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A CLASSICAL MODEL OF KALUZA-KLEIN THEORIES WITH TORSION

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ABSTRACT

Recent attempts in which torsion in the extra dimensions of a multidimensional unified theory is seen as a compactifying agent introduce spin condensates with no energy-momentum counterpart. While this is legitimate from the quantum viewpoint, it has no classical limit. We work out a Kaluza-Klein multidimensional extension of an Einstein-Cartan theory with a totally antisymmetric torsion on the spacetime as well as on the internal dimensions, providing a classical background in which more general compactification mechanisms could be analysed. The dimensional reduction of the field equation gives an Einstein-Cartan-Yang-Mills theory. A sufficient condition for the dimensional reduction of a multidimensional model in the presence of torsion is specified.

1. INTRODUCTION

In this paper we work out a simple model of Kaluza-Klein theories with nonvanishing and totally antisymmetric torsion on all the dimensions of a multidimensional extended spacetime, here a principal fibre bundle (U_D,π,U_4,G), where G is a compact and semisimple Lie group (here and afterwards we denote with U_n an n-dimensional metrical space with metric tensor γ and metricity condition $\nabla\gamma = 0$; in our notations V_n is a U_n with vanishing torsion). We derive and discuss the field equations of this model.

The usual 5-dimensional Kaluza-Klein theory (KK) was born in the 1920s[1], furtherly developed by Einstein, Bergmann, Jordan and Thiry[2] and, after some formally and dimensionally generalized theories of KK type in the 1960s, the latter became physically interesting when the first real applications for Yang-Mills fields were known. In the present geometrical language of fibre bundles we can consider KK as the prototype of the modern Multidimensional Unified Theories (MUTs) in their standard version[3,4], where (V_D,π,V_4,G) is the D-dimensional extended spacetime (G is a nonabelian structural group). In these theories the "extra" dimensions are invisible because they are compactified in very little volumes with a typical radius $r \sim 10^{-32}$ cm.

The standards version of MUTs leads to problems due to the multidimensionality of the internal space G: they concern the appearing of a non-vanishing and very large cosmological constant (in a theory with a Kaluza-Klein or a Jordan-Thiry metric) and the absence of a Minkowskian solution for the ordinary 4-dimensional vacuum (in a theory with a Jordan-Thiry metric). A formal solution of these problems (in a Jordan-Thiry framework) consists in introducing a nonvanishing and completely antisymmetric torsion on the internal space G[5], where torsion is linked to spin density as established by the Einstein-Cartan theory (EC) with a first order action principle[6]; this approach leads also to a spontaneous compactification of the "extra" dimensions. Now we want to discuss the physical meaning and relevance of this formal solution.

2. AN ANALYSIS OF K-K THEORIES WITH VERTICAL TORSION

The physical "translation" of the introduction of a nonvanishing torsion in a metrical theory with a first order action principle is the insertion of a matter field in the theory. In particular in our context a totally antisymmetric torsion on G is algebraically correlated to the spin density of a Dirac field[7], more precisely to a pure spin condensate completely antisymmetric in the internal coordinates, as showed in eq.(1) below. Therefore our discussion about the physical meaning of the Orzalesi-Pauri model[5] must examine the possibility of answering the following question: what is physically and how can exist a pure spin condensate (on the internal space)?

Our notations about indices of tensorial quantities defined on a D-dimensional manifold (principal fibre bundle) are: capital Latin indices refer to the whole manifold (that is to say M, N, ... = 1, ..., D), Greek ones refer to the usual 4 dimensions of spacetime (μ, ν, ... = 1, ..., 4) and small Latin ones refer to the remaining N = D-4 dimensions (i, j, ... = 5, ..., D).

It must be stressed that in the Orzalesi-Pauri scheme, extending the standard version of MUTs, a quantum viewpoint can enter in a natural way[8,9]. Indeed in this context, if $\underset{\sim}{S}$ is the torsion tensor defined by eq.(14), one deals with the expectation value $\bar{S}_{ij}{}^{k}$ in the quantum-mechanical vacuum (q-vacuum), that is on the background needed for quantization. Then, since

$$S_{ijk} = S_{[ijk]} = \tilde{S}_{[ijk]} = \frac{8\pi G_D}{c^3} \tau_{[ijk]} = \frac{8\pi G_D}{c^3} \tau_{ijk} \qquad (1)$$

($\underset{\sim}{S}$ is the modified torsion tensor defined by eq.(15), G_D is the D-dimensional gravitational constant defined later on in terms of Newton's constant and $\underset{\sim}{\tau}$ is the spin density tensor defined by eq.(40)) and

$$\tau^{ijk} = \tau^{[ijk]} \sim \bar{\psi} \gamma^{[i} \gamma^{j} \gamma^{k]} \psi , \qquad (2)$$

where ψ is a Dirac field, the requirement that $\bar{S}_{ij}{}^{k} \neq 0$ is equivalent to the requirement that $<\bar{\psi} \ldots \psi>_o \neq 0$. In substance in this quantum view the existence of a ψ_{cl}, a classical solution of the field equations, is not required; in fact here we assume simply

