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A SATISFACTORY FORMALISM FOR MAGNETIC MONOPOLES BY CLIFFORD ALGEBRAS. (°)

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Abstract - We approach the problem of electromagnetism with magnetic monopoles by the physically interesting and mathematically powerful formalism of Clifford Algebras, which provide a natural language for Minkowski space-time (Dirac algebra) and Euclidean space (Pauli algebra). We construct a Lagrangian and Hamiltonian formalism for interacting monopoles, which overcomes many of the long-standing difficulties that are known to plague the approaches till now developed.

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While the events of a two-dimensional Minkowski space-time M(1,1) are sufficiently well represented by ordinary Complex Numbers, when dealing with the events of the four-dimensional Minkowski space-time $M_4 \equiv M(1,3)$ one has of course to look for hypercomplex numbers or, more generally, for the elements of a Clifford Algebra. For instance, the Dirac algebra is known to be the Clifford Algebra constructed on M_4, and can therefore be regarded as the natural algebra[1] for M_4. From this point of view, the Dirac vectors γ_μ are nothing but a basis of unit vectors for Minkowski space-time, such that $\gamma_o{}^2 = +1$; $\gamma_i{}^2 = -1$; and $\gamma_\mu \cdot \gamma_\nu = 0$ for $\mu \neq \nu$ (that is to say, they are four pseudo-orthonormal vectors). Let us moreover recall that the Pauli algebra is a subalgebra of Dirac's: It is enough to choose a time-direction γ_o in the Dirac algebra in order to single out the Pauli subalgebra, in the sense that for every Dirac vector x in M_4

$$x\gamma_o = x^\mu \sigma_\mu = t + x\vec{\sigma}_x + y\vec{\sigma}_y + z\vec{\sigma}_z \,, \qquad (1)$$

where $\sigma_o \equiv \mathbb{1}$ and $\vec{\sigma}_i$ are the ordinary Pauli matrices $(i = 1,2,3)$.[≠1] Incidentally, we prefer to use the boldface type for $\vec{\sigma}_i$ since in Pauli algebra they are a basis for the three-dimensional space R_3; we shall analogously set in boldface all the Pauli vectors (e.g., $\vec{E}$), but not the pseudo-vectors (or Pauli "bivectors").[1] Notice that by eq.(1) the generic Dirac vector x has been decomposed into a temporal part (a Pauli scalar, t) plus a spatial part (a Pauli vector, $x^i\vec{\sigma}_i$), and thus represented in the Pauli algebra. It should moreover be mentioned that we are exploiting the Clifford Algebras in terms of "multivectors", and in particular by Hestenes' language, which suits space-time quite well. Let us recall, at last, that the Clifford product $x\gamma_o$ is the sum of the internal product $x \cdot \gamma_o$ and the wedge product $x \wedge \gamma_o$.

In the present formalism, the ordinary electromagnetic field $F^{\mu\nu}$ is described by the Dirac "bivector" F whose expansion in the bivector basis is

$$F = F^{\mu\nu}\gamma_{\mu\nu} \; ; \qquad \left[\gamma_{\mu\nu} \equiv \gamma_\mu \wedge \gamma_\nu\right] \, , \tag{2}$$

where one should bear in mind that (when defining the Pauli algebra as a Dirac subalgebra) the identification $\gamma_i\gamma_o \equiv \vec{\sigma}_i$ occurs, so that the Dirac bivectors $\gamma_{\mu\nu}$ give origin to Pauli vectors and to Pauli bivectors σ_{ik}:

$$\gamma_{io} \equiv \vec{\sigma}_i \; ; \quad \gamma_{ik} \equiv \sigma_{ik} \; .$$

