+\theta_0) - r_0 \sin\theta_0], \qquad (6.34)$$

where the dependence on t' of x',y',θ_0 and r_0 is understood.

The microbunch distribution (6.34) is centered at the point (x',y') such that

$$x' \cos(\Omega\Delta t+\theta_0) - y' \sin(\Omega\Delta t+\theta_0) = r_0 \cos\theta_0, \qquad (6.35a)$$

$$x' \sin(\Omega\Delta t+\theta_0) - y' \cos(\Omega\Delta t+\theta_0) = r_0 \sin\theta_0, \qquad (6.35b)$$

which yield

$$x'_0 = r_0(t') \cos\Omega\Delta t, \quad y'_0 = - r_0(t') \sin\Omega\Delta t, \qquad (6.36)$$

as one expects.

We notice that (6.34) can be written as

$$\rho(x',y',t;t') = I\ \delta\ [x' \cos\Omega\Delta t - y' \sin\Omega\Delta t - r_0] \times$$
$$\times\ \delta\ [x' \sin\Omega\Delta t + y' \cos\Omega\Delta t], \qquad (6.37)$$

whose center, in the coordinate system (x',y'), coincides with (6.36) (see Appendix B).

Since in (6.37) the second delta function implies

$$\tan^{-1}(y'/x') = -\Omega\Delta t\ , \qquad (6.38)$$

we may re-write the argument of the first delta function as

$$x' \cos\Omega\Delta t - y' \sin\Omega\Delta t - r_0 = r' - r_0 = r - r_0$$

(see (6.30)). Firthermore, if in the second delta function we replace x', y' by (6.30), we obtain the expression of (6.37) in the old coordinate frame (x, y), namely

$$\rho(x,y,t;t') = I\delta(r-r_0(t'))\delta[x\ \sin(\Omega\Delta t-\theta_0)+y\ \cos(\Omega\Delta t-\theta_0)] \qquad (6.39)$$

The total microbunch distribution reads

$$\rho(x,y,t)) = \int_0^{t'_M} \rho(x,y,t;t')\ dt' =$$

$$= I \int_0^{t'_M} \delta(r-r_0(t'))\delta[x\ \sin(\Omega\Delta t-\theta_0)+y\ \cos(\Omega\Delta t-\theta_0)]\ dt'. \qquad (6.40)$$

Following the same scheme adopted for Case I, this integral can be evaluated applying (6.6) to the second delta function, whose argument vanishes for

$$y/x = -\tan(\Omega\Delta t - \theta_0)\ , \qquad (6.41)$$

which yields

$$\Omega\Delta t = -\theta + \theta_0 + m\pi \ , \tag{6.42}$$

where $\theta = \tan^{-1}(y/x)$ and m is a (positive or negative) integer.

Unlike what happens for the Case I, (6.42) does not allow one to determine explicitly the values of t' for which (6.42) is satisfied. This is due to the fact that now one does not know how θ_0 depends on t'. Anyway, indicating by (t'_{jm}) the set of zeros of (6.42), we get

$$\Omega(t - t'_{jm}) = -\theta + \theta_0(t'_{jm}) + m\pi \ . \tag{6.43}$$

Therefore, we can write

$$\delta\ [x \sin(\Omega\Delta t - \theta_0) + y \cos(\Omega\Delta t - \theta_0)] =$$

$$= \sum_{j,m} \frac{1}{r\ |\ \Omega + \dot{\theta}_0\ |_{t'=t'_{jm}}}\ \delta(t'-t'_{jm}) \ , \tag{6.44}$$

where $\dot{\theta}_0 = d\ \theta_0/dt'$.

Substituting (6.44) in (6.40), we easily have

$$\rho(r,\theta,t) = \sum_{j,m} \frac{I}{r\ |\ \Omega + \dot{\theta}_0(t'_{jm})\ |}\ \delta(r - r_0(t'_{jm})) \ , \tag{6.45}$$

which corresponds to the formula (6.10) previously derived.

If we suppose that $r_0(t')$ is a monotonic function of t', then a sole term corresponding to one root among the values $\{t'_{jm}\}$, say $\bar{t}'$, contributes to the summation in (6.45).

Furthermore, if $r_0(t')$ is also a continuous function with $r_0(0) = r_1$, $r_0(t'_M) = r_2$, the microbunch distribution $\rho(r,\theta,t)$ will be different from zero for all values of r belonging to the interval of extremes r_1 and r_2.

With the help of (6.45), now we shall evaluate the integral

$$\int_0^{2\pi} \rho(r,\theta,t)\ d\theta = \int_0^{2\pi} \frac{I}{r\ |\ \Omega + \dot{\theta}_0(\bar{t}')\ |}\ \delta(r - r_0(\bar{t}'))\ d\theta. \tag{6.46}$$

In doing so, we observe that (see (6.43))

$$\bar{t}' = r_0^{-1}(r) = t + \frac{1}{\Omega}\ [\theta - \theta_0(\bar{t}') + m'\pi], \tag{6.47}$$

where m' is that value of m corresponding to the zero $\bar{t}$', and $\bar{\theta}$ is the (unique) zero of the argument of the delta function in the variable θ.