$$\bar{T}_{MN} = < 0 \mid T_{MN} \mid 0 > = 0 , \qquad (3)$$

$$\bar{S}_{MN}{}^{P} = < 0 \mid S_{MN}{}^{P} \mid 0 > = 0 , \quad \text{(if at least one among } M,N,P = 1,\ldots,4), \qquad (4)$$

$$\bar{S}_{ij}{}^{k} = < 0 \mid S_{ij}{}^{k} \mid 0 > \neq 0 , \qquad (5)$$

where T_{MN} is the energy-momentum tensor of a Dirac field[8],

$$T_{MN} \sim (\nabla_M \bar{\psi}) \gamma_N \psi - \bar{\psi} \gamma_N \nabla_M \psi , \qquad (6)$$

and $\bar{T}_{MN}$ is its expectation value in the q-vacuum. In this quantum framework the previous question can therefore be easily answered: a pure spin condensate on G exist when eqs.(3-5) are valid; physically this means in particular that we require the vanishing of the expectation value $\bar{T}_{MN}$ in the q-vacuum and not the vanishing of the energy-momentum tensor T_{MN}: this fact has a great physical importance, as we will verify at once afterwards.

In this communication we want to explore how the scheme put forward by Orzalesi and Pauri should be modified in order to have a classical counterpart. It seems indeed legitimate to test how far the success of a classical description in MUTs can arrive. The previous question must then be considered in a classical framework. From this viewpoint, the field equations require a ψ_{cl} as their solution. In this new situation the model with purely vertical torsion requires

$$T_{MN} = 0 , \qquad (7)$$

$$S_{MN}{}^{P} = 0 \text{ (if at least one among } M,N,P = 1,\ldots,4) , \qquad (8)$$

$$S_{ij}{}^{k} \neq 0 , \qquad (9)$$

instead of eqs.(3-5). But the conditions (7-9) are physically inconsistent, because this means requiring the existence of "matter" carrying spin but not energy and momentum. Moreover classically one does not see how $\underset{\sim}{S}$ can be confined only in some dimensions of an extended spacetime. Thus in a classical framework the Orzalesi-Pauri scheme leads to an unphysical situation: classically a pure spin condensate has no physical reality.

Finally we must remark that here (as in refs.(8)) we "take seriously" the existence of the internal dimensions and then, in a classical view, the model with purely vertical torsion must be rejected on the ground of previous physical considerations. The situation may seem different if one considers the internal space only like a useful formal artifice; in this way, however, condition (9) appears like a formal one on an abstract space, only suitable for solving problems arising within the standard formulation of MUTs, and it is not testable, even in principle. So this way of proceeding does not lead to an increase of knowledge and, in conclusion, it seems contrary to a scientific logic.

Thus hereafter we do not consider spaces with teleparallelism, that is to say spaces with vanishing curvature and nonvanishing torsion, but we work out an extension of MUTs introducing nonvanishing T_{MN} and $\tau_{MN}{}^{P}$ in

$$U_D = U_4 \times U_N \tag{10}$$

with a Kaluza-Klein type metric having signature D-2. We want to emphasize that our procedure follows an "ansatz logic": we will not speculate about the dynamical origin of compactification.

3. THE GEOMETRY OF THE MODEL

Our notations and conventions about the differential geometry of spaces with torsion are generally those of Schouten[(10)], because they are the most commonly used in treatments about EC[(6)] (we will compare the field equations of our model with those of EC). Now we present these conventions and at the same time their link with the more intrinsic language one can use in a Koszul approach to differential geometry[(11)].

In a given basis (a D-bein $\{\underset{\sim}{e}_M\}_{M=1}^{D}$) of a D-dimensional manifold U_D we can define the covariant derivative of a basic vector in terms of its components either as

$$\nabla_{\underset{\sim}{e}_M} \underset{\sim}{e}_N = \Gamma_{MN}{}^{P}\, \underset{\sim}{e}_P \tag{11}$$

or as

$$\nabla_{\underset{\sim}{e}_M} \underset{\sim}{e}_N = \Gamma_{NM}{}^{P}\, \underset{\sim}{e}_P \,. \tag{12}$$

We want to stress that eq.(11) represents a left covariant derivative, that is to say the first index of the connection coefficients denotes the differentiating vector, while (12) is a right covariant derivative, in the same way. Of course in spaces with torsion parallel transport is determined by which of two types of covariant derivatives one chooses; we decide to use always left covariant derivatives hereafter. Moreover we deal in general with anholonomic bases: so

$$[\underset{\sim}{e}_M, \underset{\sim}{e}_N] = c_{MN}{}^P \underset{\sim}{e}_P \; ; \tag{13}$$

$c_{MN}{}^P$ is the object of anholonomity, which "measures" how much a basis differs from a coordinate one. Then we define torsion as

$$\begin{aligned} \underset{\sim}{S}(\underset{\sim}{e}_M,\underset{\sim}{e}_N) &= \frac{1}{2}\,(\nabla_{\underset{\sim}{e}_M} \underset{\sim}{e}_N - \nabla_{\underset{\sim}{e}_N} \underset{\sim}{e}_M + [\underset{\sim}{e}_M, \underset{\sim}{e}_N]) = \\ &= \frac{1}{2}\,(\Gamma_{MN}{}^P - \Gamma_{NM}{}^P + c_{MN}{}^P)\, \underset{\sim}{e}_P = S_{MN}{}^P \underset{\sim}{e}_P \; . \end{aligned} \tag{14}$$