As a consequence, we can decompose F in Pauli algebra <u>either</u> as $F = \vec{E} + H$ (in which case H is a Pauli pseudo-vector), <u>or</u> as[2]:

$$F = \vec{E} + i\vec{H} \, , \tag{3}$$

in which case $\vec{E}$ and $\vec{H}$ are Pauli vectors. Following eq.(3), let us identify the ordinary magnetic field with the pseudo-vector $i\vec{H}$. This is equivalent to write $\vec{E} = \frac{1}{2}(F - F^*)$ and $i\vec{H} = \frac{1}{2}(F + F^*)$, with

$$F^* \equiv -\vec{E} + i\vec{H} \, , \tag{4}$$

the star operation being the space-inversion in Pauli algebra, i.e. the operator carrying $\sigma_i \to -\sigma_i$. The ordinary Maxwell equations will then read

$$\partial\!\!\!/ F = J_e \, , \tag{5}$$

where $\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$ is the square-root of the ordinary D'Alambertian operator, and $J_e \equiv J_e^{\ \mu}\gamma_\mu$ is a Dirac vector (representing an electric charge 4-current) whose representation in Pauli algebra is:

$$J_e\gamma_o = \rho_e + \vec{J}_e \tag{6}$$

$\rho_e \equiv J_e \cdot \gamma_o$ being the electric charge density (a Pauli scalar) and $\vec{J}_e \equiv J_e \wedge \gamma_o$ the electric charge 3-current (a Pauli vector). By Clifford multiplication on the left of eq.(5) by γ_o one gets the Maxwell equations in the ordinary differential form $\left[\text{remember that } -\vec{\nabla}\wedge \equiv i\vec{\nabla}\times, \text{ so that e.g. } i\vec{\nabla}\wedge\vec{H} = \text{rot}\,\vec{H}\right]$:

$$\vec{\nabla}\cdot\vec{E} + (\partial_o\vec{E} + i\vec{\nabla}\wedge\vec{H}) + i(\partial_o\vec{H} + \vec{\nabla}\wedge\vec{E}) + i\vec{\nabla}\cdot\vec{H} = \rho_e - \vec{J}_e \, . \tag{7}$$

Let us pay attention to the pseudo-scalar unit of the Dirac algebra, $\gamma_5 \equiv \gamma_o\wedge\gamma_1\wedge\gamma_2\wedge\gamma_3 \equiv \gamma_o\gamma_1\gamma_2\gamma_3$, and observe that, from eq.(3),

$$\hat{F} \equiv \gamma_5 F = i\vec{E} - \vec{H} \, , \tag{8}$$

so that the multiplication by γ_5 corresponds to the fundamental "duality exchange" $\vec{H} \to \vec{E}$; $\vec{E} \to -\vec{H}$. For Maxwell equations from eq.(5) we obtain $\not{\partial}\hat{F} = \gamma_5 J_e$, so that the roles of electric and magnetic field get interchanged when we pass from the vector J_e to the pseudo-vector $\hat{J} \equiv \gamma_5 J_e$. Let us recall, incidentally, that $\gamma_5 \to i$ when going from Dirac to Pauli algebra.

At last, let us introduce the Dirac vector A (electromagnetic potential) such that[1] $F \equiv \not{\partial} A = \not{\partial}\cdot A + \not{\partial}\wedge A$, and therefore

$$\not{\partial}\cdot A = 0 ; \qquad \not{\partial}\wedge A = F ,$$

which yield the standard definitions $\vec{E} = -\partial_o A - \vec{\nabla}\Phi$; $\vec{H} = -i\vec{\nabla}\wedge\vec{A} \equiv \vec{\nabla}\times\vec{A}$, provided that $A\gamma_o = A\cdot\gamma_o + A\wedge\gamma_o \equiv \Phi + \vec{A}$. The ordinary Maxwell equations (5) assume the form

$$\not{\partial}^2 A = J_e ; \qquad \not{\partial}\cdot A = 0 . \tag{9}$$

But ordinary Maxwell equations are not fully satisfactory and symmetrical enough, as wellknown. For instance, it is trivial to realize that in the r.h.s. of eq.(7) there are lacking a pseudo-scalar density $-i\rho_m$ and a pseudo-vectorial (= bivectorial) current $i\vec{J}_m$. Such two quantities form a Dirac pseudo-vector $\gamma_5 J_m \equiv \gamma_5(\rho_m + \vec{J}_m)\gamma_o = \gamma_5\gamma_o(\rho_m - \vec{J}_m)$. By associating[3] those terms, as usual, with magnetic monopole currents, the completed Maxwell equations write in the Dirac algebra :

$$\not{\partial} F = \bar{J} ; \qquad \left[\bar{J} \equiv J_e + \gamma_5 J_m\right] \tag{10}$$

with $J_m\gamma_o \equiv \rho_m + \vec{J}_m$; $J_e\gamma_o \equiv \rho_e + \vec{J}_e$; while, in Pauli's, the r.h.s. of eq.(7) takes on the form $\rho_e - \vec{J}_e + i(\rho_m - \vec{J}_m)$. Notice that $\not{\partial}\cdot\bar{J} = 0$ since $\not{\partial}\cdot\bar{J} = \not{\partial}\cdot J_e - \gamma_5\not{\partial}\cdot J_m$.