Thus, we can write

$$\delta(r-r_0(\bar{t}')) = \frac{1}{\left| \frac{dr_0(t')}{d\theta} \right|_{\bar{t}'=r_0^{-1}(r)}} \delta(\theta-\bar{\theta}), \tag{6.48}$$

which leads to the relation (see (6.46))

$$\int_0^{2\pi} \rho(r,\theta,t)\, d\theta = \frac{I}{r \left| \Omega+\dot{\theta}_0(\bar{t}') \right|_{\bar{t}'=r_0^{-1}(r)} \left| \frac{dr_0(t')}{d\theta} \right|_{\bar{t}'=r_0^{-1}(r)}} =$$

$$= \frac{I}{r \left| \dot{r}_0[r_0^{-1}(r)] \right|}, \tag{6.49}$$

where (6.43) has been used. We can then exploit this expression to calculate the total number of particles and the average density inside a circular ring of radii r and $r+\Delta r$, centered at the origin of the phase space frame.

The average density is given by

$$P(r) = \frac{I}{2\pi r \left| \dot{r}_0[r_0^{-1}(r)] \right|} . \tag{6.50}$$

Comparing (6.50) with (6.19). we deduce that the results (6.22) and (6.27) can also be achieved in this case in which the microbunches are injected along a curve, instead of along the x-axis of the phase space, provided that x_0 is replaced by r_0. Moreover, we remark that the corresponding formulae depend on the painting variables, $x_0(t')$ and $y_0(t')$, only through $r_0(t')$.

7. MODEL OF UNIFORM PAINTING IN THE PENDULUM-LIKE CASE

Here we abandon the harmonic motion approximation and deal with a model of uniform painting in the case where the synchronous oscillations are pendulum-like.

In this circumstance the particle motion is described by the Vlasov equation (2.19), which reduces to (2.5) for small oscillations.

Our model will be built up following the pattern of Sec. 6. In doing so, it is convenient to introduce a new set of coordinates, defined in terms of the phase space variable x and y as follows:

$$\sin\frac{x}{2} = k\ \mathrm{sn}\ (\beta,k), \tag{7.1a}$$

$$y = 2k\ \mathrm{cn}\ (\beta,k), \tag{7.1b}$$

where k^2 coincides with the energy of a particle of phase space coordinates (x,y), and

$$\beta = \mathrm{tn}^{-1}\left[\frac{2\sin\frac{x}{2}}{y}, k\right] . \tag{7.2}$$

From (7.1b) we get

$$\dot{y} = \frac{d}{dt}\ [2\ k\ \mathrm{cn}\ (\beta,k)] = -2\ k\ \mathrm{sn}\ (\beta,k)\ \mathrm{dn}\ (\beta,k)\ \dot{\beta}. \tag{7.3}$$

Comparing (7.3) with

$$\dot{y} = -\ \Omega\ \sin x \tag{7.4}$$

and taking account of (7.1a), we find

$$\beta - \Omega\Delta t = \mathrm{const.} \tag{7.5}$$

Consequently, if

$$\rho(x,y,t = t';t') = \rho_0(x,y;t') \tag{7.6}$$

denotes the initial microbunch distribution in the (x,y) phase space, the time evolution of (7.6) in the coordinate frame (k,β) reads (see (2.52))

$$\tilde{\rho}\ (k,\beta,t>t';t') = \rho_0\ \{2\ \sin^{-1}\ [k\ \mathrm{sn}\ (\beta - \Omega\Delta t,k)],$$
$$2k\ \mathrm{cn}\ (\beta - \Omega\Delta t,k);t'\}. \tag{7.7}$$

Now let us consider an initial microbunch distribution of the form (6.1). Keeping in mind (7.7), for any $t>t'$ we have

$$\tilde{\rho}\ (k,\beta,t;t') = I\ \delta\ \{2\ \sin^{-1}\ [k\ \mathrm{sn}\ (\beta - \Omega\Delta t,k)]- x_0(t')\}\ \times$$
$$\times\ \delta\ [2k\ \mathrm{cn}\ (\beta - \Omega\Delta t,k)]. \tag{7.8}$$

