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STRUCTURE CONSTANTS OF THE (A,D) MINIMAL c<1 CONFORMAL MODELS

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ABSTRACT

The primary fields structure constants of the (A,D) minimal $c<1$ series are computed explicitly. Various inperpretations of the fusion algebras are discussed.

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1. The BPZ [1] $c=1-6/[m(m+1)]<1$ minimal series of local (unitary) conformal field theories constructed explicitly on the plane in [2] describe a field content corresponding to the (A,A) modular invariants on the torus [3]. In [4] we have built local (i.e. symmetric under permutation of the fields) euclidean 4-point functions which generate the (A,D) series. The structure constants of the primary fields can be read off from these functions, but we feel it is worthwile presenting the results in a more explicit form.

For any $m+1 \geqslant 6$, even, the $(A_{m-1}, D_{\frac{m+3}{2}})$ fusion algebra of Vir x $\overline{\text{Vir}}$ primary fields has the general structure [4]

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 , \qquad \mathcal{A}_i \times \mathcal{A}_j = \mathcal{A}_{i+j} , \quad i,j = 0,1 \mod 2 \tag{1a}$$

where the subalgebra $\mathcal{A}_0$ is a subalgebra of the (A_{m-1}, A_m) scalar algebra

$$\mathcal{A}_0 = \{ \varphi_A \,|\, \Delta_A = \bar{\Delta}_A = \Delta_{n'n} , \quad n \text{ - odd} \} \tag{1b}$$

with real structure constants D_{IJ}^{K} computed in [2], while

$$\mathcal{A}_1 = \{ \hat{\varphi}_{\hat{F}} \,|\, \hat{F}=(F,\bar{F}),\ \Delta_F = \Delta_{k'k},\ \Delta_{\bar{F}} = \Delta_{k'\, m+1-k},\ k=\tfrac{m+1}{2} \ (\text{mod } 2) \} \tag{1c}$$

contains $(m-1)/2$ new scalar fields along with nonzero, integer spin $s(\hat{F}) = \Delta_F - \Delta_{\bar{F}}$ fields (mod 2 $s(\hat{F})$ depends only on k). In (1b,c) $1 \leqslant n', k' \leqslant m-1$, $1 \leqslant n,k \leqslant m$ and to select the nonequivalent fields it is enough to take $1 \leqslant n', k' \leqslant (m-1)/2$ (see [1]).

The set of scalars generates the whole algebra $\mathcal{A}$. For m+1=6 mod 4 ((A, D_{2p+2}) series in the notation of [3]) the scalars of dimension $\Delta_{k' \frac{m+1}{2}}$ double. The detailed fusion rules factorize to the A- and D-type $\widehat{su}(2)$ fusion rules [5], reflecting the factorization of the crossing matrix. However, the structure constants (and hence the normalized fusion matrices of [6]) do not factorize and so the problem does not reduce simply to that of the $\widehat{su}(2)$-case [7].

For $m \geqslant 6$, even, there is an analogous (D,A) fusion algebra.

Let $\varphi_A \in \mathcal{A}_0$, $A=(n'n)$, $\hat{\varphi}_{\hat{C}} \in \mathcal{A}_1$, $\hat{C}=(C,\bar{C})=[(k'k),(k'\ m+1-k)]$. The structure constants are extracted from the leading singular behaviour of a pair of functions

$$\langle \varphi_A(x_1)\hat{\varphi}_{\hat{C}}(x_2)\varphi_A(x_3)\hat{\varphi}_{\hat{C}}(x_4)\rangle \approx \sum_{\hat{F}} \frac{(\hat{D}^{\hat{F}}_{A\hat{C}})^2 \langle \hat{\varphi}_{\hat{F}}(x_2)\, \hat{\varphi}_{\hat{F}}(x_4)\rangle}{(z_{12}z_{34})^{\Delta_A+\Delta_C-\Delta_F}(\bar{z}_{12}\bar{z}_{34})^{\Delta_A+\Delta_{\bar{C}}-\Delta_{\bar{F}}}}+\dots \tag{2a}$$