We have to look, now, for a new potential $\bar{A}$ such that

$$F = \not{\partial}\bar{A} . \tag{11}$$

Preliminarly, given the "second" current J_m , it will be possible [cf. eqs.(9)] to find out a "second" potential B such that $\not{\partial}^2 B = J_m$; $\not{\partial}\cdot B = 0$; so that we are surely allowed to write down the "completed" Maxwell equations in the form

$$\not{\partial}^2\bar{A} = \bar{J} ; \qquad \not{\partial}\cdot\bar{A} = 0 \tag{12}$$

by setting:

$$\overline{A} \equiv A + \gamma_5 B \; ; \qquad \overline{J} \equiv J_e + \gamma_5 J_m \; . \tag{12'}$$

Notice that also B and J_m are Dirac vectors (whilst $\gamma_5 B$ and $\gamma_5 J_m$ are Dirac pseudo-vectors, i.e. Dirac trivectors). It is easy to realize that the generic duality transformation $\exp(\gamma_5 \phi)$ corresponds to a rotation by ϕ in the space $M_4 \oplus \hat{M}_4$, with $\hat{M}_4 \equiv \gamma_5 M_4$, since from eqs.(10) one gets $\partial\!\!\!/ [\exp(\gamma_5 \phi) F] = [\exp(\gamma_5 \phi)] (J_e + \gamma_5 J_m) = J_e \cos\phi - J_m \sin\phi + \gamma_5 (J_e \sin\phi + J_m \cos\phi)$. Notice, moreover, that by eq.(2) and the expansion $\overline{A} = \overline{A}_\nu \gamma^\nu$ (where, as we saw before, $\overline{A}_\nu \equiv A_\nu + \gamma_5 B_\nu$), we can infer from eq.(11) that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \varepsilon_{\mu\nu\rho\sigma} \partial^\rho B^\sigma \quad , \tag{13}$$

which are nothing but the Cabibbo-Ferrari relations[3]. The derivation of eq.(13) stands on the important fact that the space of Dirac bivectors is (globally) invariant under the action of γ_5. Actually, γ_5 acts in Dirac algebra as the Hodge star operator[1]: in particular, $\gamma_5 \gamma^\mu \wedge \gamma^\nu = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \gamma_\rho \wedge \gamma_\sigma$.

A last comment: in Minkowski space-time[≠2] we are used to associate J_e with a current of (electrically charged) particles endowed with a Dirac-vector velocity and whose world-line events are Dirac vectors. Then, eqs. (10),(12) suggest associating $\gamma_5 J_m$ with a current of (magnetically charged) monopoles, endowed now with a Dirac-pseudovector velocity and whose world-events are Dirac pseudovectors; so that magnetic poles are expected to populate a second Minkowski space $\hat{M}_4 \equiv \gamma_5 M_4$ orthogonal to M_4 (quantity γ_5 operating a 90° rotation in $M_4 \oplus \hat{M}_4$). However, in our 3-dimensional space R_3 we are able to detect both Pauli vectors and pseudo-vectors (e.g., both $\vec{E}$ and $i\vec{H}$): instructed by such a circumstance, we may be led[4] to identify M_4 with $\hat{M}_4$, and the operation γ_5 would thus transform vectors of M_4 into pseudovectors of the same M_4 (even if, actually, monopoles have not yet been detected).