By integrating (7.8) over all the injection times $t' \in (0, t'_M)$, we obtain the total microbunch distribution

$$\tilde{\rho}\,(k,\beta;t) = \int_0^{t'_M} \tilde{\rho}\,(k,\beta;t,t')\,dt' =$$

$$= I \int_0^{t'_M} \delta\,\{2\,\sin^{-1}[k\,sn\,(\beta-\Omega\Delta t,k)] - x_0(t')\} \times$$

$$\times\ \delta\,[2k\,cn\,(\beta - \Omega\Delta t,k)]\,dt'. \qquad (7.9)$$

The integral (7.9) can be evaluated applying first the property (6.6) in the second delta function. We have

$$\delta\,[2k\,cn\,(\beta - \Omega\Delta t,k)] =$$

$$= \sum_j \frac{\delta(t'-t'_j)}{2k\Omega\,|\,sn(\beta-\Omega\Delta t,k)dn(\beta-\Omega\Delta t,k)\,|_{t'=t'_j}}\,, \qquad (7.10)$$

where $\{t'_j\}$ is the set of zeros of cn $(\beta-\Omega\Delta t,k)$, namely

$$t'_j = t - \frac{1}{\Omega}\,[\beta - (2j + 1)\,K],\ \ j = 0,\ \pm\,1,\ \pm\,2,\ \ldots, \qquad (7.11)$$

and $K = K(k^2)$ is the quarter-period of the Jacobi elliptic functions, i.e.

$$K\,(k^2) = \int_0^{\pi/2} \frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}\,. \qquad (7.12)$$

Hereafter, resorting to the same assumptions made on the painting function $x_0(t')$ under the harmonic approximation, we pursue our calculation taking into account only one zero, say

$$t'_m = t - \frac{1}{\Omega}\,[\beta - (2m + 1)\,K]\,. \qquad (7.13)$$

Then, Eq. (7.10) becomes

$$\delta\,[2k\,cn\,(\beta - \Omega\Delta t,k)] = \frac{\delta(t'-t'_m)}{2k\Omega\,|\,sn(K,k)\,dn(K,k)\,|} =$$

$$= \frac{\delta\,(t'-t'_m)}{2\Omega k\sqrt{1-k^2}}\,. \qquad (7.14)$$

Substitution from (7.14) into (7.9) yields

$$\rho(k,\beta,t) = \frac{I}{2\Omega k\sqrt{1-k^2}} \int_0^{t'_M} \delta\,\{2\,\sin^{-1}[k\ \mathrm{sn}\,(\beta-\Omega\Delta t,k) - x_0(t')\} \times$$

$$\times\ \delta(t'-t'_m)\,dt' = \frac{1}{2\Omega k\sqrt{1-k^2}}\,\delta[2\,\sin^{-1}k - x_0(t')]. \qquad (7.15)$$

In what follows, we need to know the value of the integral

$$\int_0^{4K} \rho(k,\beta,t)\,d\beta =$$

$$= \frac{1}{2\Omega k\sqrt{1-k^2}} \int_0^{4K} \delta\,[2\,\sin^{-1}k - x_0(t'_m)]\,d\beta\ . \qquad (7.16)$$

To calculate (7.16), we take first into account (6.6), which gives

$$\delta\,[2\,\sin^{-1}k - x_0(t'_m)] = \frac{\delta(\beta-\beta_0)}{\left|\frac{dx_0(t')}{d\beta}\right|_{\beta=\beta_0}}\ , \qquad (7.17)$$

where β_0 indicates the root of the equation

$$2\,\sin^{-1}\,k - x_0(t'_m) = 0. \qquad (7.18)$$

Second, since (see (7.2))

$$k = \sin\frac{x_0(t')}{2} \qquad (7.19)$$

and

$$\cos\frac{x_0}{2}\,\frac{dx_0}{d\beta} = 2\,\frac{d}{d\beta}\,\sin\frac{x_0}{2}\ , \qquad (7.20)$$

by virtue of the well known property

$$f(x)\,\delta(x - a) = f(a)\,\delta(x - a), \qquad (7.21)$$

where a is a constant, we can write

$$\int_0^{4K} \rho(k,\beta,t)\,d\beta = \frac{1}{4\Omega k}\int_0^{4K} \frac{\delta(\beta-\beta_0)}{\left|\frac{d}{d\beta}\,X_0(t')\right|_{\beta=\beta_0}}\,d\beta =$$

$$= \frac{1}{4k} \int_0^{4K} \frac{\delta(\beta-\beta_0)}{|\ \dot{X}_0[X_0^{-1}(k)]\ |} \, d\beta =$$

$$= \frac{1}{4k\ \ |\dot{X}\ [X_0^{-1}(k)]\,|} \ . \tag{7.22}$$

In deriving (7.22), we have put

$$k = \sin \frac{x_0(t'_m)}{2} = X_0(t'_m) \ , \tag{7.23}$$

and have used

$$\frac{dX_0(t')}{d\beta}\Bigg|_{\beta=\beta_0} = \frac{dX_0(t')}{dt'} \frac{dt'}{d\beta}\Bigg|_{t'=t'_m,\ \beta=\beta_0} =$$

$$= -\frac{1}{\Omega} \frac{dX_0(t')}{dt'}\Bigg|_{t'=t'_m,\ \beta=\beta_0} = -\frac{1}{\Omega}\,\dot{X}_0[X_0^{-1}(k)],$$

where

$$t'_m = X_0^{-1}\,(k) \ . \tag{7.24}$$

Furthermore, we notice that (7.22) has been obtained assuming for X_0 the same hypothesis already made on x_0 for the harmonic painting in the computation of the integral (6.12).