$$\langle \varphi_A(x_1)\varphi_A(x_2)\hat{\varphi}_{\hat{C}}(x_3)\hat{\varphi}_{\hat{C}}(x_4)\rangle \approx \sum_{F'} \frac{D^{F'}_{AA}\,\hat{D}^{F'}_{\hat{C}\hat{C}}\,\langle \varphi_{F'}(x_2)\,\varphi_{F'}(x_4)\rangle}{(x^2_{12})^{2\Delta_A-\Delta_{F'}}z_{34}^{2\Delta_C-\Delta_{F'}}\bar{z}_{34}^{2\Delta_{\bar{C}}-\Delta_{\bar{F}'}}}+\dots , \tag{2b}$$

suppressing the contribution of the descendent fields. Here $x=(x^1,x^2)\in\mathbb{R}_2$, $z=x^1+ix^2$.

We choose

$$D^{\mathbf{1}}_{FF}=1\,,\quad \hat{D}^{\mathbf{1}}_{\hat{F}\hat{F}}=(-1)^{s(\hat{F})} \tag{3}$$

consistently with the choice of the normalization of the 2-point function

$$\langle \hat{\varphi}_{\hat{F}}(x_1)\hat{\varphi}_{\hat{F}}(x_2)\rangle = d_{\hat{F},\hat{F}}\, z_{12}^{-2\Delta_F}\,\bar{z}_{12}^{-2\Delta_{\bar{F}}}\,,\quad d_{\hat{F},\hat{F}}=e^{i\pi s(\hat{F})}\,,\quad d_{F,F}=1. \tag{4}$$

The reason for this choice will become clear below.

The relative constant of (2a) and (2b) is fixed by locality and assuming (3) we have, using the explicit expressions in [4]:

$$\langle \varphi_A(x_1)\hat{\varphi}_{\hat{C}}(x_2)\varphi_A(x_3)\hat{\varphi}_{\hat{C}}(x_4)\rangle = \sum_{\hat{F}} (-1)^{s(\hat{F})}\, J_F(A,C,A,C;Z)\, J_{\bar{F}}(A,\bar{C},A,\bar{C};\bar{Z})$$

$$= f(z_{ij},\alpha_k)f(\bar{z}_{ij},\bar{\alpha}_k)\sum_{j',j}(-1)^{j-\frac{n+1}{2}+s(\hat{C})}\, S_{j'j}(a,b,c)\,S_{n'+1-j'\,j}(a,b,\bar{c})\, \tilde{I}^{n'n}_{j'j}(a,b,c;z)\,\tilde{I}^{n'n}_{n'+1-j'\,j}(a,b,\bar{c};\bar{z}) \tag{5}$$

where $\tilde{I}^{n'n}_{j'j}(z)=I^{n'n}_{j'j}(z)/N_{j'j}\approx z^{-2\alpha_1\alpha_2-(\Delta_A+\Delta_C-\Delta_F)}[1+O(z)]$ are the normalized contour integrals defined in [2], $z=z_{12}z_{34}/z_{13}z_{24}$; $f(z_{ij},\alpha_k)=[z(1-z)]^{2\alpha_1\alpha_2}z_{13}^{-2\Delta_A}z_{24}^{-2\Delta_C}$; $\alpha_1=\alpha_3=\bar{\alpha}_1=\bar{\alpha}_3=\alpha_{nn}$, $\alpha_2=2\alpha_0-\alpha_{k'k}$, $\bar{\alpha}_2=\alpha_{k'\,m+1-k}$, $a=b=2\alpha_-\alpha_1$, $c=2\alpha_-\alpha_2$, $\bar{c}=2\alpha_-\bar{\alpha}_2$, $-\alpha_-=$ $=m\,2\alpha_0=[m/(m+1)]^{1/2}$, $F=(p'p)$, $p=k-n+2j-1$, $p'=k'-n'+2j-1$, $\Delta(\alpha)=\alpha(\alpha-2\alpha_0)$.