Lagrangian formalism: Let us define in Dirac algebra the "tilde" operator as follows:

$$D_r \equiv d_1 \wedge d_2 \wedge \ldots \wedge d_r \; ; \qquad \tilde{D}_r \equiv d_r \wedge \ldots \wedge d_2 \wedge d_1 \; ,$$

so that, e.g., $\tilde{F} = -F$ and $\tilde{\overline{J}} = J_e - \gamma_5 J_m$. It is then natural to assume as interaction Lagrangian-density:

$$L_{int} = \tilde{\bar{J}}\cdot\bar{A} = J_e\cdot A + J_m\cdot B + \gamma_5(J_e\cdot B - J_m\cdot A) \qquad (14)$$

in which there explicitly appear the crossed interactions between J_e and B, and between J_m and A. It is essential to notice that L_{int} is invariant under the gauge symmetry $A \to A + \partial\!\!\!/\chi_A$; $B \to B + \partial\!\!\!/\chi_B$, that is to say

$$\bar{A} \to \bar{A} + \partial\!\!\!/\bar{\chi} ,$$

quantities χ_A , χ_B being Dirac scalars and $\bar{\chi} \equiv \chi_A + \gamma_5\chi_B$ (in fact, $\partial\!\!\!/\cdot\bar{J} = 0$). More precisely, let us choose the field (total) Lagrangian-density

$$L = -\tfrac{1}{2}F\cdot F + \tilde{\bar{J}}\cdot\bar{A} = \tfrac{1}{2}\tilde{F}\cdot F + \tilde{\bar{J}}\cdot\bar{A} , \qquad (15)$$

where the potential $\bar{A}$ and its derivative $\partial_\mu\bar{A}$ play the role of generalized coordinate and generalized velocity, respectively[5]. Notice that also the total Lagrangian-density, eq.(15), is gauge invariant.

In the case when the potentials are only those generated by the currents $\bar{J}$ themselves, we can make recourse to the Euler-Lagrange (EL) equations for the Lagrangian (15):

$$\left[\partial_\mu \frac{\partial}{\partial(\partial_\mu\bar{A})} - \frac{\partial}{\partial\bar{A}}\right] L = 0 . \qquad (16)$$

It is immediate to see that they yield the (completely symmetrical) Maxwell equations; in fact, since

$$\frac{\partial L}{\partial\bar{A}} = \tilde{\bar{J}} ; \qquad \frac{\partial L}{\partial(\partial_\mu\bar{A})} = -F\gamma^\mu ,$$

eqs.(16) yield $\partial_\mu(-F\gamma^\mu) = \tilde{\bar{J}}$, that is to say $\partial\!\!\!/F = \bar{J}$, or

$$\partial\!\!\!/^2\bar{A} = \bar{J} . \qquad (12'')$$

We therefore succeeded in defining a Lagrangian (leading to the "completed" Maxwell equations) in which both electric and magnetic charges do interact with both potentials A and B. It should be observed that L is not a scalar, but the sum of a Dirac scalar and a Dirac pseudoscalar; nevertheless we were able to extend[5] the variational principle to our approach: notice that eq.(16) is just an "extremum condition" (in the sense that it requires the Lagrangian to vanish under the action of the EL differential operator, $\mathcal{O}_{EL}$).

<u>Hamiltonian formalism</u>: To pass to the field Hamiltonian-density, for "didactic" purposes let us temporarily adopt the customary tensorial language, by taking advantage of the relation

$$\tfrac{1}{2}\tilde{F}\cdot F \equiv \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\gamma_5\, \varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \; . \tag{17}$$

We may follow, now, the ordinary formalism of classical field-theory. Our EL equations (16) yield

$$\partial_\nu\left[\partial_\mu\bar{A}_\alpha \frac{\partial L}{\partial(\partial_\nu\bar{A}_\alpha)} - \delta_\mu{}^\nu L\right] = 0 \; ,$$

which lead us to assume as energy-momentum tensor the conserved quantity

$$T^{\mu\nu} \equiv \partial^\mu\bar{A}_\alpha \frac{\partial L}{\partial(\partial_\nu\bar{A}_\alpha)} - g^{\mu\nu}L \tag{18}$$

that simply <u>generalizes</u> the standard expression $H=\vec{p}\cdot\vec{V} - L$ (remember that our "generalized coordinates" are $\bar{A}$ and $\partial_\mu\bar{A}$).