At this stage, let us consider a surface S in the phase space (x,y) delimited by two closed curves corresponding to the phase paths of particles of energy $k^2$and $k^2 + \Delta k^2$, respectively.

The number of particles, $\Delta N(k)$, contained in S, is

$$\Delta N(k) \int_S \rho(x,y,t)\ dx\ dy =$$

$$= \int_{S'} \rho(k,\beta,t)\ \ |\,J\,|\ \ dk\ d\beta \simeq \frac{I\Delta k}{|\,\dot{X}_0[X_0^{-1}(k)]\,|} \ , \tag{7.25}$$

where $\Delta k << k$, J is the Jacobian of the transformation (7.2) (see Appendix C), and (7.22) has been used.

The area $\Delta A(k)$ of the phase space ring S is expressed by

$$\Delta A(k) = 4 \int_0^{4K} d\beta \int_k^{k+\Delta k} k' \; dk' \simeq 16 \; K(k^2) \; k\Delta k, \tag{7.26}$$

where A(k) denotes the area of the phase space region D, including the origin, whose contour is the phase path of a particle of energy k^2.

Thus we can define the average density of particles inside the ring S as follows:

$$P\,(k) = \frac{\Delta N(k)}{\Delta A(k)} = \frac{I}{16k \;\; K(k^2)|\dot{X}_0[X_0^{-1}(k)]|} \; . \tag{7.27}$$

Now, in analogy with the harmonic painting treated in Sec.6, we look for the painting function $X_0(t')$ which permits 1) a uniform density inside the region D, and 2) a uniform line density on a finite interval of the x-axis.

1) In order to have a uniform density, it must be

$$\Delta N\,(k) = \gamma\,[A\,(k + \Delta\,k) - A\,(k)\,]\;, \tag{7.28}$$

where γ is a constant of proportionality. With the help of (7.25), (7.28) yields

$$X_0^{-1}(k) = \frac{\gamma}{I}\,A\,(k), \tag{7.29}$$

apart from an additive constant.

Keeping in mind that $X_0^{-1}(k)$ is the time at which a microbunch is injected on the x-axis at the position $x_0(t')$, from (7.29) we deduce

$$t' = \frac{\gamma}{I}\,A\,(k). \tag{7.30}$$

We remark that under the approximation of elliptic phase paths, A (k) coincides with the area of an ellipse of semiaxes x_0 and 2 sin $(x_0/2)$ (see (7.19)).

Therefore, (7.30) gives

$$t' \sim 2\pi \frac{\gamma}{I}\, x_0 \,\sin\,(x_0/2), \tag{7.31}$$

that is just the painting law employed by Colton /8,9/. In other words, to fill uniformly an elliptic region centered at the origin of the phase space, the injection must satisfy the painting law (7.31).

2) In the following, we shall regard the average density (7.27) as a local distribution, in which the dependence on the phase space variables (x,y) appears through the expression of the energy k (see (7.1)). On the basis of this consideration, let us define the line density

$$p\ (x)\ =\ 2\int_0^{y_M} P(k)\ dy\ , \tag{7.33}$$

where $(0,y_M)$ is the interval in which P(k) is different from zero for a given value of x.

By means of (7.1), (7.33) can be written as

$$p\ (x)\ =\ 4\int_{\sin\frac{x}{2}}^{k_M} \frac{kP(k)}{\sqrt{k^2-\sin^2\frac{x}{2}}}\ dk\ , \tag{7.34}$$

where

$$k_M\ =\ \sin\frac{x_0(t'_M)}{2}\ \equiv\ \sin\frac{R}{2}\ . \tag{7.35}$$

We point out that (7.34) resembles (4.4). Therefore, the problem of finding an expression for P(k) such that p(x) be uniform, is similar to that tackled in the case of the harmonic painting. As a consequence, P(k) should be given by

$$P\ (k)\ =\ \frac{s}{k_M\sqrt{k_M^2-k^2}}\ , \tag{7.36}$$

where s is a suitable constant. Substitution from (7.36) into (7.27) yields the relation

$$|\dot{X}_0\ [X_0^{-1}(k)]|\ =\ \frac{Ik_M}{16s}\ \frac{1}{K(k^2)}\ \sqrt{(k_M/k)^2-1}\ , \tag{7.37}$$

where $c\ =\ \frac{Ik_M^2}{16s}$.