The constants

$$S_{j'j}(a,b,c) = \sqrt{\frac{\gamma_{j'j}(a,b,c)}{\gamma_{n'n}(a,c,b)}\,\frac{\mathcal{N}_{j'j}(a,b,c)}{\mathcal{N}_{n'n}(a,c,b)}} = S_{n'+1-j'\,n+1-j}(b,a,d) \tag{6a}$$

reproduce the Dotsenko - Fateev (DF) structure constants

$$S^2_{j'j}(a,a,c)=(D^F_{AC})^2\,,\qquad S^2_{j'\,n+1-j}(a,a,\bar{d})=(D^{\bar{F}}_{A\bar{C}})^2\,. \tag{6b}$$

Here $\gamma_{j'j}(a,b,c) = \gamma_{j}(a,b,c)\gamma_{j'}(a',b',c')$ are the expansion coefficients computed in [2]; $a'=-a\alpha_-^{-2}$, etc., $d = 2\alpha_-(2\alpha_c-\alpha_2)$, $\bar{d}=2\alpha_-(2\alpha_c-\bar{\alpha}_2)$, $c-\bar{c}=\bar{d}-d=1$ mod 2, $\bar{d'}-c'=0$ mod 2, $d'-\bar{c'}=0$ mod 2. The coefficients $\gamma_j(a,b,c)$ do not change if some of the parameters a,b,c change by an integer.

Comparing (5) with (2) and accounting for (4) we obtain

$$(\hat{D}^{\hat{F}}_{A\hat{C}})^2 = S_{jj'}(a,a,c)\, S_{j'\, n+1-j}(a,a,\bar{d}) \, . \tag{7a}$$

All considerations up to here can be repeated without essential changes for the general minimal series, where m is replaced by p/(q-p), p,q - coprime integers, q - even. For the main series (q=p+1) a direct but tedious check shows that for values of the parameters consistent with the chiral fusion rules the r.h.s. of (7a) is positive and hence the constants $\hat{D}^{\hat{F}}_{A\hat{C}}$ are real. We obtain

$$(\hat{D}^{\hat{F}}_{A\hat{C}})^2 = D^{F}_{AC}\, D^{\bar{F}}_{\bar{A}\bar{C}} = (\hat{D}^{\hat{\bar{F}}}_{A\hat{\bar{C}}})^2 \tag{7b}$$

since the DF constants D^{F}_{AC} and $D^{\bar{F}}_{A\bar{C}}$ can be chosen to be of the same sign.

From the locality of the 3-point functions one obtains various relations for their normalization coefficients d^{F}_{AC}, which in our integer spin case are symmetric for even permutations of the triple (F,A,C). For the structure constants $D^{F}_{AC} = d^{F}_{AC}/d_{F,F}$ (cf.(4)) these relations read

$$\hat{D}^{\hat{F}}_{A\hat{C}} = (-1)^{s(\hat{F})+s(\hat{C})}\hat{D}^{\hat{F}}_{\hat{C}A} = (-1)^{s(\hat{F})+s(\hat{C})}\hat{D}^{\hat{C}}_{\hat{F}A} = \hat{D}^{\hat{C}}_{A\hat{F}} = (-1)^{s(\hat{C})}\hat{D}^{A}_{\hat{F}\hat{C}} = (-1)^{s(\hat{F})}\hat{D}^{A}_{\hat{C}\hat{F}} \, . \tag{8}$$