By following Landau's procedure[6], we can construct at this point the <u>symmetrical</u> tensor

$$\Theta^{\mu\nu} \equiv T^{\mu\nu} - \partial_\alpha F^{\alpha\mu}\bar{A}^\nu \; , \tag{18'}$$

sometimes called the Belinfante tensor, that we adopt as (symmetrical) Generalized Hamiltonian[6,7]. For instance, the role of the "old" Hamiltonian —representing the energy-density— is played by

$$H \equiv \Theta^{oo} = \tfrac{1}{2}(\vec{E}^2 + \vec{H}^2) - \tilde{\bar{J}}\cdot\bar{A} \tag{19}$$

where the first addendum is the energy-density of the <u>field</u>, and the second one is that of the <u>interaction</u>.

It should be observed, moreover, that the components Θ^{oi} yield the Pointing vector

$$\Theta^{oi} = \vec{E}\times\vec{H} \; , \tag{20}$$

while the further components yield other conserved quantities.

Let us explicitly notice that in eq.(19) the field energy-density is the <u>correct</u> one. As to the interaction energy-density (cf. eq.(15)), it contains scalar ($J_e\cdot A$ and $J_m\cdot B$) and pseudoscalar parts ($J_e\cdot\gamma_5 B$ and $\gamma_5 J_m\cdot A$), since the magnetic current $\gamma_5 J_m$ and the "second potential" $\gamma_5 B$ are just pseudoscalars.

Finally, let us briefly mention how eqs.(17)÷(20) would look like in Clifford Algebra. Equation (18) would write

$$T^{\mu\nu} = P^{\mu}V^{\nu} - g^{\mu\nu}L \,, \tag{21a}$$

where:

$$P^{\mu} \equiv \frac{\partial L}{\partial(\partial_{\mu}\overline{A})} = -F\gamma^{\mu}\,; \tag{21b}$$

$$V^{\nu} \equiv \partial^{\nu}\overline{A} = \gamma^{\nu}\cdot\partial\!\!\!/\overline{A} = \tfrac{1}{2}(\gamma^{\nu}\partial\!\!\!/\overline{A} - \partial\!\!\!/\overline{A}\gamma^{\nu})\,, \tag{21c}$$

and we took advantage of the fact that $\partial\!\!\!/\overline{A} = F$ is a Dirac bivector. Therefore:

$$T^{\mu\nu} = S^{\mu}\cdot\gamma^{\nu} - g^{\mu\nu}\tilde{J}\cdot\overline{A} + \left[S^{\mu}\wedge\gamma^{\nu} + \tfrac{1}{2}\tilde{F}(\gamma^{\mu}\wedge\gamma^{\nu})F\right] \tag{22}$$

where we set

$$2\,S^{\mu} \equiv F\gamma^{\mu}F = -\tilde{F}\gamma^{\mu}F \tag{23}$$

so that $S^{\mu}\cdot\gamma^{\nu}$ is nothing but the Belinfante tensor. One can pass (in the case $\overline{J} = 0$) from $T^{\mu\nu}$ to $\Theta^{\mu\nu} \equiv S^{\mu}\cdot\gamma^{\nu}$ by neglecting[5,7] in the r.h.s. of eq.(22) the square-bracket (i.e., the zero-divergence part). Observe at last that S^{μ} has the form of a probability current (if we identify $F \equiv \psi$) for the field photons[4].

In conclusion, by Clifford Algebras we produced a (gauge invariant) Lagrangian formalism for electromagnetism with electric and magnetic charges, both interacting with both the potentials A and B; and a consistent Hamiltonian formalism, which naturally yields correct results.

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FOOTNOTES

#1 Or, rather, the Pauli matrices are a representation of the $\vec{\sigma}_i \equiv \gamma_i\gamma_o$. In fact, the set $\vec{\sigma}_i$ under Clifford multiplication satisfies the same algebra as the Pauli matrices.

#2 Attention should be paid, however, to the fact that our Minkowski space--time —when regarded as the base of our "Clifford bundle"— has nothing to do with the subspaces M_4 and $\hat{M}_4$ contained in our fibers $\Pi^{-1} \sim C_{1,3}$ (even if the latter are isomorphic to the former).