Now the modified torsion tensor $\tilde{S}_{MN}{}^P$ is definible as

$$\tilde{S}_{MN}{}^P = S_{MN}{}^P + 2\,\delta^P_{[M}\, S_{N]Q}{}^Q \; ; \tag{15}$$

it reduces to $S_{MN}{}^P$ when torsion is totally antisymmetric. Finally we can define the contorsion tensor as

$$K_{MN}{}^P = -\,S_{MN}{}^P - S^P{}_{MN} + S_N{}^P{}_M \qquad (= -\,K_M{}^P{}_N) \; . \tag{16}$$

Now it is not difficult to prove that our connection $\Gamma_{MN}{}^P$ can be expressed as

$$\Gamma_{MN}{}^P = \overset{\circ}{\Gamma}_{MN}{}^P - K_{MN}{}^P \quad , \tag{17}$$

where

$$\begin{aligned} \overset{\circ}{\Gamma}_{MN}{}^P &= \frac{1}{2}\,\gamma^{PQ}\,(\partial_N\gamma_{MQ} + \partial_M\gamma_{NQ} - \partial_Q\gamma_{MN}) + \\ &+ \frac{1}{2}(c_{NM}{}^P + \gamma^{PQ}\gamma_{MR}c_{NQ}{}^R + \gamma^{PQ}\gamma_{NR}c_{MQ}{}^R) \end{aligned} \tag{18}$$

is the Levi-Civita connection (hereafter we use the symbol $\circ$ over quantities built in terms of $\overset{\circ}{\Gamma}$ and with vanishing torsion). Moreover we define the Riemann tensor $\underset{\sim}{R}$ as

$$\begin{aligned} \underset{\sim}{R}\,(\underset{\sim}{e}_M,\underset{\sim}{e}_N)\underset{\sim}{e}_P &= (\nabla_{\underset{\sim}{e}_M}\nabla_{\underset{\sim}{e}_N} - \nabla_{\underset{\sim}{e}_N}\nabla_{\underset{\sim}{e}_M} + \nabla_{[\underset{\sim}{e}_M,\underset{\sim}{e}_N]})\underset{\sim}{e}_P = \\ &= \nabla_{\underset{\sim}{e}_M}(\Gamma_{NP}{}^R\underset{\sim}{e}_R) - \nabla_{\underset{\sim}{e}_N}(\Gamma_{MP}{}^R\underset{\sim}{e}_R) + c_{MN}{}^R\nabla_{\underset{\sim}{e}_R}\underset{\sim}{e}_P = \\ &= [2(\partial_{[M}\Gamma_{N]P}{}^Q + \Gamma_{[N|P|}{}^R\,\Gamma_{M]R}{}^Q) + c_{MN}{}^R\Gamma_{RP}{}^Q]\underset{\sim}{e}_Q = R_{MNP}{}^Q\,\underset{\sim}{e}_Q \end{aligned} \tag{19}$$

and so the Ricci tensor is

$$R_{MN} = R_{PMN}{}^P = 2(\partial_{[P}\Gamma_{M]N}{}^P + \Gamma_{[M|N|}{}^R\,\Gamma_{P]R}{}^P) + c_{PM}{}^R\,\Gamma_{RN}{}^P \; , \tag{20}$$

where square brackets have the usual commutative meaning: for instance $A_{[M|N|}B_{P]} = \frac{1}{2}(A_{MN}B_P - A_{PN}B_M)$.

Let us come to the formulation of our model. Its geometry is the same one described in refs.(4), except that for 3 points: 1)here we deal with an extended spacetime with torsion; 2) we always consider left covariant derivatives; 3) as a consequence of point (2) and our definitions of torsion and contorsion, also the decomposition formulae (17-20) are unlike the analogous ones in refs.(4). Our extended spacetime is a principal fibre bundle (U_D,π,U_4,G) with a rule of horizontality (bundle connection) and G is a compact and semisimple Lie group; then a basis of vector fields on U_D is

$$\{\hat{\underset{\sim}{V}}_\mu,\overset{*}{\underset{\sim}{V}}_i\},\quad \mu = 1,\ldots,4, \qquad i = 5,\ldots,D, \tag{21}$$

where $\hat{\underset{\sim}{V}}_\mu$ are the horizontal liftings of the fields $\underset{\sim}{V}_\mu$, which constitute a basis on U_4, and $\overset{*}{\underset{\sim}{V}}_i$ are the fundamental fields of the Lie algebra $\mathfrak{X}(U_D)$, induced by a basis of N left invariant vector fields $\underset{\sim}{V}_i$ of the Lie algebra $L(G)$. (We remark that $\forall\ p\in U_D$ the fields $\overset{*}{\underset{\sim}{V}}_i$ are tangent to the fibre of $x=\pi(p)$ and therefore they are vertical). The commutation rules relating to the basis (21) are

$$[\hat{\underset{\sim}{V}}_\mu,\hat{\underset{\sim}{V}}_\nu] = c_{\mu\nu}{}^k\overset{*}{\underset{\sim}{V}}_k = -F_{\mu\nu}{}^k\overset{*}{\underset{\sim}{V}}_k \tag{22}$$

$$[\hat{\underset{\sim}{V}}_\mu,\overset{*}{\underset{\sim}{V}}_i] = 0 \quad , \tag{23}$$

$$[\overset{*}{\underset{\sim}{V}}_i,\overset{*}{\underset{\sim}{V}}_j] = c_{ij}{}^k\overset{*}{\underset{\sim}{V}}_k = f_{ij}{}^k\overset{*}{\underset{\sim}{V}}_k \quad . \tag{24}$$