We have further information about the sign of $\hat{D}^{F}_{\hat{C}\hat{C}}$ from the explicit expression for $D^{F}_{AA}\,\hat{D}^{F}_{\hat{C}\hat{C}}$ obtained from the 4-point function (2b) (or equivalently from the $(x_1,x_3) \to (x_2,x_4)$ channel of (5))

$$\langle \varphi_A(x_1)\varphi_A(x_2)\hat{\varphi}_{\hat{C}}(x_3)\hat{\varphi}_{\hat{C}}(x_4)\rangle = \sum_F (-1)^{e(F)+s(\hat{C})} J_F(A,A,C,C;Z)\, J_F(A,A,\bar{C},\bar{C};\bar{Z}) \tag{9}$$

$$\left(= (-1)^{s(\hat{C})} \sum_{\hat{F}} J_F(C,A,A,C;1-Z)\, J_{\bar{F}}(\bar{C},A,A,\bar{C};1-\bar{Z}) \right),$$

$$e(F) = \left\{ \begin{array}{ll} s(\hat{F}) + \frac{m-1}{4} & \text{for } m+1=6 \bmod 4 \\ s(\widehat{F+1})+1 + \frac{m+1}{4} & \text{for } m+1=8 \bmod 4 \end{array} \right\} \bmod 2 \, . \tag{10a}$$

(Here F+1 is a notation for (p' p+1)'. The conformal blocks in (9) are realized by the DF contour integrals analogously to (5); 1-Z denotes (z_3, z_2, z_1, z_4).) In the special case when $\hat{\varphi}_{\hat{C}}$ is a scalar , $\hat{C}=(C,C)$, we get

$$\hat{D}^{F}_{\hat{C}\hat{C}} = (-1)^{e(F)} D^{F}_{CC} . \tag{10b}$$

The two (A,D) series -for m+1=8 mod 4 $((A,D_{2p+1}))$ and m+1=6 mod 4 $((A,D_{2p+2}))$ - are treated here on equal footing. In particular, the fusion rules (cf. 1a) for both cases are

$$\begin{aligned} \varphi_{(F_1,F_1)} \times \hat{\varphi}_{(F_2,\bar{F}_2)} &= \sum_i \hat{\varphi}_{(F_i,\bar{F}_i)} \\ \hat{\varphi}_{(F_1,\bar{F}_1)} \times \hat{\varphi}_{(F_2,\bar{F}_2)} &= \sum_i \varphi_{(F_i,F_i)} \end{aligned} \tag{11}$$

where the r.h.s. is described according to the standard chiral (left) fusion rules. The mapping $\sigma(F) = \bar{F}$ for F = (k'k) , $k=\frac{m+1}{2}$ mod 2 (cf.1c) can be extended in the case m+1=8 mod 4 to $\sigma(P)=P$, if P =(k'k), k - odd. Then it provides an automorphism of the chiral fusion rules [8] of the corresponding (A,A) fusion algebra. Yet the scalar fields $\hat{\varphi}_{\hat{F}}$, $\hat{F}=(F,F)$, cannot be identified with their DF counterparts, as is clear from (7,10,11). As we shall see below, the case m+1=6 mod 4, which corresponds to diagonalizable modular invariants in [3], admits also other interpretations than the one considered up to now.

2. We can interpret the conformal invariant function in (5) as the correlation $\langle \varphi^*_A \hat{\varphi}^*_{\hat{C}} \varphi_A \hat{\varphi}_{\hat{C}} \rangle$ and , accordingly, the function in (4) as the 2-point function $\langle \hat{\varphi}^*_{\hat{F}} \hat{\varphi}_{\hat{F}} \rangle$ with apriori different normalizations. The DF correlations admit a similar interpretation. Here * denotes the hermitian conjugation with respect to the euclidean sesquilinear form (,), which we assume to realize our fields averages. From the hermiticity of the form and locality of 2- and 3-point functions one has $\overline{d_{A,B}} = d_{A^*,B^*}$, $\overline{D^{C}_{AB}} = D^{C^*}_{A^* B^*}$ (we use now a unified notation to cover all cases). Starting with the DF -case, the Osterwalder - Schrader (OS)