From (7.37) we have

$$\int_{k}^{k_M} \frac{k' K(k'^2)\, dk'}{\sqrt{1-\left[\frac{k'}{k_M}\right]^2}} = c(t'_M - t') . \tag{7.38}$$

For practical purposes, it would be useful to exploit the following series representation for $K(k^2)$ (see Ref. 4, p. 905):

$$K(k^2) = F\left[\frac{\pi}{2}, k\right] = \frac{\pi}{2}\, {}_2F_1\left[\frac{1}{2}, \frac{1}{2}; 1; k^2\right] =$$

$$= \frac{\pi}{2}\left\{1+\left[\frac{1}{2}\right]^2 k^2 + \left[\frac{1.3}{2.4}\right]^2 k^4 + \ldots + \left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n} + \ldots\right\}, \tag{7.39}$$

where ${}_2F_1$ denotes the Gauss hypergeometric function.

Eq. (7.38) formally provides the expression for the inverse of the painting function $X_0(t')$ in order to have a uniform line density on a finite interval of the x-axis of the phase space. We remark that for $k_M^2 << 1$, (7.38) reproduces just the corresponding filling law (6.27) already found in the case of harmonic painting. This arises taking into account only the first term of the series expansion (7.39).

In doing this approximation, one can determine the constant s appearing in (7.36), which turns out to be $\frac{1}{8\pi}$.

Substituting the first three terms of (7.39) in (7.38), we arrive at the approximate expression

$$\sqrt{1-\frac{k^2}{k_M^{\,2}}}\left[1 + \frac{1}{6} k_M^{\,2} + \frac{3}{40} k_M^{\,4} + \frac{1}{4}\left(\frac{1}{3} + \frac{3}{20} k_M^{\,2}\right) k^2 + \right.$$

$$\left. + \frac{9}{320} k^4\right] \simeq I\,(t'_M - t') . \tag{7.40}$$

In Figs. 1 and 2 we report the values of $I(t'_M - t')$ vs. x_0 calculated from (6.27), (7.38) and (7.40) for two different values of R, precisely R = 2.0 and R = 3.0.

We observe that for values of R smaller than 2, the approximation (7.40) is better than that found for R = 2. We deduce that formula (7.40) constitutes a very good approximation of the exact law (7.38) in a large range of values of R.

CONCLUSIONS

We have outlined an analytical treatment of the longitudinal painting problem in circular accelerators with high intensity currents. The particle beam behavior is described using the Vlasov equation with the harmonic approximation and for pendulum-like motion, respectively. In both these cases, in which space charge effects are neglected, starting from an initial microbunch distribution of the delta function type we develop a model of filling a limited region in the two dimensional phase space (x,y). We also determine a painting law in such a way as to get a uniform linear projection density. In doing so, we assume that a great number of microbunches is injected turn by turn via a continuous injection mechanism.

For the pendulum-like motion, a general painting law is obtained which includes the corresponding one holding in the harmonic case.

We point out that in dealing with the injection of microbunches along a curve under harmonic approximation, we find a painting law which is formally the same as that derived in the case where the microbunches are injected along the x-axis. It is quite reasonable to guess that this situation may occur also in the case of the pendulum-like motion.

Concerning the arguments of Sec.3, we remark first that a very simple filling strategy leading to a stationary line density is achieved, where the painting action is left to the self synchronous motion of the microbunches. Second, in regard to the concept of the weak stationary condition, we feel that it deserves further deeper investigations for a more systematic theory of the painting.

Although many questions remain unanswered about the painting strategies in circular machines, we believe that the analytical approach developed in this paper may be a first attempt to set up a satisfactory theory of painting. An obvious next step of our program is to tackle the painting in which also space charge effects are taken into account.

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APPENDIX A

For the reader's convenience, in this Appendix we report the standard procedure to solve the pendulum equation

$$\frac{d^2x}{dt^2} + \Omega^2 \sin x = 0 \ . \tag{A1}$$

where $x = x(t)$, being t the time variable.

First, let us multiply (A1) by dx/dt and integrate the resulting equation. We get

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \Omega^2 \cos x = C. \tag{A2}$$

where C is an arbitrary constant.

If we suppose now that x_0 is the greatest displacement from the equilibrium position, then $(dx/dt)_{t=t_0} = 0$, where t_0 is such that $x(t_0) = x_0$. Thus, (A2) provides

$$C = -\Omega^2 \cos x_0 \ . \tag{A3}$$

Inserting (A3) into (A2) we obtain

$$\frac{1}{\sqrt{\cos x - \cos x_0}}\, dx = \sqrt{2}\ \Omega\, dt \ . \tag{A4}$$

At this point we introduce the quantities k and φ, defined by

$$k = \sin \frac{x_0}{2} , \tag{A5}$$

and

$$\cos x = 1\text{-}2\ k^2 \sin^2 \varphi , \tag{A6}$$

which allows us to write

$$\cos x - \cos x_0 = 2\ k^2 \cos^2 \varphi , \tag{A7}$$

and

$$\sin x = 2\ k \sin \varphi \sqrt{1\text{-}k^2 \sin^2\varphi} . \tag{A8}$$

Substitution from (A7) and (A8) into (A4) yields

$$\frac{1}{\sqrt{1\text{-}k^2 \sin^2\varphi}} d\varphi = \Omega\ dt . \tag{A9}$$

The time τ necessary in order that the pendulum reaches the position $x = \bar{x}$ starting from the equilibrium position $x = 0$, is given by