Now the requirement for the Kaluza constraint on the D-dimensional metric γ, that is

$$L_{\overset{*}{\underset{\sim}{V}}_i}\gamma = 0 \qquad (i = 5,\ldots,D), \tag{25}$$

the γ-ortogonality condition and the successive restriction to a Killing-Cartan bi-invariant vertical metric (for which $f_{ijk} = f_{[ijk]}$) lead to work with a γ of Jordan-Thiry type, that is ($x\in U_4$)

$$\hat{\gamma}_{MN} = \left(\begin{array}{c|c} g_{\mu\nu}(x) & 0 \\ \hline 0 & \phi(x)\delta_{ij} \end{array}\right) = \hat{\gamma}_{MN}(x) \quad . \tag{26}$$

here and afterwards the symbol ^ denotes quantities referring to the lifted system $(\hat{\underset{\sim}{V}}_\mu,\overset{*}{\underset{\sim}{V}}_i) = (\hat{\partial}_\mu,\hat{\partial}_i) = \hat{\partial}_M$.

Now, in an analogous way, we can consider a Kaluza constraint on the D-dimensional contorsion; so, if $K_{MNP} = \gamma_{PR}K_{MN}{}^R$, we request

$$L_{\overset{*}{\underset{\sim}{V}}_i}K_{MNP} = 0 \qquad (i = 5,\ldots,D), \tag{27}$$

This condition leads immediately to the following decomposition formulae for the components of $K_{MNP}(x\varepsilon U_4,\ y\varepsilon G)$:

$$K_{\mu\nu\rho} = K_{\mu\nu\rho}(x) \ , \qquad (28a)$$

$$K_{\mu\nu j} = h_{\mu\nu p}(x)\ D_j{}^p(y) \ , \qquad (28b)$$

$$K_{\mu jk} = h'_{\mu pq}(x)\ D_j{}^p(y)\ D_k{}^q(y) \ , \qquad (28c)$$

$$K_{jkl} = h''_{pqr}(x)\ D_j{}^p(y)\ D_k{}^q(y)\ D_l{}^r(y) \ , \qquad (28d)$$

where $D_i{}^j$ are the elements of the adjoint representation matrix[4]. In particular it is important to remark that in this way terms square in the contorsion of the type $K_{MNP}\ K^{MNP}$ depend only on $x\varepsilon U_4$, being constant on the fibre of $x\ \forall p\varepsilon\pi^{-1}(x)\epsilon U_D$, because

$$\overset{*}{\underset{\sim}{V}}_i(K_{MNP}\ K^{MNP}) = 0 \qquad \forall\ M,N,P = 1,\dots,D \quad \forall\ i = 5,\dots,D \ , \qquad (29)$$

as we can see by a straightforward calculation using the fundamental relation[4]

$$\overset{*}{\underset{\sim}{V}}_i\ D_j{}^m = f_{ij}{}^k\ D_k{}^m \qquad (30)$$

and the bi-invariance of the vertical metric.

Hereafter we consider the simplifying condition of a totally antisymmetric torsion which is nonvanishing on all the dimensions of U_D (then $K_{MN}{}^P = -\ S_{MN}{}^P$ in eq.(16)). If we remember that the only nonvanishing coefficients $c_{MN}{}^P$ are $c_{\mu\nu}{}^k$ and $c_{ij}{}^k$, then from eqs.(17-18) and (22-28) we can make explicit the geometry of our model (with Kaluza constraints): so we obtain (in the lifted system)

$$\hat{\Gamma}_{\mu\nu}{}^{\rho} = \overset{\circ}{\Gamma}_{\mu\nu}{}^{\rho(4)} - K_{\mu\nu}{}^{\rho(4)} \ , \qquad (31a)$$

$$\hat{\Gamma}_{\mu\nu}{}^{k} = -\ \hat{\Gamma}_{\nu\mu}{}^{k} = \frac{1}{2}\ F_{\mu\nu}{}^{k} - \hat{K}_{\mu\nu}{}^{k} \ , \qquad (31b)$$

$$\hat{\Gamma}_{\mu i}{}^{\rho} = \frac{1}{2}g^{\rho\nu}F_{\nu\mu}{}^{k}\ \phi\ \delta_{ik} - \hat{K}_{\mu i}{}^{\rho} \ , \qquad (31c)$$

$$\hat{\Gamma}_{i\mu}{}^{\rho} = \frac{1}{2}g^{\rho\nu}F_{\nu\mu}{}^{k}\ \phi\ \delta_{ik} - \hat{K}_{i\mu}{}^{\rho} \ , \qquad (31d)$$