(reflection) positivity (equivalently, Wightman positivity in M_2) of the 2-point functions requires $d_{A^*,A}$ - real , positive (and $\Delta > 0$ which is ensured in the main series). Hence the choice $d_{A^*,A} = 1 = D^{\dagger}_{A^*A}$ is possible, irrespectively of the choice for $d_{A,A}$. This is also consistent with the explicit expression for the 4-point functions, since what has to be identified with $\overline{D^F_{AB}}\, D^F_{AB}$ is indeed positive for the main series. Now we can choose $d_{A^*,A} = d_{A,A} = 1$ and hence $D^F_{AB} = \overline{D^F_{AB}} = D^{F^*}_{A^*B^*}$, which implies that all DF fields for the main series can be considered to be real (hermitian) $\varphi^*_A = \varphi_A$. In our case we can choose $d_{\hat{\bar{F}}^*,\hat{F}} = d_{\hat{F},\hat{F}}$ with $d_{\hat{F},\hat{F}}$ as in (4), hence $D^{\hat{F}}_{A\hat{C}}$ are real, given by (7a) and $D^{\hat{F}}_{A\hat{C}} = \overline{D^{\hat{\bar{F}}}_{A\hat{\bar{C}}}} = D^{\hat{\bar{F}}^*}_{A\hat{\bar{C}}^*} = D^{\hat{\bar{F}}}_{A\hat{\bar{C}}}$. This implies that we can identify $\varphi_{\hat{\bar{F}}} = \varphi^*_{\hat{F}}$ and hence to consider all scalar fields, as well as the field $\hat{F} \oplus \hat{\bar{F}}$, irreducible under $O^{\uparrow}(3,1)$, as real fields. Since in our case all $\Delta, \bar{\Delta} > 0$ the choice of the sign in (4) ensures the OS - positivity of the 2-point functions for these fields . This is equivalent to the positivity of the Wightman function counterparts of (4), which can be easily checked looking at the Fourier transform

$$\hat{W}(p) = -i[\, \hat{S}(p_1, -ip^0+\varepsilon) - \hat{S}(p_1, -ip^0-\varepsilon)] \tag{12a}$$
$$= \frac{8\pi^2\, (p_1 + \frac{s}{|s|} p_0)^{2|s|}}{4^d\, \Gamma(d-s)\, \Gamma(d+s)}\, \theta(p^0)\, \theta(-p^2_M)\, (-p^2_M)^{d-|s|-1}$$

$d = \Delta + \bar{\Delta}$, $p^2_M = p^2_1 - p^2_0$; $\hat{S}(p^1, p^2)$ is the Fourier transform of the euclidean 2-point function in (4). (See also [9] for explicit formulae and a discussion of reflection positivity in the framework of the conformal theories.) To check the OS - positivity of the 4-point functions and hence, to recover unitarity of the main minimal series, one needs more information than is available at present. Consider for simplicity the case when in (5) the field $\varphi_{\hat{C}}$ is a scalar. The OS - positivity condition for the 4-point function $S(x_1, A; x_2, \hat{C}; x_3, A; x_4, \hat{C}) = S(x_2, \hat{C}; x_1, A; x_3, A; x_4, \hat{C})$ in (5) reads

$$\iint dx_1 dx_2\, \bar{f}(\Theta x_1, \Theta x_2)\, S(x_1, A; x_2, \hat{C}; x_3, A; x_4, \hat{C})\, f(x_3, x_4) > 0 \tag{12b}$$

where $f(x,y)$, $x=(x^1,x^2)\in R_{2)}$ is a suitable test function, which vanishes with all its derivatives unless $x^2, y^2 > 0$, $x \neq y$; $\theta x=(x^1,-x^2)$. The condition (12b) reduces to the OS - positivity of the 2-point functions, if all structure constants , including those resulting from the (spin) combinations of the descendent (quasi-primary) fields, are real.