$$\tau = \frac{1}{\Omega} \int_0^{\varphi_0} \frac{1}{\sqrt{1\text{-}k^2 \sin^2\varphi}} d\varphi = \frac{1}{\Omega} F(k,\varphi_0), \tag{A10}$$

where (see A6)

$$\varphi_0 = \sin^{-1} \left[\frac{1}{k} \sin \frac{\bar{x}}{2} \right], \tag{A11}$$

and $F(k,\varphi_0)$ denotes the elliptic integral of the first kind.

Now let us put

$$u = u(\varphi) = \int_0^{\varphi} \frac{1}{\sqrt{1\text{-}k^2 \sin^2\varphi'}} d\varphi', \tag{A12}$$

where φ, the amplitude of u, is denoted by

$$\varphi = \text{am } u \tag{A13}$$

By means of (A12) one can define the following Jacobi elliptic functions of modulus k:

$$\text{sn } (u,k) = \sin \varphi , \tag{A14a}$$

$$\text{cn } (u,k) = \cos \varphi = \sqrt{1\text{-sn}^2(u,k)} , \tag{A14b}$$

$$\text{dn } (u,k) = \Delta\varphi = \sqrt{1\text{-}k^2 \sin^2\varphi} , \tag{A14c}$$

$$dn\ (u,k) = sn\ (u,k)\ /\ cn\ (u,k) = \tan \varphi\ . \qquad (A14d)$$

The inverse functions of (A14)s read

$$u = sn^{-1}(\sin \varphi,k) = cn^{-1}(\cos \varphi,k) = dn^{-1}(\Delta\varphi,k) =$$
$$= am^{-1}\ (\varphi,k) = tn^{-1}\ (\tan \varphi,k) \qquad (A15)$$

By integrating (A9) and using (A15) and (A6), we obtain

$$\int_0^{\varphi} \frac{1}{\sqrt{1-k^2\sin^2\varphi'}}\, d\varphi' = \Omega(t-t_0) =$$

$$= sn^{-1}(\sin \varphi,k) = sn^{-1}\left(\frac{1}{k} \sin \frac{x}{2}\ ,k\right), \qquad (A16)$$

which finally yields the general solution of (A1), namely

$$x = 2 \sin^{-1}\ [k\ sn\ (\Omega(t-t_0),k)]\ . \qquad (A17)$$

APPENDIX B

In order ti prove that (6.34) and (6.37) have the same center (6.36), for the reader's convenience we report here some mathematical notions on the theory of distributions.

Following essentially the terse exposition of Ref. 10, a distribution $<f, \phi>$ is called regular, when it can be put into the form

$$<f, \phi> = \int_{-\infty}^{+\infty} \bar{f}\ (x)\ \phi\ (x)\ dx\ , \qquad (B1)$$

where the integral is meant in the Lebesgue sense, $\bar{f}(x)$ denotes the conjugate of f(x), the test function $\phi\ (x) \in C^{\infty}$ (i.e. $\phi\ (x) = \phi\ (x_1, x_2,\dots, x_n)$ is an infinitely continuously differentiable complex valued function defined in every point of R_n).

As is well known, the Dirac delta function is really a singular distribution. However, there exist sequences of C^{∞} functions which have distributions as their weak limits[10]. To this regard, we remind the reader that a sequence of distributions $\{f_n(x)\}(n = 1,2,\dots)$, converges to the distribution f(x), iff

$$\lim_{n\to+\infty} < f_n(x), \phi(x) > = < f(x), \phi(x) >, \quad \text{(B2)}$$

for each test function $\phi(x)$. f(x) is named the distributional (or the weak) limit of the sequence $\{f_n(x)\}$.

An important example of delta-sequence, for the one-dimensional case, is constituted by the sequence of test functions of the Gaussian type

$$f_n(x) = \frac{n}{\sqrt{\pi}} \exp(-n^2x^2) . \quad \text{(B3)}$$

Then, the Dirac delta function can be represented by

$$\lim_{n\to+\infty} < f_n(x), \phi(x) > = \phi(0) , \quad \text{(B4)}$$

for any test function $\phi(x) \in D$, where D is the space of all C^{∞}-functions $\phi(x)$ defined on R and vanishing outside some bounded region of R_1, which depends on $\phi(x)$.