$$\hat{\Gamma}_{\mu i}{}^{k} = \frac{1}{2}(\underset{\sim}{V}_{\mu}\ln\phi)\ \delta_i^k - \hat{K}_{\mu i}{}^{k} \ , \qquad (31e)$$

$$\hat{\Gamma}_{i\mu}{}^{k} = \frac{1}{2}(\underset{\sim}{V}_{\mu}\ln\phi)\ \delta_i^k - \hat{K}_{i\mu}{}^{k} \ , \qquad (31f)$$

$$\hat{\Gamma}_{ij}{}^{\rho} = -\ \frac{1}{2}g^{\rho\mu}(\underset{\sim}{V}_{\mu}\phi)\ \delta_{ij} - \hat{K}_{ij}{}^{\rho} \ , \qquad (31g)$$

$$\hat{\Gamma}_{ji}{}^{\rho} = -\frac{1}{2} g^{\rho\mu} (\nabla_{\underset{\sim}{\mu}} \phi)\, \delta_{ij} - \hat{K}_{ji}{}^{\rho} , \tag{31h}$$

$$\hat{\Gamma}_{ij}{}^{k} = - \hat{\Gamma}_{ji}{}^{k} = \frac{1}{2} f_{ji}{}^{k} + \hat{K}_{ji}{}^{k} , \tag{31i}$$

where the superscript $^{(4)}$ denotes quantities defined in the 4 usual dimensions, and

$$\begin{aligned} \hat{R}_{\mu\nu} = {} & \mathring{R}^{(4)}_{\mu\nu} - \hat{\nabla}_{\rho} K_{\mu\nu}{}^{\rho(4)} - \hat{\nabla}_{k} \hat{K}_{\mu\nu}{}^{k} - \frac{N}{2} \hat{\mathring{\nabla}}_{\nu} (\hat{\mathring{\nabla}}_{\mu} \ln \phi) - \\ & - \frac{N}{4} (\hat{\mathring{\nabla}}_{\nu} \ln \phi)(\hat{\mathring{\nabla}}_{\mu} \ln \phi) - \frac{1}{2} \phi F_{\rho\mu}{}^{i} F^{\rho}{}_{\nu i} - \\ & - K_{\lambda\nu}{}^{\rho(4)} K_{\mu\rho}{}^{\lambda(4)} - 2 \hat{K}_{\rho\nu}{}^{i} \hat{K}_{\mu i}{}^{\rho} - \hat{K}_{i\nu}{}^{j} \hat{K}_{\mu j}{}^{i} , \end{aligned} \tag{32}$$

$$\begin{aligned} \hat{R}_{\mu i} = {} & \frac{1}{2} \hat{\mathring{\nabla}}_{\rho} (\phi F^{\rho}{}_{\mu i}) - \hat{\nabla}_{\rho} \hat{K}_{\mu i}{}^{\rho} - \hat{\nabla}_{k} \hat{K}_{\mu i}{}^{k} + \\ & + \frac{N}{4} F^{\rho}{}_{\mu i} (\hat{\mathring{\nabla}}_{\rho} \ln \phi) - \frac{1}{4} (\hat{\mathring{\nabla}}^{\rho} \phi) F_{\mu\rho i} - \\ & - \hat{K}_{\lambda i}{}^{\rho} K_{\mu\rho}{}^{\lambda(4)} - 2 \hat{K}_{ji}{}^{\rho} \hat{K}_{\mu\rho}{}^{j} - \hat{K}_{ji}{}^{k} \hat{K}_{\mu k}{}^{j} , \end{aligned} \tag{33}$$

$$\begin{aligned} \hat{R}_{ij} = {} & \frac{1}{4} \delta_{ij} + \hat{\nabla}_{\mu} \hat{K}_{ji}{}^{\mu} + \hat{\nabla}_{k} \hat{K}_{ji}{}^{k} - \frac{1}{4} \phi^{2} F^{\rho}{}_{\mu j} F^{\mu}{}_{\rho i} - \\ & - \frac{1}{2} \delta_{ij} \hat{\mathring{\nabla}}_{\mu} \hat{\mathring{\nabla}}^{\mu} \phi + \left(\frac{1}{2} - \frac{N}{4}\right) \delta_{ij} (\hat{\mathring{\nabla}}_{\mu} \phi)(\hat{\mathring{\nabla}}^{\mu} \ln\phi) - \\ & - \frac{N}{2} \hat{K}_{ij}{}^{\mu} \hat{\mathring{\nabla}}_{\mu} \ln\phi - \hat{K}_{\mu j}{}^{\rho} \hat{K}_{i\rho}{}^{\mu} - 2 \hat{K}_{\mu j}{}^{k} \hat{K}_{ik}{}^{\mu} - \hat{K}_{kj}{}^{l} \hat{K}_{il}{}^{k} , \end{aligned} \tag{34}$$

where

$$\hat{\mathring{\nabla}}_{P} B_{MN}{}^{P} = \hat{\partial}_{P} B_{MN}{}^{P} + \hat{\mathring{\Gamma}}_{PR}{}^{P} B_{MN}{}^{R} - \hat{\mathring{\Gamma}}_{PM}{}^{R} B_{RN}{}^{P} - \hat{\mathring{\Gamma}}_{PN}{}^{R} B_{MR}{}^{P} , \tag{35}$$