3. Recalling that for m+1=6 mod 4 both $\hat{\varphi}_{\hat{A}}$ and φ_A , $\hat{A}=(A,A)$, $A=(k'\,\frac{m+1}{2})$, belong to the fusion algebra, we define the fields

$$\Phi^{\pm} = \frac{1}{\sqrt{2}}(\varphi_A \pm \hat{\varphi}_{\hat{A}}) \qquad \text{for } m+1 = 10 \mod 8 \tag{13a}$$

$$\Phi = \frac{1}{\sqrt{2}}(\varphi_A + i\hat{\varphi}_{\hat{A}}) \,,\ \Phi^* = \frac{1}{\sqrt{2}}(\varphi_A - i\hat{\varphi}_{\hat{A}}) \quad \text{for } m+1 = 6 \mod 8 \tag{13b}$$

Then $\langle\Phi^*_A \Phi_A\rangle = \langle\varphi_A \varphi_A\rangle, \langle\Phi^{\pm}_A \Phi^{\pm}_A\rangle = \langle\varphi_A \varphi_A\rangle$. Now (7), which holds for both (A,D) series, gets modified for m+1=6 mod 4. Indeed, using (8,10) (for A=B and a proper choice of sign for A≠B) one obtains that the products $\Phi^{\pm}_A \times \Phi^{\pm}_B$ and $\Phi_A \times \Phi_B$ ($\Phi^*_A \times \Phi^*_B$) contain only <u>even</u> (including zero) spin primary fields, while $\Phi^{\pm}_A \times \Phi^{\mp}_B$ and $\Phi^*_A \times \Phi_B$ contain <u>odd</u> spin fields along with scalar fields. The scalar content in the first case includes $\Phi^{\pm}_F$ and Φ^*_F (Φ_F), respectively, with structure constants modified by a factor $\sqrt{2}$. The correlation functions admit a diagonal form, reminiscent of the diagonal modular invariants of [3], with the factor 2 appearing in front of the contribution of the fields $\Phi^{\pm}_F$ or $\Phi^*_F(\Phi_F)$. Since $(\hat{D}^F_{\hat{A}\hat{B}})^2 = (D^F_{AB})^2$ for $s(\hat{A})=0=s(\hat{B})$, in computing the correlation functions of the fields (13) we postulate that $\langle\hat{\varphi}_{\hat{A}}\,\hat{\varphi}_{\hat{B}}\,\hat{\varphi}_{\hat{A}}\,\hat{\varphi}_{\hat{B}}\rangle = \langle\varphi_A\,\varphi_B\,\varphi_A\,\varphi_B\rangle$.

According to the interpretation in [8,10] one can identify each of these diagonal terms, accomodating representations F, $\bar{F}$, $\hat{F}$, and $\hat{\bar{F}}$ with the same (mod 1) dimensions Δ , $\bar{\Delta}$, as the product of the conformal blocks of an extended chiral algebra. The fusion rules of the resulting (extended) (A,D) series diagonalize. Clearly, using the basis with the linear combinations in (13), instead of the initial scalar fields , already changes the homogeneous fusion algebra (1a). Accordingly the formulae for the structure constants (7) change due to cancellations and appearance of the factors $\sqrt{2}$, $1/\sqrt{2}$, which can be easily traced ; $1/\sqrt{2}$ modifies the constants for a triple containing just one of the fields in (13). (Modifications also appear due to the factors $-1, \pm i$, in the definition (13).)

Let us give an example. Take $m+1=6$. Denoting $\Phi_r = \Phi_{r3}$, $r=1,2$ and identifying $\Phi_{5-r} = \Phi_r$, we have

$$\Phi_{r_1} \times \Phi_{r_2} = \sum_r \Phi_r^*$$

$$\Phi_{r_1} \times \Phi_{r_2}^* = \sum_r (\varphi_{r1} + \varphi_{r5} + \hat{\varphi}_{\hat{r1}} + \hat{\varphi}_{\hat{r5}}) \tag{14}$$

$$\Phi_{r_1} \times \varphi_{r_2 1} = \sum_r \Phi_r \,, \quad \Phi^*_{r_1} \times \varphi_{r_2 1} = \sum_r \Phi^*_r \,, \text{ etc.}$$