From (B4) we have

$$\lim_{n\to+\infty} < f_n(x) = \delta(x). \quad \text{(B5)}$$

In the two-dimensional case, the Dirac delta function can be defined by[11]

$$\lim_{n\to+\infty} < f_n(x_1, x_2), \phi(x_1, x_2) > = \phi(0,0), \quad \text{(B6)}$$

for any test function $\phi(x_1, x_2) \in D$, where $f_n(x_1,x_2)$ is given by

$$f_n(x_1,x_2) = \frac{n^2}{\pi} \exp[-n^2(x_1^2 + x_2^2)] . \quad \text{(B7)}$$

Eq. (B6) tells us that

$$\lim_{n\to+\infty} < f_n(x_1, x_2) = \delta(x_1, x_2) = \delta(x_1)\, \delta(x_2) . \quad \text{(B8)}$$

It turns out that the two-dimensional delta function (B8) is invariant under rotations.

We shall resort below to the following operation. Let A be a non singular linear transformation of the independent variables

$x = (x_1, x_2, \ldots, x_n)$. The formula

$$<f\ (A\ x),\ \phi(x)> \ = \ \frac{1}{|A|}\ <f(x),\ \phi(A^{-1}x)>, \qquad \text{(B9)}$$

holds, where A^{-1} is the inverse transformation and $|A|$ the absolute value of the determinant of A.

Now let us introduce two systems of cartesian coordinates, say (X,Y) and (X',Y'), where (X',Y') are rotated counterclockwise through an angle θ_0. The coordinates of a generic point (X,Y) in the first frame are transformed into the coordinates (X',Y') in the second frame by the relations

$$X' = X \cos \theta_0 + Y \sin \theta_0 , \qquad \text{(B10a)}$$

$$Y' = X \sin \theta_0 + Y \cos \theta_0 . \qquad \text{(B10b)}$$

By means of (B7) and (B8), we may define in the (X,Y)-plane a delta function centered at the point $(X_0, Y_0) = (r_0 \cos \theta_0, r_0 \sin \theta_0)$, where r_0 is a positive real number.

Keeping in mind (B9), under the rotation (B10) the two-dimensional delta function $\delta(X - X_0)\ \delta(Y - Y_0)$ is transformed into the two-dimensional delta function $\delta(X' - r_0)\ \delta(Y')$, whose center takes place at the point $(X'_0, Y'_0) = (r_0, 0)$ of the rotated frame (X',Y').

If we identify X and Y with

$$X = x' \cos (\Omega \Delta t + \theta_0) - y' \sin (\Omega \Delta t + \theta_0), \qquad \text{(B11a)}$$

$$Y = x' \sin (\Omega \Delta t + \theta_0) + y' \cos (\Omega \Delta t + \theta_0), \qquad \text{(B11b)}$$

(B10) provides

$$X' = x' \cos \Omega \Delta t - y' \sin \Omega \Delta t, \qquad \text{(B12a)}$$

$$Y' = x' \sin \Omega \Delta t + y' \cos \Omega \Delta t. \qquad \text{(B12b)}$$

Therefore, the expression (6.37), regarded as a distribution in the (X',Y') plane, can be obtained from (6.34) under the rotation (B10).

In the coordinate systems (X,Y) and (X',Y') the distributions (6.34) and (6.37) are centered at distinct points. However, we point out that in the coordinate system (x',y'), (6.34) and

(6.37) turn out to have the same center, which is located at the point $(x_0, y'_0) = (r_0 \cos \Omega\Delta t, -r_0 \sin \Omega\Delta t)$ (see (B11) and (B12)).

Since

$$\{[x' \cos(\Omega\Delta t + \theta_0) - y' \sin(\Omega\Delta t + \theta_0)] - r_0 \cos\theta_0\}^2$$

$$+ \{[x' \sin(\Omega\Delta t + \theta_0) + y' \cos(\Omega\Delta t + \theta_0)] - r_0 \sin\theta_0\}^2 =$$

$$= [(x' \cos \Omega\Delta t - y' \sin \Omega\Delta t) - r_0]^2 =$$

$$= (x' - r_0 \cos \Omega\Delta t)^2 + (y' + r_0 \sin \Omega\Delta t)^2, \quad \text{(B13)}$$

by virtue of (B7) and (B8) it follows that (6.34) and (6.37) represent the same delta function.

APPENDIX C

Here we calculate the Jacobian of the transformation (7.1), namely:

$$J = \begin{vmatrix} \frac{\partial x}{\partial k} & \frac{\partial x}{\partial \beta} \\ \frac{\partial y}{\partial k} & \frac{\partial y}{\partial \beta} \end{vmatrix} . \quad \text{(C1)}$$

To this aim, exploiting the derivatives of the Jacobi elliptic functions in regard to their modulus k (Ref.12), we have

$$\frac{\partial x}{\partial k} = \frac{2}{\sqrt{1-k^2 sn^2(\beta,k)}} \frac{d}{dk} [k \; sn(\beta,k)] =$$

$$= \frac{2}{\sqrt{1-k^2 sn^2(\beta,k)}} \left[sn(\beta,k) - k \; cn(\beta,k) dn(\beta,k) \frac{dF}{dk} \right], \quad \text{(C2)}$$

$$\frac{\partial x}{\partial \beta} = 2k \; cn(\beta,k), \quad \text{(C3)}$$

$$\frac{\partial y}{\partial k} = 2 \; cn(\beta,k) + 2 \; k \frac{d}{dk} cn(\beta,k) = 2 \; cn(\beta,k)$$

$$+ 2 \; k \; sn(\beta,k) \; dn(\beta,k) \frac{dF}{dk} , \quad \text{(C4)}$$

$$\frac{\partial y}{\partial \beta} = -2k \ \mathrm{sn}(\beta,k) \ \mathrm{dn}(\beta,k), \tag{C5}$$

where F denotes the elliptic integral of the first kind. The insertion of (C2) - (C5) in (C1) provides J = -4k.