$$\begin{aligned} \hat{\nabla}_{P} B_{MN}{}^{P} = {} & \hat{\partial}_{P} B_{MN}{}^{P} + \hat{\Gamma}_{PR}{}^{P} B_{MN}{}^{R} - \hat{\Gamma}_{PM}{}^{R} B_{RN}{}^{P} - \hat{\Gamma}_{PN}{}^{R} B_{MR}{}^{P} = \\ = {} & \hat{\mathring{\nabla}}_{P} B_{MN}{}^{P} + \hat{K}_{PM}{}^{R} B_{RN}{}^{P} + \hat{K}_{PN}{}^{R} B_{MR}{}^{P} \end{aligned} \tag{36}$$

and $\delta_{ij} = - f_{ik}{}^{l} f_{jl}{}^{k}$ (it is our Killing-Cartan vertical metric). Therefore the scalar $\hat{R}$ (which, of course, equals R) is

$$\begin{aligned} \hat{R} = {} & \mathring{R}^{(4)} - \frac{1}{4} \phi F_{\rho\mu}{}^{i} F^{\rho\mu}{}_{i} + \frac{N}{4} \phi^{-1} - N \hat{\mathring{\nabla}}^{\mu} \hat{\mathring{\nabla}}_{\mu} \ln\phi + \\ & + \frac{N(1-N)}{4} (\hat{\mathring{\nabla}}^{\mu} \ln\phi)(\hat{\mathring{\nabla}}_{\mu} \ln\phi) - \hat{K}_{P}{}^{MN} \hat{K}_{MN}{}^{P} = = \hat{R}^{(\text{JT-MUTs})} - \hat{K}_{P}{}^{MN} \hat{K}_{MN}{}^{P} . \end{aligned} \tag{37}$$

4. DIMENSIONAL REDUCTION AND FIELD EQUATIONS

Following the example of EC we define (λ_D is the D-dimensional cosmological constant, $\hat{\gamma} = \det(\hat{\gamma}_{MN})$)

$$A_g = \frac{c^3}{16\pi G_D} \int (R - \lambda_D) \sqrt{|\hat{\gamma}|}\, d^D x \tag{38}$$

as the action of the D-dimensional spacetime continuum,

$$A_m = \frac{1}{cV_G} \int L_m(\psi,\ \partial\psi,\ \underset{\sim}{\gamma},\ \partial\underset{\sim}{\gamma},\ \underset{\sim}{S})\, d^D x \tag{39}$$

as the action of the matter field; we also use

$$\sqrt{|\hat{\gamma}|}\ \tau_P^{\ NM} = \phi^{N/2} \sqrt{|g|}\ \tau_P^{\ NM} = \frac{\delta L_m}{\delta K_{MN}^{\ \ \ P}} \tag{40}$$

to define the spin density tensor $\tau_P^{\ NM}$, and

$$\sqrt{|\hat{\gamma}|}\ \overset{\circ}{T}{}^{MN} = \phi^{N/2} \sqrt{|g|}\ \overset{\circ}{T}{}^{MN} = 2\ \frac{\delta L_m}{\delta \gamma_{MN}} \tag{41}$$

to define the symmetric energy-momentum tensor $\overset{\circ}{T}{}^{MN}$; finally the energy-momentum tensor of our model is

$$T^{MN} = \overset{\circ}{T}{}^{MN} + c\ \nabla_P\ (\tau^{MNP})\ . \tag{42}$$

If we now apply the Kaluza constraints (on $\underset{\sim}{\gamma}$ and $\underset{\sim}{K}$), in particular $R = R(x)$ and so on, through a simple dimensional reduction, eq.(38) becomes ($\hat{\gamma}=\phi^N g=\phi^N \det(g_{\mu\nu})$, $V_G = \int d^N x < +\infty$ and $G = G_D/V_G$)

$$A_g = \frac{c^3}{16\pi G_D} \int (R - \lambda_D)\ \phi^{N/2} \sqrt{|g|}\ d^4x\ d^N x =$$

$$= \frac{c^3}{16\pi G_D} \int \Big(\overset{\circ}{R}{}^{(4)} - \lambda_D + \frac{N}{4}\phi^{-1} - \frac{1}{4}\ \phi F_{\mu\nu}^{\ \ \ K}\ F^{\mu\nu l}\delta_{kl} -$$

$$- N\ \hat{\overset{\circ}{\nabla}}{}^{\mu}\ \hat{\overset{\circ}{\nabla}}_{\mu}\ \ln\phi + \frac{N(1-N)}{4}\ (\hat{\overset{\circ}{\nabla}}{}^{\mu}\ \ln\phi)\ (\hat{\overset{\circ}{\nabla}}_{\mu}\ \ln\phi) -$$

$$- \hat{K}_P{}^{MN} \hat{K}_{MN}{}^P) \, \phi^{N/2} \sqrt{|g|} \, d^4x \, . \qquad (43)$$

We stress that in eq.(43) a Yang-Mills Lagrangian density appears ($- \frac{\phi}{4} F_{\mu\nu}{}^k F^{\mu\nu l} \delta_{kl}$); if $x \in U_4$ and $y \in G$, then $F = F(x)$ and $\mathcal{F} = \mathcal{F}(x,y)$: here the replacement of $\mathcal{F}$ in terms of F occurs by application of

$$\mathcal{F}_{\mu\nu}{}^k(x,y) = F_{\mu\nu}{}^m(x) \, D^{-1}{}_m{}^k(y) \qquad (44)$$

and

$$\delta_{kl} = D_k{}^i \, D_l{}^j \, \delta_{ij} \quad , \qquad (45)$$

where $D_i{}^j$ are the elements of the usual adjoint representation matrix.