The sums in (14) run by two from $|r_1 - r_2| + 1$ to $r_1 + r_2 - 1$, t.e. have one or two different terms. If the fields φ_{r5}, $\hat{\varphi}_{\hat{r1}}$, $\hat{\varphi}_{\hat{r5}}$ are treated as descendants of φ_{r1}, the fusion rules (14) recover exactly the neutral fusion subalgebra of the $\mathbb{Z}_3$ - model of [11]. Indeed, the field Φ_1 can be identified with the product of the parafermionic fields $\psi(z)\,\overline{\psi}(\bar{z})$, $\Phi_1^* = \psi^+(z)\,\overline{\psi}^+(\bar{z})$, while $\Phi_2 = \sigma_1 = \phi^{(1)}_{[1,1]}$, $\Phi_2^* = \sigma_2 = \sigma_1^+$, and $\varphi_{11} = \mathbb{1}$, $\varphi_{21} = \varepsilon = \phi^{(2)}_{[0,0]}$ (see [11] for notation).

The interpretation of the even (A,D) series discussed here extends to non-unitary series as well, however, the fields Φ_A and Φ_A^* will be independent, not hermitian conjugated fields.

The set of local correlation functions constructed in [4] is sufficient to recover the spin combinations listed in the (A,D) modular invariants and to provide a field interpretation described by the set of fusion rules in (1,11) (or, equivalently, by the fusion algebra generated when passing to the linear combinations in (13)). A question arises are these functions the only local functions, yielding the (A,D) content. Recent results [10,7] (for the $\widehat{su}(2)$-case) suggest that there might be more possibilities. Consider the simplest example in [10] when $m+1 = 14$. The combinations (5,9) and (9,5) ($k'=1$) which appear in the product $\varphi_7 \times \hat{\varphi}_{\hat{7}}$ (or $\Phi_7 \times \Phi_7^*$) could be also obtained by effectively doubling the scalar (5,5). Indeed, it can be checked explicitly that there is a crossing invariant (under, say $x_1 \leftrightarrow x_3$) and monodromy invariant around $z_{12}=0$ and $z_{23}=0$, nondiagonal 4-point function, which replaces the DF diagonal combination <(5,5)(5,5)(5,5)(5,5)>. It yields (9,5) and (5,9) and reproduces the same modification by $1/\sqrt{2}$ of the structure constants as in [7]. However, we were not able to find a field interpretation, say, considering it as a correlation of two (or more) different (5,5) fields, to be consistent with all locality (symmetry) constraints. Note that (5,9), (9,5), have odd spin, so that the product $\mathbb{B}_{12}\,\bar{\mathbb{B}}_{12}$ of left and right braid transformations (see below) is not trivial, although $\mathbb{B}^2_{12}\,\bar{\mathbb{B}}^2_{12} = \mathbb{1}$. *)

4. To prove the crossing invariance of the correlation (5) one has to exploit the fact that a change by an integer of any of the parameters a,b,c of the crossing matrix $\alpha_{jk}(a,b,c)$ keeps its matrix elements invariant up to a sign. (The parameter $d = -a-b-c-2\alpha_-^2(n-2)+2(n'-2)$ is a function of a,b,c, so that the charge conservation condition is maintained.) The exact form of these sign factors has been fixed in [4b] using an identity ((A.13)) for the crossing matrices. It results when connecting by analytic continuation the three fundamental bases at $z = 0,1,\infty$. As conjectured in [4b] this identity turns out to be equivalent to the Yang - Baxter equation for the braid matrices transforming the full euclidean chiral 4-point conformal blocks (see [12]-[14]). In the Yang - Baxter equation for the 4-point function in the standard basis the braid matrix $\mathbb{B}_{12}$ is diagonal

$$B_{12}(\varepsilon)(=(\mathbb{B}^{IJ}_{AC}(\varepsilon))_{AC}) = e^{i\pi\varepsilon\,[2\alpha_1\alpha_2 + A^{n'n}_{j'j}(a+c)]} \tag{15a}$$

$$= e^{-i\pi\varepsilon\,[\Delta_A + \Delta_C - \Delta_J(A,C)]}$$

where $A^{n'n}_{j'j}(a+c) = A^{n'}_{j'}(a'+c') + A^{n}_{j}(a+c) - 2(j-1)(j'-1)$, $A^{n}_{j} = (j-1)[1+a+c+(j-2)\alpha_-^2]$ (see [2])). The nondiagonal braid matrix