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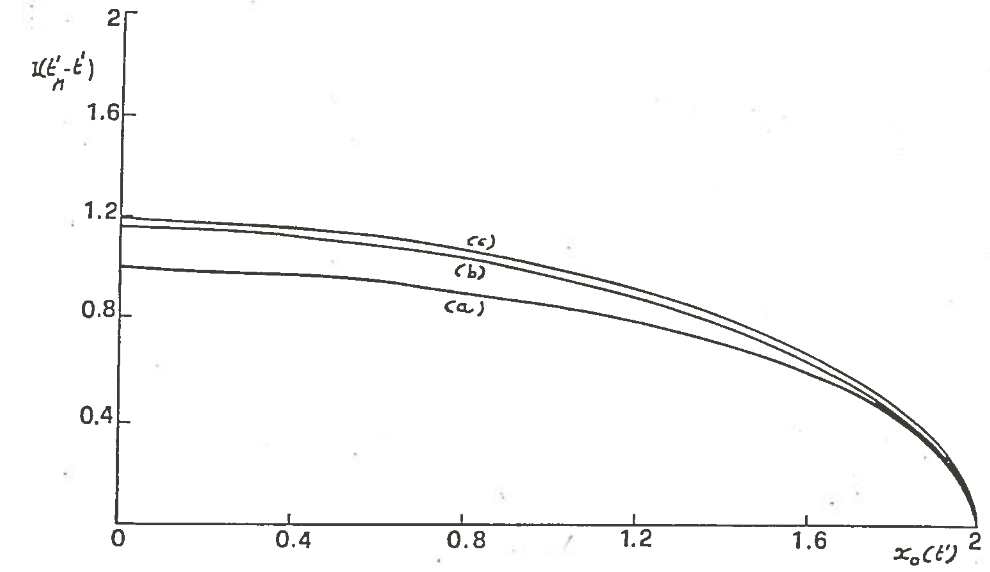
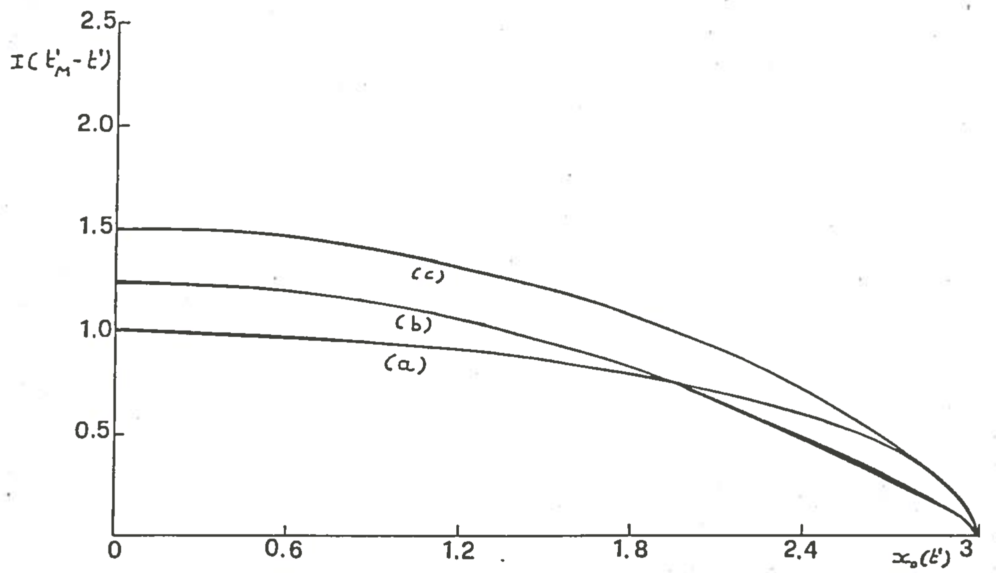


Fig. 1: Behavior of the injection function $x_0(t')$ for the harmonic case [Eq.(6.27), curve (a)] and the pendulum case under the approximation (7.40) [curve (b)]. Curve (c) represents the exact painting law (7.38) for the pendulum-like case. Elsewhere we have chosen



R = 2.0.

Fig. 2: Behavior of the same painting laws pictured in Fig.1 corresponding to the choice R = 3.0.