However we impose the Kaluza constraints at a later step, on the field equations: the difference is substantial. So, by applying a Hilbert variational principle to $A_{g+m} = A_g + A_m$, we can impose:

$$\frac{\delta A_{g+m}}{\delta \hat{\gamma}_{MN}} = 0 \, , \qquad \frac{\delta A_{g+m}}{\delta \hat{S}_{MN}{}^P} = 0 \quad , \qquad (46)$$

obtaining in this way the following field equations:

$$\hat{R}_{MN} - \frac{1}{2} \hat{\gamma}_{MN}(R - \lambda_D) = \frac{8\pi G_D}{c^4} \hat{T}_{MN} \, , \qquad (47a)$$

$$\hat{S}_{MNP} = - \hat{K}_{MNP} = \frac{8\pi G_D}{c^3} \hat{\tau}_{MNP} \, . \qquad (47b)$$

We now isolate in (47a) the Riemannian terms, use (47b) to replace torsion in terms of spin densities and carry out a dimensional reduction making use of conditions (26-27). By setting ($\overset{\circ}{R}{}^{(G)}_{ij} = \frac{1}{4} \delta_{ij}$, $\overset{\circ}{R}{}^{(G)} = \frac{N}{4}$),

$$\int d^N y \, \hat{T}_{MN}(x,y) = \hat{T}_{MN}(x) \quad \text{and} \quad \int d^N y \, \hat{\tau}_{MNP}(x,y) = \hat{\tau}_{MNP}(x) \, ,$$

we find the three following equations:

$$\overset{\circ}{R}{}^{(4)}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \overset{\circ}{R}{}^{(4)} =$$

$$\begin{aligned}
&= \frac{1}{2}\,\phi\, F_{\rho\mu}{}^{i}\, F^{\rho}{}_{\nu i} + \frac{N}{2}\,\hat{\mathring{\nabla}}_{\nu}\,(\hat{\mathring{\nabla}}_{\mu}\ln\phi) + \frac{N}{4}\,(\hat{\mathring{\nabla}}_{\nu}\ln\phi)(\hat{\mathring{\nabla}}_{\mu}\ln\phi) + \\
&+ \frac{1}{2}\, g_{\mu\nu}\Big(-\lambda_{D} + \frac{N}{4}\,\phi^{-1} - \frac{1}{4}\,\phi\, F_{\rho\lambda}{}^{i}\, F^{\rho\lambda}{}_{i} - \\
&- N\,\hat{\mathring{\nabla}}^{\rho}\,\hat{\mathring{\nabla}}_{\rho}\ln\phi + \frac{N(1-N)}{4}\,(\hat{\mathring{\nabla}}^{\rho}\ln\phi)\,(\hat{\mathring{\nabla}}_{\rho}\ln\phi)\Big) + \frac{8\pi G}{c^{4}}\,\hat{\mathring{T}}_{\mu\nu} + \\
&+ \Big(\frac{8\pi G}{c^{3}}\Big)^{2}\Big(\tau_{\lambda\nu}{}^{\rho(4)}\,\tau_{\mu\rho}{}^{\lambda(4)} + 2\hat{\tau}_{\rho\nu}{}^{i}\,\hat{\tau}_{\mu i}{}^{\rho} + \\
&+ \hat{\tau}_{i\nu}{}^{j}\,\hat{\tau}_{\mu j}{}^{i} - \frac{1}{2}\, g_{\mu\nu}\,\hat{\tau}_{P}{}^{MN}\,\hat{\tau}_{MN}{}^{P}\Big)\,,
\end{aligned} \tag{48}$$

$$\begin{aligned}
&\hat{\mathring{\nabla}}_{\rho}\,(\phi\, F^{\rho}{}_{\mu i}) = -\frac{N}{2}\, F^{\rho}{}_{\mu i}\,(\hat{\mathring{\nabla}}_{\rho}\ln\phi) + \frac{1}{2}\,(\hat{\mathring{\nabla}}^{\rho}\phi)\, F_{\mu\rho i} + \frac{16\pi G}{c^{4}}\,\hat{\mathring{T}}_{\mu i} + \\
&+ 2\Big(\frac{8\pi G}{c^{3}}\Big)^{2}\Big(\hat{\tau}_{\lambda i}{}^{\rho}\,\tau_{\mu\rho}{}^{\lambda(4)} + 2\,\hat{\tau}_{ji}{}^{\rho}\,\hat{\tau}_{\mu\rho}{}^{j} + \hat{\tau}_{ji}{}^{k}\,\hat{\tau}_{\mu k}{}^{j}\Big)
\end{aligned} \tag{49}$$