$$\mathbb{B}_{23}(\varepsilon)\ (=(\mathbb{B}^{AD}_{CB}(\varepsilon))_{JT}) = e^{i\pi\varepsilon[\Delta_A + \Delta_D - \Delta_J(A,C) - \Delta_T(A,B)]}\ \underset{\sim}{\alpha}_{jt}(c,b,a)\ \underset{\sim}{\alpha}_{j't'}(c',b',a') \tag{15b}$$

is recovered by any of the two sides of the identity in [4b], when the phases coming from the prefactors are taken into account. (More precisely, one has to use the identity for $\underset{\sim}{\alpha}_{(j'j)(t't)} = \underset{\sim}{\alpha}_{jt}\, \underset{\sim}{\alpha}_{j't'}$, which completely factorizes). Here J and T are expressed by (a,c;j'j) and (a,b;t't) using the formulae of [2] and $\underset{\sim}{\alpha}_{jt}(a,b,c) = [\gamma_j(a,b,c)/\gamma_t(b,a,c)]^{1/2} \alpha_{jt}(a,b,c)$ $(\gamma_j(a,b,c) = \gamma_j(c,b,a))$ are the DF crossing matrices satisfying

$$\underset{\approx}{\alpha}\, {}^t\underset{\sim}{\alpha} = \mathbb{1} \;, \quad \underset{\approx}{\alpha}(a,b,c)\, \underset{\approx}{\alpha}(b,a,c) = \mathbb{1} \,. \tag{16}$$

Conversely, the crossing matrix can be expressed up to an overall phase factor by the l.h.s. (or r.h.s.) of the Yang - Baxter equation.

$$e^{i\pi\varepsilon[\Delta(d)-\Delta(a)-\Delta(b)-\Delta(c)]}\, \underset{\approx}{\alpha}(a,b,c) = \mathbb{B}_{12}(\varepsilon)\, \mathbb{B}_{23}(\varepsilon)\, \mathbb{B}_{12}(\varepsilon) \tag{17}$$

The braid -matrices defined in (15) satisfy automatically the conformal condition of [14b]. For (15b) it follows from the second equality in (16) and it is explicit for (15a).

There has been given a general algorithm for the computation of the matrices $\alpha_{jk}(a,b,c)$ in [2] and explicit expressions have been found for k=1 (or j=1) and the cases related to these by symmetries. Presumably, the formulae for the 6j-symbols derived in the context of the quantum group $sl_q(2)$ representations [15] (for $q^N = 1$) provide an explicit expression in the general case. There is a full correspondence between the set of polynomial identities of [6] for g=0 and the set of identities for the quantum 6j - symbols, derived in [15] from the corresponding relations for the quantum Clebsch - Gordan coefficients. Note that (A.13) in [4b] can be also rewritten as

$$\sum_K (\mathbb{B}^{AD}_{CB}(\varepsilon))_{JK}\, (F^{AD}_{BC})_{KT}\, e^{i\pi\varepsilon[\Delta_B + \Delta_C - \Delta(A,D)]_T} = (F^{AD}_{CB})_{JT} \tag{18}$$

-an identity which can be derived in the framework of [6] and which has an analogue in [15]. Here $\left(F^{AB}_{CD}\right)_{JT} = \underset{\sim}{\alpha}_{(j'j)(t't)}(c,b,a)$ corresponds to the 6j - symbols $\begin{Bmatrix} C & A & J \\ B & D & K \end{Bmatrix}^{RW}$ in [15].

* Because of the same reason we get, e.g., only even spins in the product $\Phi_A \times \Phi_A$ of two undistinguishable fields in disagreement with [7]. We think the reason for this and other disagreements, if we try to extend the results in [7] to the minimal theories, is the unfounded basic argument in the proof in [7] about the invariance of the braid and crossing (fusion) matrices.

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