and

$$\begin{aligned}
&\mathring{R}^{(G)}_{ij} - \frac{1}{2}\,\delta_{ij}\,\mathring{R}^{(G)} = \frac{1}{4}\,\phi^{2}\, F^{\rho}{}_{\mu j}\, F^{\mu}{}_{\rho i} + \frac{1}{2}\,\delta_{ij}\,\hat{\mathring{\nabla}}_{\mu}\,\hat{\mathring{\nabla}}^{\mu}\phi - \\
&- \frac{N^{2} - 3N + 4}{8}\,\delta_{ij}\,(\hat{\mathring{\nabla}}_{\mu}\phi)(\hat{\mathring{\nabla}}^{\mu}\ln\phi) + \\
&+ \frac{\phi}{2}\,\delta_{ij}\Big(-\lambda_{D} + \frac{N}{4}\,\phi^{-1} - \frac{1}{4}\,\phi\, F_{\rho\mu}{}^{k}\, F^{\rho\mu}{}_{k} - N\hat{\mathring{\nabla}}^{\mu}\hat{\mathring{\nabla}}_{\mu}\ln\phi\Big) + \\
&+ \frac{8\pi G}{c^{4}}\Big(\hat{\mathring{T}}_{ij} - \frac{cN}{2}\,\hat{\tau}_{ij}{}^{\mu}\,\hat{\mathring{\nabla}}_{\mu}\ln\phi\Big) + \\
&+ \Big(\frac{8\pi G}{c^{3}}\Big)^{2}\Big(\hat{\tau}_{\mu j}{}^{\rho}\,\hat{\tau}_{i\rho}{}^{\mu} + 2\,\hat{\tau}_{\mu j}{}^{k}\,\hat{\tau}_{ik}{}^{\mu} + \hat{\tau}_{kj}{}^{l}\,\hat{\tau}_{il}{}^{k} - \frac{1}{2}\,\phi\,\delta_{ij}\,\hat{\tau}_{P}{}^{MN}\,\hat{\tau}_{MN}{}^{P}\Big)\,.
\end{aligned} \tag{50}$$

As a first comment about these three field equations we observe that the limit of MUTs with a JT metric (JT-MUTs) is obtained (compare with refs.(2,3)); moreover the EC combined field equation is[6]

$$\overset{\circ}{R}{}^{(4)}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \overset{\circ}{R}{}^{(4)} = \frac{8\pi G}{c^4} \overset{\circ}{T}_{\mu\nu} +$$

$$+ \left(\frac{8\pi G}{c^3}\right)^2 [- 4 \tau_{\mu}{}^{\lambda}{}_{[\rho} \tau_{|\nu|}{}^{\rho}{}_{\lambda]} - 2\tau_{\mu}{}^{\lambda\rho} \tau_{\nu\lambda\rho} + \tau^{\lambda\rho}{}_{\mu} \tau_{\lambda\rho\nu} +$$

$$+ \frac{1}{2} g_{\mu\nu} (4 \tau_{\sigma[\rho}{}^{\lambda} \tau^{\sigma\rho}{}_{\lambda]} + \tau^{\sigma\lambda\rho} \tau_{\sigma\lambda\rho})] \tag{51}$$

and therefore, considering $\tau_{\mu\nu\rho} = \tau_{[\mu\nu\rho]}$ in eq.(51), one can see easily that also the EC limit of our model is the correct one.

We end by making some final remarks: first of all eq.(49) vouches the covariant nonconservation of Yang-Mills fields; secondly we want to stress explicitly that the model presented here maintains one of the basic properties of a physical theory, because the metricity condition is still valid: in fact

$$\nabla_P \gamma_{MN} = \overset{\circ}{\nabla}_P \gamma_{MN} + K_{PM}{}^R \gamma_{RN} + K_{PN}{}^R \gamma_{MR} = K_{PMN} + K_{PNM} = 0. \tag{52}$$

Moreover, one must consider that, because of the algebraic structure of the field equation (47b), the "torsionic interaction" is a contact one, that is torsion does not propagate outside the spinning matter as a "torsionic wave"; so the terms $\underset{\sim}{\tau}.\underset{\sim}{\tau}$ are nonvanishing only inside the spinning matter. In spite of this, there exists an effect of matter spin on outside spacetime, its influence on the metric tensor; in fact expliciting the left hands of eqs.(48) and (51) we obtain the same second order differential operator acting on $g_{\mu\nu}$ in the study of gravitational waves in general relativity: in comparison with Einstein's theory of gravitation the difference is in the right hand side, the source. Now also a change involving only $\underset{\sim}{\tau}$, with a constant $\overset{\circ}{T}_{\mu\nu}$, implies a change in the metric tensor that propagates in the outside spacetime as ordinary gravitational waves.

5. CONCLUSIONS

In conclusion, we have worked out an explicit model in which torsion has spacetime together with vertical components.

The model avoids the criticism made initially about the Orzalesi-Pauri model, i.e. the fact that it requires in a classical framework presence of "matter" carrying spin but not energy and momentum. From the formal point of view, we analysed the specific form the Kaluza constraint on the torsion must assume in order that a dimensional reduction can be carried out in a simple way.

The dynamics of the model (Eqs. 48, 49, 50) requires further analysis and is under investigation.

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